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The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions,* that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

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The COPLEY MEDAL to Sir JOSEPH DALTON HOOKER, K.C.S.I., F.R.S., for his services to Botanical Science as an investigator, author, and traveller.

A ROYAL MEDAL to Professor HENRY NOTTIDGE MOSELEY, F.R.S., for his numerous Researches in Animal Morphology, and especially his Investigations on Corals and on Peripatus.

A ROYAL MEDAL to Colonel ALEXANDER ROSS CLARKE, R.E., C.B., for his Comparison of Standards of Length and Determination of the Figure of the Earth.

The DAVY MEDAL to JOHN A. R. NEWLANDS for his Discovery of the Periodic Law of the Chemical Elements.

The Bakerian Lecture, "On the Dissociation of some Gases by the Electric Discharge," was delivered by Professor J. J. THOMSON, F.R.S.

The Croonian Lecture, "On *Parieasaurus bombidens* (OWEN), and the Significance of its Affinities to Amphibians, Reptiles, and Mammals," was delivered by Professor H. G. SEELEY, F.R.S.

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(A.)

VOL. 178.

LIST OF ILLUSTRATIONS.

PHILOSOPHICAL TRANSACTIONS.

I. *On the Luni-Solar Variations of Magnetic Declination and Horizontal Force at Bombay, and of Declination at Trevandrum.*

By CHARLES CHAMBERS, *F.R.S., Superintendent of the Colába Observatory, Bombay.*

Received March 24,—Read April 1, 1886.

[PLATES 1–5.]

IN the early attempts to investigate the influence of the moon upon terrestrial magnetism, the observations dealt with extended over periods so limited that little was possible beyond determining the average character of the lunar diurnal variation. This was mainly because magnetic disturbance tends—and especially in extra-tropical regions—to mask the minute variations that depend upon the moon. The series of observations made at the Colába Observatory, Bombay, and discussed in the present paper, extending over twenty-five years in the case of the declination and over twenty-six and a half years in the case of the horizontal force, possesses therefore the double advantage of being originally affected by only the relatively small disturbances of a tropical station, and of being lengthy enough to secure an approximate elimination of such disturbance as is involved in it, even by combination of portions only of the whole body of observations.

2. The instruments used at Colába were made by GRUBB, of Dublin, and are like those described in the report (of 1840) of the Committee of Physics of the Royal Society, the magnets being fifteen inches long. An account of them and of their history will be found in the 'Appendices to the Bombay Magnetical and Meteorological Observations, 1879 to 1882,' pages [84] and [138]: and to this account reference may be made for particulars as to the adjustments and determination of scale coefficients of both the declination and horizontal force magnetometers, and as to the determination of the temperature coefficient of the latter instrument. The following extract is, however, given in full, the matter of it being essential to an understanding of the principal object of this paper.

3. *Method of Reduction of the Observations.*—The mode of treatment adopted is that which was introduced by General Sir EDWARD SABINE, and which is described by him in the 'Proceedings of the Royal Society,' vol. 10, pp. 624–626, in the following words:—"The hourly directions of the magnet are entered in monthly

Tables, having the days of the month in successive horizontal lines, and the hours of the day in vertical columns. The 'means' of the entries in each vertical column indicate the mean direction of the magnet at the different hours of the month to which the Table belongs, and have received the name of 'first normals' On inspecting any such monthly Table, it is at once seen that a considerable portion of the entries in the several columns differ considerably from their respective means or first normals, and must be regarded as 'disturbed observations.' The laws of their relative frequency and amount of disturbance in different years, months, and hours are then sought out by separating for that purpose a sufficient body of the most disturbed observations, computing the amount of departure in each case from the normal of the same month and hour, and arranging the amounts in annual, monthly, and hourly tables. In making these computations, the first normals require to be themselves corrected by the omission in each vertical column of the entries noted as disturbed, and by taking fresh means, representing the normals of each month and hour after this omission, and therefore uninfluenced by the larger disturbances. These new means have received the name of 'final normals,' and may be defined as being the mean directions of the magnet in every month and every hour after the omission from the record of every entry which differed from the mean by a certain amount either in excess or in defect

"In this process there is nothing indefinite, and nothing arbitrary save the assignment of the particular amount of difference from the normal which shall be held to constitute the measure of a large disturbance, and which, for distinction's sake, we may call 'the separating value.' It must be an amount which will separate a sufficient body of disturbed observations to permit their laws to be satisfactorily ascertained, but in other respects its precise value is of minor significancy; and the limits within which a selection may be made, without materially affecting the results, are usually by no means narrow, for it has been found experimentally on several occasions that the ratios by which the periodical variations of disturbance in different years, months, and hours are characterised and expressed do not undergo any material change by even considerable differences in the amount of the separating value. The separating value must necessarily be larger at some stations than at others, because the absolute magnitude of the disturbance variation itself is very different in different parts of the globe, as well as its comparative magnitude in relation to the more regular solar diurnal variation, but it must be a constant quantity throughout at one and the same station, or it will not truly show the relative proportion of disturbance in different years and different months." The words "directions of the magnet" in the extract must be taken when applied to the Colába declination observations to imply those directions as expressed either by the original scale readings or those readings converted into minutes of easterly declination; and, when applied to the horizontal force observations, to imply the directions as expressed by the original scale readings after reduction to a uniform temperature.

4. The period of the declination observations is from 1846·0 to 1871·0, and that of the horizontal force observations from 1846·5 to 1873·0. With the exception of Sundays and eight or ten complete days in each year, the observations were taken continuously at hourly intervals throughout these periods. The entries in the monthly declination Tables of the years 1846 to 1865 were in minutes of arc, in those of the years 1866 to 1870 in original scale readings, and the "separating value" made

use of was 1'·4 or its equivalent 204 scale divisions. The separating value adopted for the horizontal force was the equivalent in scale reading of ·000334 of a C G.S. unit of force.

5. Continuing now the extract—from the monthly Tables already described in it, "all observations having been thrown out which deviated from the 'Final Normal' of each hour by more than the amount of the separating value, new Tables were formed in which each observation was substituted by its excess above the Final Normal of its own hour; or rather, by that excess, *plus* a constant round number Practically, the Final Normal was diminished by the round number, and the difference being then taken between the number found and each observation in the same hour column, all the differences have the *positive* sign, and thus the inconvenience of dealing with *positive* and *negative* numbers is avoided. On the new Tables the observation at the solar hour of each day which was nearest to the time of the moon's crossing the meridian of Bombay (from East to West) was marked with a figure 0 to indicate that that observation must be placed to the 0th hour of the lunar day in Tables having the lunar hours, from 0 to 23, marked in consecutive order at the top of the several columns If only twenty-three observations intervened between two marked ones, they were entered in the Table consecutively as forming a complete lunar day; but, as twenty-four observations generally intervened, the two which were in the nearest correspondence to the same lunar hour were combined together, and the mean of the two treated as a single observation." The times of New Moon, First Quarter, Full Moon, and Last Quarter—taken from the Nautical Almanac, and duly corrected for difference of longitude between Greenwich and Bombay, or Göttingen*—were also marked on the margin of the sheet opposite to the several corresponding solar days, for the groups of days, two before and two after these times respectively, the moon was regarded as at the several quarters; and during the intervening days the designations of the moon's phases were one-eighth, three-eighths, five-eighths and seven-eighths respectively. The entries in the Tables of lunar differences, thus marked, were now distributed, in full lunar days, amongst thirty-two new Tables, called lunar abstracts, of which there was one for each eighth of a cycle of the moon's phases in each quarter of the year. The whole of a lunar day's differences were entered in the lunar abstracts under that variety of phase to which the greater half of the day belonged. Whenever the number of undisturbed observations on a lunar day was less than twelve the whole day's differences were rejected. When all the differences of each category had been entered on their respective lunar Tables, the hourly means were taken on these Tables, and the excess was then found of each of these means above the mean of all the hours; finally, these excesses were converted into force equivalents. The series of hourly excesses thus found may, in a sufficiently extended inquiry, be taken to represent the lunar diurnal variation of declination or horizontal force when the sun and moon have the positions indicated by that particular Table.

6. The following Tables show, for each magnetic element, the converted hourly excesses found on each of the thirty-two lunar abstracts:—†

* Until 1866·0 the records were made at hours of Göttingen Astronomical time; since that date, at hours of Bombay Civil time All the results tabulated or graphically represented in this paper have reference, however, to the lunar day or the solar astronomical day at the place of observation

† The calculations having been made also for the summer and winter half-years, and for the full year, their results for these periods are also shown in the Tables

TABLE I.—Declination at Bombay, showing the Mean Lunar Diurnal Variation at New Moon and One-eighth Phase in each Quarter of the Year.

Phase of moon.	New moon.							One-eighth.						
Bombay lunar hours.	November to January	February to April	May to July	August to October	April to September	October to March	Year.	November to January.	February to April	May to July	August to October	April to September	October to March	Year
0	+ ·000016	000000	− 000012	− 000021	− 000018	+ 000010	− 000004	+ ·000006	− 000002	− 000030	− 000007	− 000019	+ 000002	− 000008
1	+ 000018	− ·000005	− 000018	− ·000017	− 000023	+ 000012	− ·000006	+ 000014	− 000002	− 000012	− 000007	− 000007	+ 000003	− 000002
2	+ ·000008	− 000006	− ·000008	− ·000009	− ·000010	+ 000002	− 000004	+ 000014	+ 000004	− 000004	+ 000015	+ 000001	+ 000013	+ 000008
3	+ ·000006	− ·000004	− ·000002	+ 000013	+ 000003	+ ·000003	+ ·000003	+ 000005	+ ·000006	+ 000006	+ 000010	+ 000008	+ 000005	+ 000006
4	− 000004	− 000004	− ·000002	+ 000021	+ 000008	− ·000004	+ 000003	− 000004	+ 000005	− ·000001	+ 000007	+ 000008	+ 000001	+ 000002
5	− 000008	− 000004	+ ·000017	+ 000015	+ 000015	− ·000006	+ 000005	− 000003	− ·000002	+ 000004	+ 000002	+ 000002	− 000002	000000
6	− ·000012	− ·000007	+ 000014	+ 000008	+ ·000010	− 000009	+ 000001	− 000004	− 000002	− 000002	000000	− 000001	− 000004	− 000002
7	− 000011	− ·000011	+ ·000003	+ 000004	+ ·000001	− 000009	− 000003	− 000005	− 000001	000000	000000	000000	− 000004	− 000001
8	− 000006	− ·000011	+ ·000002	+ ·000006	+ 000002	− 000007	− ·000002	− ·000002	− 000004	000000	+ 000001	000000	− 000003	− 000001
9	− ·000003	− 000005	+ ·000002	+ 000001	+ ·000002	− ·000004	− ·000001	− 000003	+ 000001	000000	− 000003	− 000001	− 000002	− 000001
10	− ·000001	+ ·000001	+ ·000003	+ 000002	+ ·000002	·000000	+ 000001	000000	+ ·000005	000000	− 000002	− 000002	+ 000003	+ 000001
11	+ 000004	+ 000003	+ ·000002	+ ·000001	+ 000001	+ ·000003	+ ·000003	+ 000003	+ 000008	− 000001	− 000004	− 000003	+ 000005	+ 000002
12	+ 000004	+ ·000016	+ ·000008	− 000002	+ 000002	+ ·000008	+ ·000005	+ ·000004	+ ·000006	+ 000001	000000	− 000001	+ 000006	+ 000003
13	+ 000006	+ 000004	− ·000001	− 000001	− 000001	+ ·000005	+ 000002	+ 000006	+ 000001	− 000003	+ 000003	− 000003	+ 000006	+ 000002
14	+ 000004	+ 000010	000000	+ 000003	+ 000001	+ ·000007	+ ·000004	+ 000006	000000	− 000002	+ 000003	− 000003	+ 000006	+ 000002
15	+ 000005	+ 000012	+ ·000002	000000	+ ·000001	+ ·000009	+ 000005	+ 000005	− 000001	+ 000006	+ 000004	+ 000003	+ 000004	+ 000004
16	+ 000001	+ 000001	+ 000005	+ 000004	+ 000003	+ 000002	+ ·000003	+ 000001	− 000004	+ 000014	+ 000001	+ 000010	− 000004	+ 000003
17	− 000004	+ ·000004	+ ·000010	+ 000005	+ 000008	− ·000001	+ 000004	− ·000008	− 000002	+ 000014	+ 000013	+ 000015	− 000006	+ 000005
18	− 000016	+ 000003	+ ·000012	+ 000003	+ ·000010	− ·000010	+ 000001	− 000008	− 000009	+ 000021	+ 000006	+ 000014	− 000008	+ 000003
19	− ·000009	+ 000007	+ ·000012	+ ·000008	+ 000014	− 000005	+ ·000005	− 000010	+ 000005	+ 000010	− 000004	+ 000006	− 000005	+ ·000001
20	− 000001	− ·000003	− 000003	+ 000006	+ 000004	− 000006	000000	− 000006	− ·000003	+ 000010	− 000003	+ 000006	− 000007	000000
21	− ·000008	+ ·000003	− ·000012	− 000012	− 000011	− 000005	− ·000007	− 000004	·000000	− 000002	− 000003	− 000002	− 000003	− 000002
22	− ·000006	+ 000005	− ·000009	− 000022	− 000013	− 000002	− 000007	+ ·000001	− 000005	− 000022	− 000010	− 000017	− 000001	− ·000009
23	+ ·000008	− ·000006	− 000018	− 000018	− ·000021	+ ·000008	− 000008	+ 000001	− 000013	− 000017	− 000013	− 000021	− 000001	− 000010
Number of lunar days' observations in abstract	239	250	264	249	520	482	1002	198	198	200	210	408	398	806

TABLE II.—Declination at Bombay, showing the Mean Lunar Diurnal Variation at First Quarter and Three-eighths Phase in each Quarter of the Year.

Phase of moon.	First quarter						
Bombay lunar hours.	November to January	February to April	May to July	August to October	April to September	October to March	Year
0	+ 000018	+ 000003	− 000010	− 000002	− 000007	+ 000011	+ 000002
1	+ 000012	+ 000003	− 000003	000000	− 000002	+ 000007	+ 000002
2	+ 000006	+ 000004	+ 000001	+ 000001	000000	+ 000004	+ 000002
3	− 000002	− 000002	+ 000003	+ 000002	+ 000002	− 000002	000000
4	+ 000002	− 000002	000000	+ ·000006	− 000001	+ 000003	+ 000001
5	− 000001	− 000004	+ 000001	− 000004	− 000001	− 000004	− 000002
6	− 000006	+ 000001	000000	− 000003	− 000002	− 000003	− 000002
7	− 000002	+ 000001	− 000003	− 000008	− 000004	− 000003	− ·000003
8	− 000006	000000	− 000005	− 000008	− 000005	− 000005	− 000005
9	− 000009	− 000002	− 000008	− 000007	− 000004	− 000008	− 000006
10	− 000005	+ 000001	− 000005	− 000006	− 000005	− 000003	− 000004
11	+ 000004	+ 000011	− 000008	− 000011	− 000008	+ 000005	− 000002
12	+ 000006	+ 000011	− 000012	− 000009	− 000009	+ 000006	− 000001
13	+ 000006	+ 000001	− 000010	− 000002	− 000009	+ 000006	− 000002
14	+ 000006	− ·000001	− 000006	+ 000006	000000	+ 000001	+ 000001
15	+ 000008	− 000004	+ 000004	+ 000014	+ 000007	+ 000003	+ 000005
16	− 000002	− 000004	+ 000020	+ ·000025	+ 000015	+ 000002	+ 000008
17	− 000004	− 000016	+ 000013	+ 000019	+ 000014	− 000009	+ 000002
18	− 000015	− 000009	+ 000013	+ 000013	+ 000011	− 000011	000000
19	− 000015	− 000004	+ 000011	+ 000001	+ 000008	− 000012	− 000002
20	− 000026	− 000005	+ 000013	+ 000001	+ 000009	− 000017	− 000004
21	000000	+ 000004	+ 000002	− 000006	− 000001	000000	− 000001
22	+ 000013	− 000001	− 000010	− 000009	− 000011	+ 000006	− 000003
23	+ 000016	+ 000004	− 000009	− 000001	− 000008	+ 000012	+ 000002
Number of lunar days' observations in abstract	254	262	264	260	522	518	1040

Phase of moon.	Three-eighths						
Bombay lunar hours.	November to January	February to April	May to July	August to October	April to September	October to March	Year
0	+ 000008	+ 000001	− 000001	− 000001	− 000001	+ 000004	+ 000002
1	+ 000005	000000	000000	− 000001	− 000001	+ 000003	+ 000001
2	− 000001	− 000002	− 000005	+ 000007	− 000001	+ 000001	000000
3	− 000002	− 000003	− 000004	+ 000004	− 000002	000000	− 000001
4	− 000002	− 000007	− 000007	+ 000002	− 000004	− 000003	− 000003
5	− 000010	− 000012	− 000008	− 000005	− 000006	− 000010	− 000008
6	− 000011	− 000007	− 000005	− 000002	− 000003	− 000009	− 000006
7	− 000011	− 000005	− 000003	+ 000002	− 000001	− 000007	− 000004
8	− 000010	+ 000003	000000	+ 000003	+ 000003	− 000004	− 000001
9	− 000007	+ 000004	− 000004	− 000004	− 000002	− 000004	− 000003
10	− 000002	+ 000011	− 000004	− 000012	− 000002	− 000001	− 000002
11	+ 000001	+ 000001	− 000010	− 0000 3	− 000009	− 000001	− 000005
12	+ 000011	+ 000001	− 000014	− 000007	− 000011	+ 000006	− 000003
13	+ 000022	+ 000007	− 000013	− 0.0007	− 000009	+ 000014	+ 000002
14	+ 000011	+ 000008	− 000009	+ 000005	− 000004	+ 000011	+ 000004
15	+ 000020	− 000009	+ 000001	+ 000001	000000	+ 000007	+ 000003
16	+ 000006	− 000003	+ 000010	+ 000011	+ 000009	+ 000004	+ 000006
17	− 000006	+ 000011	+ 000024	+ 000016	+ 000020	+ 000003	+ 000011
18	− 000002	+ 000008	+ 000030	+ 000008	+ 000019	+ 000004	+ 000012
19	− 000019	+ 000002	+ 000019	+ 000007	+ 000013	− 000007	+ 000003
20	− 000011	+ 000001	+ 000006	− 000005	+ 000002	− 000006	− 000002
21	− 000004	− 000001	− 000002	− 000001	− 000001	− 000002	− 000002
22	+ 000001	− 000007	− 000004	− 000002	− 000004	− 000001	− 000002
23	+ 000008	− 000002	− 000002	000000	000000	000000	000000
Number of lunar days' observations in abstract	199	198	213	212	417	405	822

TABLE III.—Declination at Bombay, showing the Mean Lunar Diurnal Variation at Full Moon and Five-eighths Phase in each Quarter of the Year

Phase of moon	Full moon.							Five-eighths.						
Bombay lunar hours.	November to January.	February to April.	May to July	August to October.	April to September	October to March	Year.	November to January.	February to April	May to July	August to October	April to September	October to March	Year
0	+ ·000008	·000000	− ·000004	+ ·000002	·000000	+ ·000004	+ ·000002	+ ·000001	+ ·000004	− ·000003	·000000	− ·000001	+ ·000002	+ ·000001
1	+ ·000004	− ·000002	+ ·000002	·000000	+ ·000001	+ ·000001	+ ·000001	+ ·000008	+ ·000009	·000000	+ ·000003	+ ·000003	+ ·000007	+ ·000005
2	+ ·000001	− ·000003	+ ·000003	+ ·000005	+ ·000002	·000000	+ ·000001	+ ·000008	+ ·000002	− ·000004	+ ·000006	+ ·000001	+ ·000004	+ ·000003
3	− ·000006	− ·000007	− ·000001	+ ·000004	·000000	− ·000005	− ·000002	+ ·000010	− ·000007	− ·000001	+ ·000007	+ ·000005	·000000	+ ·000002
4	− ·000007	− ·000006	+ ·000001	+ ·000001	+ ·000001	− ·000006	− ·000003	+ ·000001	− ·000006	+ ·000013	+ ·000005	+ ·000010	− ·000003	+ ·000004
5	− ·000009	− ·000007	+ ·000008	+ ·000005	+ ·000005	− ·000006	·000000	− ·000009	− ·000005	+ ·000015	+ ·000019	+ ·000016	− ·000006	+ ·000006
6	− ·000015	− ·000001	+ ·000006	+ ·000007	+ ·000009	− ·000010	− ·000001	− ·000024	− ·000006	− ·000001	+ ·000009	+ ·000002	− ·000012	− ·000005
7	− ·000011	+ ·000009	+ ·000012	+ ·000009	+ ·000014	− ·000006	+ ·000004	− ·000017	+ ·000004	+ ·000005	+ ·000004	+ ·000004	− ·000007	− ·000001
8	− ·000018	+ ·000004	+ ·000010	+ ·000005	+ ·000007	− ·000007	·000000	− ·000013	·000000	− ·000001	+ ·000001	− ·000001	− ·000006	− ·000003
9	− ·000017	+ ·000006	+ ·000003	− ·000002	+ ·000001	− ·000005	− ·000003	− ·000015	+ ·000007	− ·000005	− ·000002	− ·000005	− ·000003	− ·000004
10	+ ·000001	+ ·000009	− ·000018	− ·000010	− ·000011	+ ·000001	− ·000005	− ·000010	·000000	− ·000013	− ·000017	− ·000016	− ·000004	− ·000010
11	+ ·000009	+ ·000004	− ·000015	− ·000020	− ·000016	+ ·000005	− ·000005	+ ·000008	− ·000002	− ·000020	− ·000006	− ·000015	+ ·000004	− ·000005
12	+ ·000008	+ ·000001	− ·000020	− ·000015	− ·000015	+ ·000002	− ·000006	+ ·000013	− ·000006	− ·000021	− ·000012	− ·000018	+ ·000005	− ·000006
13	+ ·000012	− ·000005	− ·000015	− ·000014	− ·000014	+ ·000004	− ·000005	+ ·000018	− ·000005	− ·000007	− ·000011	− ·000010	+ ·000007	− ·000001
14	+ ·000020	− ·000003	− ·000002	− ·000003	·000000	+ ·000007	+ ·000003	+ ·000015	+ ·000003	+ ·000001	+ ·000001	+ ·000003	+ ·000007	+ ·000005
15	+ ·000010	+ ·000002	+ ·000009	+ ·000011	+ ·000008	+ ·000008	+ ·000008	+ ·000003	+ ·000001	+ ·000006	− ·000001	+ ·000004	+ ·000001	+ ·000002
16	− ·000005	+ ·000003	+ ·000009	+ ·000017	+ ·000011	+ ·000001	+ ·000006	·000000	+ ·000006	+ ·000006	+ ·000004	+ ·000006	+ ·000001	+ ·000004
17	− ·000007	·000000	+ ·000009	+ ·000016	+ ·000010	− ·000001	+ ·000005	+ ·000006	·000000	+ ·000006	− ·000003	+ ·000003	+ ·000001	+ ·000003
18	+ ·000001	− ·000005	+ ·000001	+ ·000007	+ ·000001	+ ·000001	+ ·000001	+ ·000002	− ·000001	+ ·000005	·000000	+ ·000004	·000000	+ ·000002
19	+ ·000006	− ·000002	− ·000003	− ·000007	− ·000006	+ ·000003	− ·000001	− ·000001	− ·000010	+ ·000001	+ ·000003	·000000	− ·000004	− ·000002
20	+ ·000006	− ·000003	·000000	− ·000006	− ·000002	+ ·000001	·000000	− ·000004	− ·000002	·000000	+ ·000003	·000000	− ·000001	− ·000001
21	+ ·000003	− ·000001	+ ·000004	− ·000002	+ ·000001	+ ·000002	+ ·000001	+ ·000002	+ ·000003	+ ·000007	− ·000002	+ ·000004	+ ·000001	+ ·000003
22	+ ·000002	− ·000004	+ ·000001	− ·000003	·000000	− ·000001	− ·000001	− ·000001	+ ·000002	+ ·000005	− ·000001	+ ·000003	·000000	+ ·000001
23	+ ·000004	− ·000001	+ ·000004	+ ·000001	+ ·000004	+ ·000001	+ ·000002	+ ·000003	+ ·000006	+ ·000002	+ ·000001	+ ·000002	+ ·000004	+ ·000003
Number of lunar days' observations in abstract	256	228	254	241	493	486	979	188	192	207	207	406	388	794

TABLE IV.—Declination at Bombay, showing the Mean Lunar Diurnal Variation at Last Quarter and Seven-eighths Phase in each Quarter of the Year.

Phase of moon	Last quarter							Seven-eighths						
Bombay lunar hours.	November to January.	February to April	May to July	August to October	April to September	October to March.	Year	November to January	February to April	May to July	August to October	April to September	October to March	Year
0	+ 000010	+ 000003	− 000007	− 000010	− 000008	+ 000006	− 000001	+ 000015	+ 000008	− 000016	− 000015	− 000013	+ 000010	− 000002
1	+ 000009	− 000002	− 000015	− 000009	− 000014	+ 000005	− 000004	+ 000017	+ 000001	− 000016	− 000004	− 000008	+ 000008	− 000001
2	+ 000011	− 000006	− 000006	− 000007	− 000011	+ 000007	− 000002	+ 000015	− 000001	− 000003	+ 000001	− 000001	+ 000008	+ 000003
3	+ 000014	− 000008	+ 000010	− 000001	+ 000004	+ 000003	+ 000004	+ 000007	− 000005	+ 000007	+ 000009	+ 000005	+ 000005	+ 000005
4	000000	− 000016	+ 000010	+ 000004	+ 000004	− 000005	− 000001	+ 000006	− 000005	+ 000003	+ 000023	+ 000007	+ 000007	+ 000007
5	−·000011	+ 000009	000000	+ 000019	+ 000008	000000	+ 000004	− 000007	+ 000005	+ 000001	+ 000016	+ 000009	− 000001	+ 000004
6	− 000018	− 000003	+ 000010	+ 000015	+ 000012	− 000011	+ 000001	− 000017	+ 000001	+ 000010	+ 000019	+ 000012	− 000005	+ 000003
7	− 000025	− 000002	+ 000010	+·000032	+ 000015	− 000008	+ 000003	− 000017	− 000003	+ 000007	+ 000003	+ 000007	− 000012	− 000003
8	− 000029	− 000008	+ 000010	+ 000005	+ 000003	− 000014	− 000005	− 000013	− 000006	+ 000001	− 000010	− 000003	− 000010	− 000007
9	− 000014	+ 000002	000000	− 000005	− 000002	− 000007	− 000005	− 000004	− 000006	− 000006	− 000004	− 000005	− 000005	− 000005
10	000000	− 000001	− 000017	− 000009	− 000013	− 000001	− 000007	− 000001	+ 000003	− 000008	− 000005	− 000005	000000	− 000003
11	+·000010	+ 000001	− 000011	− 000009	− 000008	+ 000003	− 000003	− 000001	+ 000004	− 000002	− 000002	− 000002	+ 000001	− 000001
12	+ 000011	+ 000004	− 000004	+ 000001	− 000002	+ 000008	+ 000003	000000	+ 000007	− 000001	+ 000004	+ 000002	+ 000004	+ 000002
13	+ 000007	+ 000003	+ 000001	− 000001	000000	+ 000004	+ 000002	+ 000002	000000	+ 000002	− 000002	+ 000001	+ 000001	000000
14	+ 000005	+ 000004	+ 000001	+·000005	+ 000003	+ 000004	+ 000003	+ 000002	+ 000008	+ 000007	000000	+ 000004	+ 000005	+ 000004
15	+ 000001	− 000001	− 000001	+ 000003	− 000002	+ 000002	000000	000000	+ 000002	+ 000003	+ 000006	+ 000005	+ 000002	+ 000003
16	+ 000003	+ 000006	+ 000001	·000000	+ 000001	+ 000003	+ 000002	− 000004	+ 000008	000000	+ 000002	+ 000003	000000	+ 000001
17	000000	+ 000006	+ 000001	− 000002	000000	+ 000002	+ 000001	− 000002	+ 000001	+ 000001	− 000003	+ 000001	− 000002	− 000001
18	− 000003	+ 000001	+ 000002	−·000001	+ 000002	− 000003	− 000001	+ 000002	− 000005	+ 000003	− 000006	+ 000001	− 000003	− 000002
19	000000	− 000004	+ 000002	− 000007	000000	− 000005	− 000002	+ 000002	− 000003	+ 000004	− 000005	+ 000001	− 000001	− 000001
20	+·000001	− 000001	+ 000003	− 000001	+ 000003	− 000003	000000	− 000003	− 000001	+ 000012	− 000004	+ 000006	− 000004	+ 000001
21	+ 000006	+ 000002	+ 000006	− 000006	+ 000003	− 000001	+ 000001	− 000004	− 000003	+ 000005	000000	+ 000001	000000	000000
22	+ 000010	+ 000001	+ 000005	− 000006	+ 000001	+ 000004	+ 000002	000000	− 000001	− 000011	− 000012	− 000011	+ 000001	− 000006
23	+ 000011	+ 000002	− 000003	−·000009	− 000005	+ 000005	000000	+ 000005	− 000009	− 000005	− 000022	− 000014	000000	− 000008
Number of lunar days' observations in abstract	247	255	258	256	515	501	1016	204	198	199	200	402	399	801

TABLE V.—Horizontal Force at Bombay, showing the Mean Lunar Diurnal Variation at New Moon and One-eighth Phase in each Quarter of the Year.

Phase of moon.	New moon.							One-eighth.						
Bombay lunar hours.	November to January.	February to April.	May to July.	August to October.	April to September	October to March.	Year.	November to January.	February to April.	May to July.	August to October.	April to September	October to March.	Year.
0	− ·000008	·000000	− ·000007	− ·000009	− ·000008	− ·000003	− ·000006	+ ·000004	− ·000003	+ ·000013	− ·000011	+ ·000005	− ·000005	·000000
1	− ·000016	− ·000005	− ·000009	− ·000001	− ·000010	− ·000006	− ·000008	− ·000009	− ·000007	+ ·000002	− ·000015	− ·000004	− ·000012	− ·000008
2	− ·000028	− ·000009	− ·000009	− ·000003	− ·000007	− ·000017	− ·000012	− ·000004	·000000	+ ·000001	− ·000025	− ·000007	− ·000008	− ·000008
3	− ·000031	− ·000017	− ·000010	− ·000011	− ·000013	− ·000020	− ·000017	− ·000015	− ·000004	− ·000009	− ·000019	− ·000012	− ·000012	− ·000012
4	− ·000019	− ·000021	− ·000008	− ·000013	− ·000012	− ·000017	− ·000015	+ ·000004	− ·000010	− ·000007	− ·000008	− ·000009	− ·000002	− ·000006
5	− ·000010	− ·000009	+ ·000001	·000000	− ·000001	− ·000007	− ·000004	·000000	− ·000008	− ·000007	− ·000009	− ·000012	− ·000001	− ·000006
6	− ·000007	− ·000007	+ ·000005	− ·000005	+ ·000001	− ·000007	− ·000003	− ·000007	− ·000011	− ·000003	− ·000014	− ·000012	− ·000006	− ·000009
7	·000000	− ·000002	+ ·000006	− ·000009	− ·000002	·000000	− ·000001	− ·000009	− ·000016	+ ·000002	− ·000006	− ·000005	− ·000010	− ·000008
8	·000000	·000000	·000000	− ·000006	·000000	− ·000003	− ·000002	− ·000001	− ·000002	− ·000001	+ ·000001	− ·000002	·000000	− ·000001
9	+ ·000006	+ ·000008	− ·000002	+ ·000003	+ ·000001	+ ·000005	+ ·000003	+ ·000002	+ ·000004	− ·000002	+ ·000008	+ ·000004	+ ·000002	+ ·000003
10	+ ·000005	+ ·000006	− ·000010	·000000	− ·000005	+ ·000006	− ·000001	+ ·000006	− ·000002	·000000	+ ·000005	·000000	+ ·000004	+ ·000002
11	+ ·000001	+ ·000001	·000000	− ·000003	+ ·000001	− ·000001	·000000	+ ·000003	− ·000005	+ ·000006	+ ·000002	+ ·000002	+ ·000001	+ ·000001
12	− ·000005	+ ·000006	+ ·000002	+ ·000003	+ ·000005	− ·000002	+ ·000001	− ·000001	− ·000001	− ·000007	+ ·000007	− ·000004	+ ·000002	− ·000001
13	− ·000005	+ ·000007	+ ·000003	+ ·000001	+ ·000004	− ·000003	·000000	− ·000008	− ·000002	− ·000003	+ ·000001	− ·000003	− ·000004	− ·000003
14	− ·000006	− ·000001	+ ·000005	+ ·000003	+ ·000004	− ·000004	·000000	− ·000013	·000000	− ·000016	+ ·000002	− ·000003	− ·000007	− ·000007
15	− ·000002	− ·000002	+ ·000006	+ ·000002	+ ·000005	− ·000002	+ ·000001	− ·000010	− ·000007	− ·000012	+ ·000006	− ·000005	− ·000007	− ·000006
16	− ·000001	− ·000007	+ ·000008	− ·000001	+ ·000006	− ·000007	·000000	− ·000007	− ·000007	− ·000004	+ ·000003	− ·000004	− ·000004	− ·000004
17	+ ·000003	− ·000003	+ ·000008	+ ·000010	+ ·000009	·000000	+ ·000005	− ·000015	− ·000003	− ·000003	+ ·000004	·000000	− ·000009	− ·000005
18	+ ·000012	+ [illegible]	+ ·000010	+ ·000018	+ ·000013	+ ·000008	+ ·000010	+ ·000007	− ·000002	− ·000002	− ·000010	+ ·000001	+ ·000006	+ ·000003
19	+ ·000016	+ ·000007	+ ·000010	+ ·000004	+ ·000013	+ ·000006	+ ·000009	+ ·000012	+ ·000008	+ ·000001	+ ·000008	+ ·000005	+ ·000009	+ ·000007
20	+ ·000023	+ ·000013	·000000	+ ·000001	+ ·000006	+ ·000013	+ ·000009	+ ·000024	+ ·000027	+ ·000010	+ ·000003	+ ·000018	+ ·000019	+ ·000016
21	+ ·000031	+ ·000017	·000000	+ ·000016	+ ·000007	+ ·000025	+ ·000015	+ ·000019	+ ·000029	+ ·000027	+ ·000013	+ ·000021	+ ·000022	+ ·000022
22	+ ·000024	+ ·000010	− ·000009	·000000	− ·000002	+ ·000016	+ ·000006	+ ·000019	+ ·000019	+ ·000010	+ ·000006	+ ·000013	+ ·000014	+ ·000013
23	+ ·000019	+ ·000012	− ·000002	− ·000004	− ·000004	+ ·000017	+ ·000006	+ ·000008	+ ·000011	+ ·000012	+ ·000018	+ ·000015	+ ·000009	+ ·000012
Number of lunar days' observations in abstract	245	240	274	251	527	483	1010	202	196	198	218	402	412	814

TABLE VI. Horizontal Force at Bombay, showing the Mean Lunar Diurnal Variation at First Quarter and Three-eighths Phase in each Quarter of the Year.

Phase of moon.	First quarter							Three-eighths						
Bombay lunar hours.	November to January	February to April	May to July.	August to October	April to September	October to March	Year	November to January	February to April	May to July	August to October	April to September	October to March	Year
0	+ 000005	000000	− 000002	− 000002	+ 000001	+ 000001	000000	+ 000004	+ 000009	000000	000000	+ 000002	+ 000004	+ 000003
1	+ 000005	+ 000006	000000	+ 000006	+ 000001	+ 000009	+ 000004	+ 000011	− 000002	+ 000002	+ 000005	+ 000007	+ 000002	+ 000004
2	+ 000007	+ 000001	− 000010	− 000004	− 000008	+ 000005	− 000002	− 000001	− 000005	+ 000010	− 000003	+ 000008	− 000008	000000
3	− 000002	+ 000001	− 000009	− 000007	− 000009	+ 000002	− 000004	+ 000013	− 000004	− 000002	− 000002	+ 000008	− 000001	+ 000001
4	+ 000012	− 000007	− 000004	− 000006	− 000007	+ 000005	− 000002	+ 000003	+ 000001	− 000001	− 000005	− 000002	+ 000001	− 000001
5	+ 000009	− 000010	− 000004	− 000012	− 000007	000000	− 000004	+ 000005	+ 000008	− 000001	+ 000007	+ 000004	+ 000006	+ 000005
6	+ 000002	− 000002	+ 000004	− 000016	− 000003	− 000002	− 000003	+ 000011	+ 000012	+ 000004	+ 000009	+ 000007	+ 000011	+ 000009
7	+ 000005	000000	− 000002	− 000011	− 000005	+ 000001	− 000003	+ 000015	+ 000012	− 000002	+ 000008	+ 000003	+ 000013	+ 000008
8	+ 000004	− 000004	− 000005	− 000002	− 000004	+ 000002	− 000002	+ 000019	+ 000017	− 000005	+ 000010	+ 000003	+ 000017	+ 000010
9	+ 000010	− 000002	− 000001	+ 000001	000000	+ 000006	+ 000002	+ 000014	+ 000010	000000	+ 000013	+ 000008	+ 000010	+ 000009
10	+ 000010	+ 000004	000000	− 000002	+ 000001	+ 000006	+ 000003	+ 000016	+ 000013	+ 000007	+ 000014	+ 000011	+ 000014	+ 000012
11	+ 000011	+ 000004	− 000002	− 000007	− 000003	+ 000007	+ 000001	+ 000013	+ 000010	+ 000007	+ 000008	+ 000009	+ 000010	+ 000010
12	+ 000006	+ 000001	+ 000004	− 000004	000000	+ 000004	+ 000001	− 000009	− 000001	− 000001	+ 000003	− 000003	− 000001	− 000002
13	− 000004	− 000008	− 000008	− 000003	− 000006	− 000002	− 000005	− 000024	− 000009	+ 000003	− 000010	− 000008	− 000011	− 000010
14	− 000015	− 000009	− 000010	− 000008	− 000010	− 000009	− 000011	− 000025	− 000022	− 000002	− 000019	− 000014	− 000020	− 000017
15	− 000029	− 000022	− 000013	− 000006	− 000011	− 000023	− 000018	− 000029	− 000025	+ 000001	− 000021	− 000013	− 000023	− 000018
16	− ·000022	− 000015	− 000001	+ 000009	+ 000005	− 000020	− 000008	− 000025	− 000014	− 000002	− 000015	− 000006	− 000021	− 000014
17	− 000018	− 000016	− 000012	+ 000011	− 000002	− 000015	− 000009	− 000027	− 000001	− 000018	− 000004	− 000011	− 000015	− 000013
18	− 000014	− 000002	− 000001	+ 000014	+ 000008	− 000009	− 000001	− 000007	− 000003	− 000011	− 000007	− 000010	− 000005	− 000007
19	− 000010	+ 000004	+ 000006	+ 000022	+ 000013	− 000002	+ 000005	− 000001	− 000009	− 000007	+ 000004	− 000004	− 000002	− 000003
20	+ 000010	+ 000009	+ 000012	+ 000002	+ 000008	+ 000009	+ 000008	+ 000009	− 000004	− 000002	+ ·000009	+ 000004	+ 000003	+ 000003
21	+ 000010	+ 000027	+ 000019	+ 000012	+ 000018	+ 000016	+ 000016	+ 000006	− 000004	+ 000004	− 000003	000000	+ 000002	+ 000001
22	+ 000010	+ 000017	+ 000020	+ 000016	+ 000020	+ 000011	+ 000015	+ 000006	+ 000004	+ 000003	+ 000001	+ 000001	+ 000007	+ 000004
23	+ 000009	+ 000015	+ 000007	+ 000001	+ 000009	+ 000008	+ 000008	+ 000014	+ 000015	+ 000005	000000	+ 000006	+ 000011	+ 000008
Number of lunar days' observations in abstract	263	252	272	268	541	514	1055	205	198	220	209	416	416	832

TABLE VII.—Horizontal Force at Bombay, showing the Mean Lunar Diurnal Variation at Full Moon and Five-eighths Phase in each Quarter of the Year.

Phase of moon	Full moon.							Five-eighths						
Bombay lunar hours.	November to January	February to April.	May to July.	August to October	April to September	October to March.	Year.	November to January.	February to April.	May to July	August to October	April to September	October to March	Year
0	+ ·000002	− ·000006	·000000	− ·000008	+ ·000001	− ·000007	− ·000003	·000000	− ·000018	+ ·000001	− ·000002	·000000	− ·000009	− ·000004
1	+ ·000006	− ·000008	− ·000003	·000000	− ·000001	− ·000001	− ·000001	− ·000003	− ·000009	− ·000003	− ·000004	− ·000002	− ·000006	− ·000004
2	− ·000006	− ·000007	− ·000003	− ·000010	− ·000005	− ·000008	− ·000006	− ·000012	− ·000007	− ·000005	− ·000007	− ·000004	− ·000010	− ·000007
3	− ·000007	− ·000008	− ·000003	− ·000005	− ·000003	− ·000007	− ·000005	− ·000019	− ·000007	+ ·000004	− ·000003	·000000	− ·000012	− ·000006
4	− ·000003	·000000	+ ·000003	− ·000004	+ ·000002	− ·000004	− ·000001	− ·000012	− ·000007	+ ·000005	− ·000005	·000000	− ·000009	− ·000004
5	− ·000001	− ·000003	− ·000002	+ ·000007	+ ·000005	− ·000004	+ ·000001	·000000	·000000	+ ·000015	+ ·000004	+ ·000005	+ ·000006	+ ·000006
6	+ ·000001	+ ·000009	·000000	+ ·000003	+ ·000004	+ ·000002	+ ·000003	+ ·000014	− ·000007	+ ·000002	− ·000002	− ·000001	+ ·000005	+ ·000002
7	+ ·000002	+ ·000010	+ ·000003	− ·000001	+ ·000002	+ ·000005	+ ·000004	+ ·000029	+ ·000020	+ ·000007	+ ·000011	+ ·000006	+ ·000027	+ ·000017
8	+ ·000019	+ ·000011	+ ·000002	+ ·000005	+ ·000003	+ ·000016	+ ·000009	+ ·000036	+ ·000021	+ ·000005	+ ·000012	+ ·000006	+ ·000031	+ ·000019
9	+ ·000025	+ ·000015	+ ·000009	+ ·000018	+ ·000011	+ ·000022	+ ·000017	+ ·000033	+ ·000010	+ ·000001	+ ·000019	+ ·000008	+ ·000024	+ ·000016
10	+ ·000027	+ ·000014	+ ·000001	+ ·000010	+ ·000004	+ ·000022	+ ·000013	+ ·000023	+ ·000020	− ·000002	+ ·000009	+ ·000006	+ ·000019	+ ·000013
11	+ ·000011	+ ·000006	+ ·000003	+ ·000006	+ ·000001	+ ·000011	+ ·000006	+ ·000010	+ ·000023	− ·000006	·000000	− ·000002	+ ·000015	+ ·000007
12	− ·000007	+ ·000009	+ ·000007	− ·000004	+ ·000005	− ·000002	+ ·000001	+ ·000005	+ ·000017	− ·000009	+ ·000011	+ ·000004	+ ·000007	+ ·000006
13	− ·000008	− ·000001	+ ·000004	− ·000002	·000000	− ·000004	− ·000002	+ ·000007	+ ·000005	− ·000011	+ ·000014	− ·000001	+ ·000009	+ ·000004
14	− ·000020	− ·000011	− ·000007	− ·000004	− ·000011	− ·000009	− ·000010	− ·000011	+ ·000002	− ·000009	− ·000005	− ·000009	− ·000003	− ·000006
15	− ·000024	− ·000009	− ·000004	− ·000003	− ·000007	− ·000012	− ·000010	− ·000023	·000000	− ·000007	− ·000007	− ·000007	− ·000012	− ·000009
16	− ·000013	− ·000007	− ·000002	− ·000004	− ·000004	− ·000009	− ·000006	− ·000017	+ ·000003	− ·000007	− ·000004	− ·000005	− ·000009	− ·000006
17	− ·000003	− ·000001	·000000	− ·000003	− ·000002	− ·000001	− ·000001	− ·000015	− ·000005	− ·000002	− ·000006	− ·000002	− ·000012	− ·000007
18	− ·000007	− ·000009	− ·000002	− ·000006	− ·000004	− ·000007	− ·000005	− ·000017	− ·000004	+ ·000006	− ·000003	+ ·000001	− ·000011	− ·000005
19	− ·000001	− ·000009	+ ·000002	·000000	+ ·000004	− ·000007	− ·000001	− ·000016	− ·000018	+ ·000004	− ·000017	− ·000003	− ·000019	− ·000011
20	− ·000001	− ·000010	·000000	− ·000001	·000000	− ·000005	− ·000003	− ·000005	− ·000009	+ ·000006	− ·000009	+ ·000002	− ·000010	− ·000004
21	− ·000002	+ ·000005	− ·000008	− ·000003	− ·000002	− ·000001	− ·000002	− ·000007	− ·000010	+ ·000001	− ·000008	− ·000002	− ·000009	− ·000005
22	+ ·000004	·000000	− ·000005	− ·000001	− ·000005	+ ·000004	·000000	+ ·000004	− ·000008	+ ·000004	− ·000003	− ·000001	− ·000001	·000000
23	+ ·000009	·000000	− ·000003	+ ·000005	− ·000001	+ ·000007	+ ·000003	− ·000001	− ·000006	+ ·000001	− ·000006	− ·000002	− ·000004	− ·000003
Number of lunar days' observations in abstract	253	230	258	241	501	481	982	197	188	209	207	412	389	801

TABLE VIII.—Horizontal Force at Bombay, showing the Mean Lunar Diurnal Variation at Last Quarter and Seven-eighths Phase in each Quarter of the Year.

Phase of moon.	Last quarter							Seven-eighths.						
Bombay lunar hours	November to January	February to April	May to July.	August to October	April to September	October to March.	Year.	November to January	February to April	May to July	August to October	April to September	October to March	Year.
0	-·000008	- 000004	- 000007	+ 000001	- 000005	- 000004	- 000004	- 000013	- 000012	- 000012	+ ·000002	- 000010	- 000008	- 000009
1	-·000006	000000	-·000007	- 000006	- 000008	- 000003	- 000005	- 000025	- 000025	- 000018	+ ·000004	- 000011	- 000021	- 000016
2	- 000018	- 000002	- 000002	- 000001	- 000002	- 000010	- 000006	- 000045	- 000026	-·000020	- 000013	- 000019	-·000034	- 000027
3	- 000023	+ 000006	+ 000013	-·000016	+ 000005	- 000016	- 000005	- 000043	- 000021	- 000016	- 000017	- 000020	- 000030	-·000025
4	- 000023	+ 000003	000000	- 000006	+ 000003	- 000017	- 000007	- 000030	-·000029	- 000007	- 000021	- 000014	- 000030	- 000022
5	- 000004	-·000003	+ 000009	- 000008	+ 000005	- 000008	- 000001	- 000013	- 000028	- 000017	- 000012	- 000015	- 000020	- 000017
6	+ 000014	+ 000007	- 000002	- 000011	- 000002	+ 000006	+ 000002	+ 000001	- 000007	- 000022	- 000018	- 000022	- 000001	- 000011
7	+ 000032	+ 000003	- 000004	- 000003	000000	+ 000014	+ 000007	- 000003	+ 000003	- 000011	+ 000002	- 000003	- 000002	- 000003
8	+ 000028	- 000005	+ 000010	+ 000002	+ 000008	+ 000010	+ 000009	+ 000020	+ 000011	- 000004	+ 000001	- 000002	+ 000016	+ 000006
9	+ 000026	- 000002	+ 000008	+ 000006	+ 000004	+ 000015	+ 000009	+ 000020	000000	- 000002	+ 000009	+ ·000002	+ ·000011	+ 000007
10	+ ·000021	- 000003	+ 000002	- 000009	- 000003	+ 000009	+ 000003	+ 000016	000000	+ 000003	+ 000009	+ 000002	+ 000012	+ 000007
11	+ 000009	+ 000002	+ 000006	- 000001	+ 000003	+ 000005	+ 000004	+ 000017	+ 000002	000000	+ 000014	+ 000004	+ 000013	+ 000009
12	+ 000005	- 000011	- 000003	+ 000003	- 000003	000000	- 000001	+ 000017	+ 000022	+ 000007	+ 000005	+ 000009	+ 000016	+ 000012
13	- 000001	- 000003	- 000004	+ 000004	- 000002	000000	- 000001	+ 000021	+ 000022	+ 000008	+ 000008	+ 000009	+ 000020	+ 0000'4
14	- 000003	+ 000002	+ 000002	+ 000002	+ 000002	000000	+ 000001	+ 000004	+ 000019	+ 000012	+ 000007	+ 000010	+ 000010	+ 000010
15	- 000009	+ 000002	- 000001	+ 000006	+ 000001	- 000003	000000	000000	+ 000019	+ 000008	- 000001	+ 000005	+ 000007	+ 000006
16	-·000007	- 000002	+ 000002	+ 000009	+ 000002	- 000001	+ 000001	+ 000011	+ 000002	+ 000011	+ 000001	+ 000006	+ 000006	+ 000006
17	- 000011	- 000007	- 000002	+ 000004	- 000001	- 000008	- 000004	+ 000009	+ 000004	+ 000005	+ 000012	+ ·000009	+ 000006	+ 000007
18	+ 000003	000000	- 000004	+ 000003	- 000001	+ 000001	000000	000000	+ 000014	+ 000012	+ ·000004	+ 000010	+ 000004	+ 000007
19	- 000006	+ 000011	- 000006	+ 000007	000000	+ 000002	+ 000001	+ 000001	+ 000012	+ 000012	+ 000002	+ 000010	+ 000002	+ 000006
20	- 000003	+ 000010	- 000003	- 000004	- 000004	+ 000004	000000	+ 000006	+ 000008	+ 000019	+ 000005	+ 000012	+ 000007	+ 000009
21	- 000001	+ 000007	- 000004	+ 000007	- 000001	+ 000005	+ ·000002	+ ·000010	+ ·000010	+ 000017	+ 000006	+ 000012	+ ·000010	+ ·000011
22	- 000006	+ 000002	- 000004	+ 000008	- 000001	+ 000001	000000	+ 000005	+ 000007	+ ·000015	+ 000005	+ 000010	+ 000005	+ 000008
23	-·000015	- 000005	- 000005	+ 000012	- 000002	- 000005	- 000003	+ 000004	- 000006	+ 000010	- 000007	000000	+ 000001	000000
Number of lunar days' observations in abstract	255	250	266	267	532	506	1038	214	190	203	209	413	403	816

7. Curves were now constructed to represent the variations otherwise expressed by the hourly excesses for the several seasons and phases, and a consideration of them led to the formation of the hypothesis that the bulk of the phenomenon dealt with is, properly speaking, not a lunar diurnal variation, but a solar diurnal variation that depends on the relative positions of the sun and moon, a variation such that it may be expressed, for any one season, by the formula

$$f_{c2}(h)\cos 2\left(\frac{2\pi}{\mathrm{P}}t\right)+f_{s2}(h)\sin 2\left(\frac{2\pi}{\mathrm{P}}t\right),$$

where h is the hour of the solar day, P the mean period of a lunation in mean solar days, and t the age of the moon in mean solar days; and $f_{c2}(h), f_{s2}(h)$, are the observed variations at new moon and when the moon has the age one-eighth of a lunation respectively. It will presently be seen that, though the typical variations $\{f(h)\}$ of one season differ from those of another, this hypothesis holds good generally in each quarter of the year separately.

Of the two characteristics of the curves that pointed in the direction of the formula, the first was the general fact that the great movements occur in them, as in curves of the mean solar diurnal variations for full lunations, in the solar *day hours*, and the night hours are relatively quiescent; or, viewed in another aspect, the significant movements occur at all hours of the lunar day in the course of a lunation, and appear earlier and earlier as the age of the moon increases. And the second characteristic was that the curves, regarded as solar diurnal curves, have generally the same form and range at intervals of half a lunation, and opposite forms at intervals of a quarter of a lunation; and this with reference separately to each of the two magnetic elements and to each season of the year.

[7*a*. It may be explained here that in the first reductions of the Bombay declination observations the categories into which the days were divided had reference (1) to the four seasons of the year, (2) to four positions of the moon in declination, and (3) to the four quarters of the moon—*not* to eight phases as in the later reductions. A short account of a first instalment of the results of these reductions, dealing with the observations of the years 1861 to 1863, was read before the Royal Society on the 2nd February, 1872; but it was the results of these earlier reductions for the full period of twenty-five years that suggested the idea of the luni-solar variation as expressed by the formula given in the last paragraph, and the computation and curving of the lunar diurnal variations of each category for that period were not completed till the 4th March, 1873. How very definite the suggestion was may be seen from the curves of figs. 50 to 53, 55 to 58, 60 to 63, and 65 to 68, which represent the lunar diurnal variations for the sixteen principal categories. To facilitate comparison, the curves for the different phases of the same season are placed under one another, the lunar time scales are arranged so that the beginning and end of the curves should correspond approximately to the same solar time, and the force scales for first quarter

and last quarter are inverted in comparison with the force scales for new moon and full moon With the curves thus arranged, the four, occupying each vertical curve-form, and of which the fifth curve is, in each case, the type (or average), can be taken into view at a single glance, and the degree of their similarity is thus easily recognised. But similarity in these curves means the same thing as the second characteristic of the curves of lunar diurnal variations, which is expressed in words in the last paragraph as follows—the curves, regarded as solar diurnal curves, have generally the same form and range at intervals of half a lunation, and opposite forms at intervals of a quarter of a lunation. The similarity is very pronounced in all seasons except the transition one—February to April—in which the inversion of character of the variations is in progress.—23rd September, 1886.]

8. It now became an object to adapt the results already obtained, which have reference to the lunar day, to the determination of the data $f_{c2}(h)$ and $f_{s2}(h)$ of the formula; for this purpose the following process was adopted.

As the lunar day roughly approximates to the same length as the solar day, we suppose the observed lunar diurnal variation at new moon to imply with rough approximation a solar diurnal variation of the same character; and we enter the hourly excesses in a Table having solar hours, from 0 to 23, marked at the top of the columns; under these numbers we enter the excesses of full moon, placing the number belonging to the 0th hour of lunar time under the 12th hour of solar time, and we then take the sums of the two sets of numbers. Again, we enter the excesses of first quarter, placing the number belonging to the 0th hour of lunar time under the 6th hour of solar time, and under these we enter the excesses of last quarter, placing the number of the 0th hour of lunar time under the 18th hour of solar time, and we then take the sum of these two sets of numbers; next, we subtract the latter sums from the former, and divide the results by 4, calling the series of quotients the typical variation for the quarters of the moon, that is $f_{c2}(h)$. Similarly, substituting the hourly excesses of the phases one-eighth, three-eighths, five-eighths, and seven-eighths for those of new moon, first quarter, full moon, and last quarter respectively, we obtain the typical variation for the eighths phases, that is $f_{s2}(h)$. In this way, of which an example will now be given, have been obtained the typical variations shown in Tables 9 and 10, using as data the numbers in Column 2 of Tables 1 to 4 for the first line of Table 9, in Column 3 of Tables 1 to 4 for the second line of Table 9, and so on.

CALCULATION of the typical Variation of Declination for the Quarters of the

	Lunar diurnal variation for—	Bombay 0	1	2	3	4	5	6	7	8	9
a	New Moon	+ 000016	+ 000018	+ 000008	+ 000006	– 000004	– 000008	– 000012	– 000011	– 000006	– 000003
b	Full Moon	+ 000008	+ 000012	+ 000020	+ 000010	– 000005	– 000007	+ 000001	+ 000006	+ 000006	+ 000003
c	$a+b$	+ 000024	+ 000030	+ 000028	+ 000016	– 000009	– 000015	– 000011	– 000005	000000	000000
d	First Quarter	– 000015	– 000015	– 000026	000000	+ ·000018	+ 000016	+ 000018	+ 000012	+ 000006	– 000002
e	Last Quarter	– 000018	– 000025	– 000029	– ·000014	000000	+ 000010	+ 000011	+ 000007	+ 000005	+ 000001
f	$d+e$	– 000033	– 000040	– 000055	– 000014	+ 000013	+ 000026	+ 000029	+ 000019	+ ·000011	– 000001
g	$c-f$	+ 000057	+ 000073	+ 000083	+ 000030	– 000022	– 000041	– 000040	– 000024	– 000011	+ 000001
h	$\frac{1}{2}(c-f)=f_{c\cdot2}(h)$	+ 000014	+ 000018	+ 000021	+ ·000007	– 000005	– 000010	– 000010	– 000006	– 000003	000000

TABLE 9.—Showing the typical Variations $f_{c.2}(h)$ and $f_{s\,2}(h)$

	Bombay Astronomical Hours	0	1	2	3	4	5	6	7	8	9
$f_{c\cdot2}(h)$	November to January	+ ·000014	+ ·000018	+ 000021	+ 000007	– ·000005	– ·000010	– ·000010	– 000006	– 000003	·000000
	February to April	+ ·000003	– ·000001	+ 000001	– 000002	·000000	– ·000002	– ·000005	– ·000005	– ·000005	– ·000001
	May to July	– 000014	– ·000013	– 000008	+ 000001	+ 000008	+ 000011	+ ·000007	·000000	·000000	+ ·000001
	August to October	– 000010	– ·000016	– 000004	+ ·000009	+ ·000014	+ ·000010	+ ·000004	000000	– ·000001	– ·000001
$f_{s\cdot2}(h)$	November to January	– 000011	– 000005	+ 000005	+ 000009	+ ·000017	+ ·000013	+ ·000004	– 000001	·000000	– ·000002
	February to April	+ 000005	+ 000001	– 000008	– 000004	– ·000001	+ 000003	+ 000003	+ ·000004	– 000001	– ·000003
	May to July	– 000004	– ·000012	– 000015	– 000023	– 000011	– 000002	+ ·000005	+ ·000004	+ 000003	+ ·000001
	August to October	– 000004	– 000015	– 000013	– ·000011	– 000007	+ ·000008	+ 000003	+ 000004	·000000	– ·000001

TABLE 10.—Showing the typical Variations $f_{c.2}(h)$ and $f_{s\,2}(h)$

	Bombay Astronomical Hours.	0	1	2	3	4	5	6	7	8	9
$f_{c\cdot2}(h)$	November to January	– ·000004	– ·000011	– 000021	– ·000023	– ·000016	– ·000008	– ·000006	– ·000001	– ·000001	+ ·000004
	February to April	+ ·000001	– 000003	– 000006	– ·000013	– ·000010	– 000007	– ·000001	– ·000003	– ·000003	+ ·000001
	May to July	+ 000001	– ·000002	– 000009	– 000010	– 000008	– 000003	+ 000002	+ ·000003	+ ·000002	·000000
	August to October	– 000004	– 000005	– 000003	– ·000008	– ·000006	– ·000001	– ·000003	– ·000005	– 000001	·000000
[illegible]	November to January	+ ·000031	+ ·000024	+ 000014	+ ·000004	·000000	– 000011	– 000016	– 000009	– ·000011	– ·000011
	February to April	+ 000021	+ 000020	+ ·000015	+ ·000006	+ 000001	– 000001	·000000	– 000003	– ·000007	– ·000011
	May to July	+ ·000011	+ ·000004	+ ·000010	+ 000009	+ 000002	000000	– ·000004	– ·000005	– ·000003	– ·000001
	August to October	+ ·000017	+ ·000013	+ 000008	+ ·000006	– ·000002	– ·000010	– ·000008	– 000005	– 000007	– ·000005

The variations of Tables 9 and

Moon in the Quarter November to January—that is, of the Variation $f_{c\,2}(h)$.

Astronomical Hours

10	11	12	13	14	15	16	17	18	19	20	21	22	23
− 000001	+ 000004	+ 000004	+ 000006	+ 000004	+ 000005	+ 000001	− 000004	− 000015	− 000009	− 000001	− 000008	− 000006	+ 000008
+·000002	+ 000004	+·000008	+ 000004	+ 000001	− 000006	− 000007	− 000009	− 000015	− 000011	− 000018	− 000017	+ 000001	+ 000009
+ 000001	+ 000008	+ 000012	+ 000010	+ 000005	− 000001	− 000006	− 000013	− 000031	− 000020	− 000019	− 000025	−·000005	+ 000017
+ 000002	− 000061	− 000006	− 000002	− 000006	− 000009	− 000005	+ 000004	+ 000006	+ 000006	+ 000006	+ 000008	− 000002	− 000004
+ 000003	000000	− 000003	000000	+ 000001	+ 000006	+ 000010	+ 000011	+ 000010	+ 000009	+ 000011	+ 000014	000000	− 000011
+ 000005	− 000001	− 000009	− 000002	− 000005	− 000003	+ 000005	+ 000015	+ 000016	+ 000015	+·000017	+ 000022	− 000002	− 000015
− 000004	+ 000009	+ 000021	+ 000012	+ 000010	+ 000002	− 000011	− 000028	− 000047	− 000035	− 000036	− 000047	− 000003	+ 000032
− 000001	+ 000002	+ 000005	+ 000003	+ 000002	·000000	− 000003	− 000007	− 000012	− 000009	− 000009	− 000012	− 000001	+·000008

of Declination at Bombay for each Quarter of the Year

10	11	12	13	14	15	16	17	18	19	20	21	22	23
− 000001	+ 000002	+ 000005	+ 000003	+ 000002	000000	− 000003	− 000007	− 000012	− 000009	− 000009	− 000012	− 000001	+ 000008
− 000002	000000	+·000003	+ 000001	+ 000002	+ 000001	− 000002	− 000004	− 000003	+ 000004	+ 000002	+ 000006	+ 000008	+ 000001
+ 000001	+ 000001	− 000001	000000	+ 000001	000000	+ 000001	+ 000007	+ 000009	+ 000012	+·000005	− 000006	− 000014	− 000011
− 000002	+ 000002	+ 000001	+ 000003	+·000004	+ 000004	+ 000004	+ 000007	+ 000007	+ 000007	+ 000003	− 000007	− 000015	− 000019
− 000003	− 000002	000000	+ 000001	+ 000004	+ 000003	+ 000006	+ 000007	+ 000006	+ 000001	− 000006	− 000014	− 000016	− 000011
− 000003	− 000003	+ 000001	+ 000001	+ 000006	+ 000005	+ 000004	000000	− 000002	− 000005	·000000	− 000006	000000	− 000002
000000	·000000	+ 000002	+·000003	+ 000002	000000	− 000001	− 000004	+ 000001	+ 000010	+ 000011	+ 000012	+·000011	+ 000005
+·000001	− 000001	−·000004	− 000002	+ 000001	+ 000002	+ 000002	+·000002	+ 000004	+ 000007	+ 000017	+ 000009	+ 000003	− 000002

of Horizontal Force at Bombay for each Quarter of the Year.

10	11	12	13	14	15	16	17	18	19	20	21	22	23
+ 000001	+ 000003	− 000002	000000	− 000003	− 000004	− 000002	+ 000001	+ 000004	+ 000007	+ 000019	+·000027	+ 000024	+ 000013
+ 000004	+·000004	000000	− 000004	− 000003	− 000004	− 000003	− 000001	+ 000004	+ 000006	+ 000009	+ 000012	+·000009	+ 000009
− 000003	+ 000001	000000	+ 000002	+ 000002	+ 000002	+ 000004	+·000003	+ 000003	+ 000006	+ 000003	+ 000002	− 000002	+ 000001
−·000001	+ 000002	+ 000002	+ 000001	000000	−·000003	− 000003	+ 000003	+ 000005	+ 000003	+·000004	+·000014	+·000002	000000
− 000014	− 000002	− 000004	− 000001	− 000003	− 000003	− 000007	− 000012	−·000013	− 000010	− 000008	+·000011	+ 000022	+ 000032
− 000013	− 000008	− 000005	− 000003	− 000006	− 000011	− 000009	−·000008	− 000008	− 000008	− 000002	+ 000001	+ 000016	+ 000024
− 000001	− 000004	− 000002	− 000001	+ 000001	− 000005	− 000004	− 000009	− 000006	−·000005	− 000001	+ 000003	+·000006	+ 000009
−·000009	− 000008	+·000001	+ 000001	− 000006	− 000002	−·000008	−·000005	− 000004	− 000005	+ 000002	+·000001	+ 000006	+·000012

10 are curved in figs. 1 to 16.

9. Considering now the typical variation curves (figs. 1 to 16), we see that those which are the most definite and systematic in character, and have the largest range, whether of declination or horizontal force, are for the winter season, and, in the case of each type, the curves for the other seasons are definitely related to these. Thus, as to declination, between winter and summer there is an inversion of the types of both figs. 1 and 5, and the type-curves of the transition period, the spring season, have but a small range, and in them the prominent features of the winter curves are all but obliterated. Necessarily there is also a return inversion of the type in another season, and it occurs between mid-autumn and mid-winter. On the other hand, the horizontal force types (figs. 9 to 16), although of largest range in winter, are never reversed in character, but contract only to a minimum range, which is reached in the summer season.

The typical diurnal variation for November to January of horizontal force at new moon (fig. 9), and that of declination at the one-eighth phase reversed to correspond to the seven-eighths phase (fig. 5 reversed), have generally a maximum about three hours before noon, a minimum about three hours after noon, and are nearly *nil* for three or four hours before and after midnight; those of declination at new moon (fig. 1) and of horizontal force at the one-eighth phase (fig. 13) have a principal upward inflexion with a turning point about noon, two nearly equal downward inflexions with turning points about five or six hours before and after noon, and little or no departure from the mean line for three or four hours before and after midnight. For the summer and autumn seasons similar descriptions of the typical diurnal variations would apply, providing due allowance were made for the double reversal in the course of the year of the declination types.

Comparing figs. 3 and 4 with 1, and figs. 7 and 8 with 5, all declination curves, it may be noted, as a minor characteristic, that the principal inflexions occur generally somewhat earlier in summer and autumn than in winter.

10. The general similarity of character of figs. 1 and 13, representing the $f_{c2}(h)$ and $f_{s2}(h)$ data of our formula for the declination and horizontal force respectively, and the opposition of character of figs. 5 and 9, representing conversely the $f_{s2}(h)$ and $f_{c2}(h)$ data for the elements taken in the same order, call for further consideration. These relations may be translated into a statement that, in the winter season, the luni-solar variation of declination due to new moon is similar to the luni-solar variation of horizontal force due to the one-eighth phase; and the luni-solar variation of horizontal force due to new moon is similar to the luni-solar variation of declination due to the seven-eighths phase. In other words, the luni-solar variation of declination due to any phase of the moon is similar to the luni-solar variation of horizontal force due to a phase later by one-eighth of a lunation. In the summer and autumn seasons it is, on the other hand, the luni-solar variations of horizontal force that precede by one-eighth of a lunation the similar variations of declination; this will be seen by comparing figs. 11 and 12 with figs. 7 and 8, and figs. 15 and 16 with figs. 3 and 4 reversed.

The fact of harmonious relations of this kind being found to subsist between results derived from long series of observations of two *independent* instruments, we cannot but regard as strong testimony to the reality of the phenomena now brought to light; neither can we refrain from claiming for such results a modest place amongst the phenomenal laws of terrestrial magnetism that must ultimately stand in the same relation to a physical theory of terrestrial magnetism that KEPLER'S laws stand in towards the theory of gravitation.

11. At this stage the question was put—With what approach to completeness does the typical variation in each case represent the four actual variations from which it was derived, or how much of a mean lunar variation is there in these over and above the typical variation? Subtracting (say) $f_{c2}(h)$ from the observed variation at new moon and full moon, and adding it to the observed variation at first quarter and last quarter in such a way that the 0th, 6th, 12th, and 18th hours of $f_{c2}(h)$ are compared with the 0th hour of new moon, first quarter, full moon, and last quarter respectively; the 1st, 7th, 13th, and 19th hours of $f_{c2}(h)$ with the 1st hour of new moon, first quarter, &c., respectively, and so on, we obtain four sets of residual variations, each commencing with the 0th hour of the lunar day; and, taking the mean of these four, we obtain the residual lunar diurnal variation that is left after appropriately eliminating the typical variation for the four quarters. A similar procedure, using the observed variations at the eighths phases, gives corresponding residual variations for the eighths phases.

12. On curving these residuals for each quarter of the year—eight in all for each magnetic element—and comparing each curve with the corresponding typical variation curve, they were all found to be of small relative range, but most of them had a definite character, in which the principal harmonic element was that which has the lunar day for its period. In the latter fact we found a suggestion that, although our formula disposes of the bulk of the phenomena for which an expression is to be found, the addition to it of two more independent terms would not only make it mathematically more complete, but would render it further expressive of an otherwise neglected, but significant, element in the luni-solar variation. Making this addition, the formula becomes

$$f_{c1}(h)\cos\left(\frac{2\pi}{P}t\right)+f_{s1}(h)\sin\left(\frac{2\pi}{P}t\right)+f_{c2}(h)\cos 2\left(\frac{2\pi}{P}t\right)+f_{s2}(h)\sin 2\left(\frac{2\pi}{P}t\right),*$$

* If ϕ be written for the angle $\frac{2\pi}{P}t$, and θ for the angle $\frac{2\pi}{24}h$, the formula may easily be transformed into

$$F_{c1}(\phi)\cos\left(\frac{2\pi}{24}h\right)+F_{s1}(\phi)\sin\left(\frac{2\pi}{24}h\right)+F_{c2}(\phi)\cos 2\left(\frac{2\pi}{24}h\right)+F_{s2}(\phi)\sin 2\left(\frac{2\pi}{24}h\right),$$

in which $F_{c1}(\phi)$, $F_{s1}(\phi)$ are variations, of constant types, having the period of a lunation, and $F_{c2}(\phi)$, $F_{s2}(\phi)$ are variations, of constant types, having a period of half a lunation; and all these swell and contract with a wave-like oscillation—$F_{c1}(\phi)$, $F_{s1}(\phi)$ in the period of a day, and $F_{c2}(\phi)$, $F_{s2}(\phi)$ in the

each term of which is symbolical of a definite physical conception, viz., that an otherwise constant variation swells and contracts with a wave-like motion, as the age of the moon increases, between the limits $-f(h)$ and $+f(h)$.

13. If the initial new moon occur at h' hours of the initial solar day, from the beginning of which time is reckoned in solar astronomical hours, the age of the moon will become $h-h'$, the period of a lunation 24P; and for $\frac{2\pi}{\mathrm{P}}t$ may be substituted $\frac{2\pi}{24\mathrm{P}}(h-h')$: and if, further, $f(h)$ be expressed in the form

$$a_1 \cos\frac{2\pi}{24}h + b_1 \sin\frac{2\pi}{24}h + a_2 \cos 2\left(\frac{2\pi}{24}h\right) + b_2 \sin 2\left(\frac{2\pi}{24}h\right) + \&\mathrm{c},$$

our extended formula may easily be transformed into

$$\begin{aligned}
&\mathrm{A}_1 \cos\left\{\frac{2\pi}{24}h\left(1-\frac{1}{\mathrm{P}}\right)+\alpha\right\}+\mathrm{B}_1\cos\left\{\frac{2\pi}{24}2h\left(1-\frac{1}{2\mathrm{P}}\right)+\beta\right\}\\
&+\mathrm{C}_1 \cos\left\{\frac{2\pi}{24}2h\left(1+\frac{1}{2\mathrm{P}}\right)+\gamma\right\}+\mathrm{D}_1\cos\left\{\frac{2\pi}{24}h\left(1+\frac{1}{\mathrm{P}}\right)+\delta\right\}\\
&+\mathrm{A}_2 \cos\left\{\frac{2\pi}{24}h\left(1-\frac{2}{\mathrm{P}}\right)+\epsilon\right\}+\mathrm{B}_2\cos\left\{\frac{2\pi}{24}2h\left(1-\frac{1}{\mathrm{P}}\right)+\zeta\right\}\\
&+\mathrm{C}_2 \cos\left\{\frac{2\pi}{24}2h\left(1+\frac{1}{\mathrm{P}}\right)+\eta\right\}+\mathrm{D}_2\cos\left\{\frac{2\pi}{24}h\left(1+\frac{2}{\mathrm{P}}\right)+\theta\right\},
\end{aligned}$$

where the numbers A_1, B_1, C_1, D_1, &c., and the angles α, β, γ, δ, &c., are constants; that is to say, it may—inclusive of the first four terms of $f(h)$—be transformed into eight simple waves whose periods, in solar hours, are

$$24\frac{\mathrm{P}}{\mathrm{P}-1};\ 12\frac{2\mathrm{P}}{2\mathrm{P}-1};\ 12\frac{2\mathrm{P}}{2\mathrm{P}+1};\ 24\frac{\mathrm{P}}{\mathrm{P}+1};\ 24\frac{\mathrm{P}}{\mathrm{P}-2};\ 12\frac{\mathrm{P}}{\mathrm{P}-1};\ 12\frac{\mathrm{P}}{\mathrm{P}+1};\ \text{and}\ 24\frac{\mathrm{P}}{\mathrm{P}+2}.$$

Of these periods the first and sixth are the lunar day and half-day respectively; from which it follows that, even if our extended formula be a substantially correct expres-

period of half-a-day. It thus appears that θ and ϕ are reciprocally related, so that the period of either may be regarded as that of the variation of constant type, and the period of the other is then that in which the variation of constant type oscillates, whilst the complex variation of the formula remains identically the same; and this result is general so long as the number of terms in the formula is the same as the number of terms in each of the variations of constant type. The name—the luni-solar variation—has been chosen to distinguish the variation expressed by the formula as one involving the periods of a lunation and the solar day, and of sub-multiples of these periods; and the name—typical variation—has been given to the variations of constant type.

sion of the phenomena in question, there is nothing anomalous in our finding, as we do in fact, definite lunar diurnal variations from the observations of *complete* lunations. It will, indeed, be seen further on (paragraph 17) that the lunar half-day wave is one of the two most prominent, and that the lunar day wave has, generally speaking, an amplitude as great as any of the remaining five.

14 Proceeding now to find the values of $f_{c\,1}(h), f_{s\,1}(h)$, having entered the observed lunar diurnal variation for new moon in a Table having the solar hours, from 0 to 23, marked at the top of the columns, commencing with 0 hours of the lunar day, we enter underneath this the observed lunar diurnal variation for full moon, commencing with 12 hours, and then subtract the lower entries from the upper· half the difference we take to be the value of $f_{c\,1}(h)$ as derived from the new moon and full moon variations. Similarly, entering the observed lunar diurnal variation for first quarter, commencing with 18 hours, and under it the observed lunar diurnal variation for last quarter, commencing with 6 hours, and subtracting the lower numbers and dividing by 2, we obtain the value of $f_{s\,1}(h)$ as derived from the first quarter and last quarter variations. In like manner we obtain from the observed lunar diurnal variation for the one-eighth and five-eighths phases the value of $f_{c\,1}(h)\cos 45^\circ + f_{s\,1}(h)\sin 45^\circ$ or $\frac{1}{\sqrt{2}}\{f_{c\,1}(h)+f_{s\,1}(h)\}=a$ (say), and from the variations for the three-eighths and seven-eighths phases the value of $\frac{1}{\sqrt{2}}\{-f_{c\,1}(h)+f_{s\,1}(h)\}=b$ (say). combining the last two quantities, we find $\frac{1}{\sqrt{2}}(a-b)=f_{c\,1}$ and $\frac{1}{\sqrt{2}}(a+b)=f_{s\,1}$ as values derived from the variations of the eighths phases. In Tables 11 to 18 are collected together the results of these various calculations.

TABLE 11.—Declination.

	Derived from lunar diurnal variations for what phases	Typical variation	Bombay 0	1	2	3	4	5	6	7	8	9
a	New and full moon	$f_{e\,1}(h)$	+ 000004	+ 000003	− 000006	− 000002	·000000	000000	− 000000	− 000008	− 000006	− 000003
b	First and last quarters	$f_{s\,1}(h)$	+ 000001	+ 000005	+ 000001	+ 000007	+ ·000006	+ 000003	+ 000003	+ 000002	000000	− 000001
c	One and five-eighths	$\frac{1}{\sqrt{2}}\left\{f_{e\,1}(h)+f_{e\cdot1}(h)\right\}$	− 000001	+ 000005	− 000004	− 000011	− 000001	− 000001	+ 000001	− 000003	− 000006	− 000007
d	Three and seven-eighths	$\frac{1}{\sqrt{2}}\left\{-f_{e\cdot1}(h)+f_{s\cdot1}(h)\right\}$	+ 000012	+ 000005	− 000003	+ 000004	− ·000003	000000	+ 000001	− ·000001	− 000002	+ 000001
e	Eighths	$f_{e\,1}(h)$	000000	+ 000004	− 000003	− 000008	000000	− 000001	000000	− 000002	− 000004	− 000005
f		$f_{s\,1}(h)$	+ 000008	+ 000004	− 000002	+ 000003	− 000002	000000	000000	000000	− 000002	000000
g	Mean of *a* and *e*	$f_{e\,1}(h)$	+ 000002	+ 000003	− 000004	− 000005	000000	− 000001	− 000003	− 000005	− 000005	− 000001
h	Mean of *b* and *f*	$f_{s\,1}(h)$	+ 000005	+ ·000004	·000000	+ ·000005	+ 000002	+ 000002	+ ·000002	+ ·000001	− ·000001	000000

TABLE 12.—Declination.

	Derived from lunar diurnal variations for what phases	Typical variation	Bombay 0	1	2	3	4	5	6	7	8	9
a	New and full moon	$f_{e\,1}(h)$	000000	000000	− ·000001	− 000003	− 000003	− 000002	− 000001	− ·000004	− 000004	− ·000002
b	First and last quarters	$f_{s\,1}(h)$	− ·000003	− 000001	+ ·000001	+ 000001	·000000	+ ·000001	000000	·000000	·000000	000000
c	One and five-eighths	$\frac{1}{\sqrt{2}}\left\{f_{e\,1}(h)+f_{e\cdot1}(h)\right\}$	− ·000001	− 000003	− ·000008	− 000001	− ·000001	− 000008	·000000	+ ·000004	+ 000002	+ ·000002
d	Three and seven-eighths	$\frac{1}{\sqrt{2}}\left\{-f_{e\,1}(h)+f_{s\,1}(h)\right\}$	− 000005	− ·000001	− ·000002	+ 000005	+ ·000004	+ ·000004	+ ·000006	− ·000005	− ·000004	− 000003
e	Eighths	$f_{e\,1}(h)$	− ·000001	− 000002	− ·000006	− 000001	000000	− 000002	·000000	+ ·000008	+ 000001	+ 000002
f		$f_{s\,1}(h)$	− 000004	− 000001	− ·000002	+ 000004	+ 000003	+ 000008	+ 000003	− 000004	− ·000008	− ·000002
g	Mean of *a* and *e*	$f_{e\cdot1}(h)$	− 000001	− 000001	− ·000004	− 000002	− 000002	− 000002	000000	− 000001	− 000001	·000000
h	Mean of *b* and *f*	$f_{s\,1}(h)$	− 000003	− ·000001	000000	+ 000002	+ 000001	+ 000002	+ ·000001	− 000002	− 000001	− ·000001

TABLE 13.—Declination.

	Derived from lunar diurnal variations for what phases.	Typical variation.	Bombay 0	1	2	3	4	5	6	7	8	9
a	New and full moon	$f_{e\cdot1}(h)$	+ ·000004	− ·000001	− ·000003	− 000005	− 000005	+ ·000004	+ ·000006	+ 000008	+ ·000001	− ·000001
b	First and last quarters	$f_{s\,1}(h)$	+ 000001	000000	+ ·000001	+ ·000001	+ 000003	+ 000001	− ·000008	− 000002	·000000	+ ·000002
c	One and five-eighths	$\frac{1}{\sqrt{2}}\left\{f_{e\cdot1}(h)+f_{e\cdot1}(h)\right\}$	+ ·000004	− 000008	− ·000010	− ·000014	− ·000008	− 000005	− ·000002	− ·000005	− ·000001	− 000008
[illegible]	[illegible] seven-eighths.	$\frac{1}{\sqrt{2}}\left\{-f_{e\cdot1}(h)+f_{s\,1}(h)\right\}$	− ·000001	− 000001	+ 000018	+ ·000005	+ 000003	000000	+ 000002	− 000001	− 000001	− ·000003
[illegible]	[illegible]	$f_{e\cdot1}(h)$	+ ·000008	− ·000005	− ·000007	− ·000010	− 000006	− ·000003	− 000001	− 000004	·000000	− ·000002
[illegible]		$f_{e\cdot1}(h)$	− 000001	000000	+ ·000008	+ ·000004	+ ·000002	·000000	+ ·000001	− ·000001	·000000	− ·000002
[illegible]	[illegible]	[illegible]	+ 000008	− ·000003	− ·000005	− ·000008	− 000006	·000000	+ ·000002	·000006	·000000	− 000002
[illegible]	[illegible]	[illegible]	·000000	·000000	+ ·000005	+ ·000002	+ ·000003	·000000	− ·000001	− ·000001	·000000	·000000

November to January

Astronomical Hours

10	11	12	13	14	15	16	17	18	19	20	21	22	23
− 000001	·000000	− 000002	+ 000001	+ 000001	+ 000005	+ 000004	+ 000002	000000	+ 000001	+ 000008	+ 000004	− 000003	·000000
000000	000000	− 000001	− 000001	− 000003	− 000007	− 000007	− 000003	− 000002	− 000001	− 000002	− 000003	− 000001	+ 000003
− 000003	+ 000002	− 000001	000000	+ 000004	+ 000008	+ 000005	+ 000002	− 000001	+·000001	+ 000002	+ 000010	+ 000001	+ 000005
000000	000000	− 000003	+ 000001	− 000004	− 000005	− 000007	− 000004	− 000004	−·000001	− 000001	+ 000006	+ 000006	+·000001
− 000002	+ 000002	− 000001	000000	+·000003	+ 000005	+ 000004	+ 000002	000000	000000	+ 000002	+ 000007	000000	+ 000004
000000	000000	−·000002	+ 000001	− 000003	− 000003	− 000005	− 000003	− 000003	·000000	− 000001	+ 000004	+ 000004	+ 000001
− 000002	+ 000001	− 000001	000000	+ 000002	+ 000005	+ 000004	+ 000002	·000000	+ 000001	+ 000005	+·000006	− 000001	+ 000002
000000	000000	− 000002	·000000	− 000003	−·000005	− 000006	− 000003	−·000002	−·000001	− 000002	000000	+ 000001	+ 000002

February to April.

Astronomical Hours

10	11	12	13	14	15	16	17	18	19	20	21	22	23
+ 000002	+ 000002	+ 000007	+ 000003	+ 000006	+·000009	+ 000003	+ 000005	+·000002	− 000001	− 000003	− 000002	− 000002	− 000005
− 000004	− 000005	000000	+ 000002	·000000	−·000002	000000	+ 000004	+ 000004	+ 000001	+ 000002	+ 000002	+ 000006	− 000012
+ 000004	+ 000004	+ 000001	+·000009	+·000007	+·000002	−·000003	− 000003	000000	− 000005	− 000003	+ 000002	− 000002	− 000006
+·000004	− 000006	− 000003	− 000006	−·000005	000000	− 000005	+ 000001	+ 000006	+ 000007	+ 000006	− 000005	+ 000003	+·000003
+ 000003	+ 000003	+ 000001	+ 000006	+ 000005	+ 000001	− 000002	− 000002	000000	− 000003	− 000002	+ 000001	− 000002	− 000004
+ 000003	− 000004	− 000002	− 000004	− 000004	000000	− 000003	000000	+ 000004	+ 000005	+ 000004	− 000003	+ 000002	+ 000002
+ 000003	+ 000002	+·000004	+ 000005	+ 000006	+ 000005	+ 000001	+·000002	+·000001	− 000002	− 000003	000000	− 000002	− 000004
000000	− 000004	− 000001	− 000001	−·000002	−·000001	−·000002	+ 000002	+ 000004	+ 000003	+ 000003	− 000001	+ 000004	− 000005

May to July.

Astronomical Hours

10	11	12	13	14	15	16	17	18	19	20	21	22	23
+ 000001	− 000001	+ 000003	− 000001	− 000001	+ 000001	+ 000002	+ 000001	+·000003	·000000	− 000006	− 000007	+·000004	− 000001
000000	000000	− 000001	− 000002	− 000004	− 000006	− 000005	− 000002	− 000002	+ 000002	000000	− 000003	+·000005	+ 000006
·000000	+ 000006	·000000	+·000001	+ 000003	+·000006	+ 000002	+ 000007	+ 000008	− 000003	+ 000002	+·000010	+·000001	+·000008
−·000001	−·000006	− 000007	− 000006	−·000006	− 000002	− 000005	− 000005	− 000001	+ 000004	− 000003	+·000012	+·000004	+·000002
000000	+ 000004	000000	000000	+ 000002	+ 000004	+ 000001	+ 000005	+ 000005	− 000002	+ 000001	+·000007	·000000	+·000006
− 000001	− 000004	− 000005	− 000004	− 000004	− 000001	− 000003	− 000003	000000	+·000003	−·000002	+·000008	+ 000003	+·000002
+ 000001	+ 000001	+ 000002	000000	000000	+·000008	+ 000002	+ 000003	+ 000004	− 000001	− 000002	·000000	+·000002	+ 000002
−·000001	− 000002	− 000003	− 000003	− 000004	− 000004	− 000004	− 000003	−·000001	+ 000003	− 000001	+ 000003	+ 000004	+ 000004

Table 14.—Declination.

	Derived from lunar diurnal variations for what phases	Typical variation	Bombay 0	1	2	3	4	5	6	7	8	9
a	New and full moon	$f_{e\,1}(h)$	− 000003	− 000001	− 000003	+ 000001	+ 000002	000000	000000	+ 000005	+ 000006	+ 000001
b	First and last quarters	$f_{s\,1}(h)$	− 000001	− 000015	− 000002	000000	000000	+ 000004	− 000001	000000	− 000002	000000
c	One and five-eighths	$\frac{1}{\sqrt{2}}\left\{f_{e\,1}(h)+f_{s\,1}(h)\right\}$	+ 000003	+ 000009	− 000003	+ 000008	000000	+ 000004	+ 000004	000000	+ 000001	+ 000002
d	Three and seven-eighths	$\frac{1}{\sqrt{2}}\left\{-f_{e\,1}(h)+f_{s\,1}(h)\right\}$	− 000004	− 000002	− 000003	− 000003	+ 000004	+ 000009	+ 000007	+ 000003	+ 000003	− 000002
e	Eighths	$f_{e\,1}(h)$	+ 000002	+ 000006	− 000002	+ 000005	000000	+ 000003	+ 000003	000000	+ 000001	+ 000002
f	Eighths	$f_{s\,1}(h)$	− 000003	− 000002	− 000002	− 000002	− 000003	+ 000006	+ 000005	+ 000002	+ 000002	− 000002
g	Mean of *a* and *e*	$f_{e\,1}(h)$	000000	+ 000002	− 000003	+ 000003	+ 000001	+ 000001	+ 000002	+ 000003	+ 000003	+ 000002
h	Mean of *b* and *f*	$f_{s\,1}(h)$	− 000002	− 000009	− 000002	− 000001	− 000001	+ 000005	+ 000002	+ 000001	000000	− 000001

Table 15.—Horizontal Force.

	Derived from lunar diurnal variations for what phases	Typical variation	Bombay 0	1	2	3	4	5	6	7	8	9
a	New and full moon	$f_{e\,1}(h)$	000000	− ·000004	− 000004	− 000003	− 000003	− 000003	·000000	000000	000000	+ ·000004
b	First and last quarters	$f_{s\,1}(h)$	− 000014	− 000021	− ·000009	− 000008	− ·000005	000000	·000000	+ 000003	+ 000005	+ 000003
c	One and five-eighths	$\frac{1}{\sqrt{2}}\left\{f_{e\,1}(h)+f_{s\,1}(h)\right\}$	− 000014	− 000004	+ 000006	+ 000003	− 000009	+ 000009	+ 000011	+ 000015	+ 000009	+ 000011
d	Three and seven-eighths	$\frac{1}{\sqrt{2}}\left\{-f_{e\,1}(h)+f_{s\,1}(h)\right\}$	000000	000000	− 000008	− 000004	− 000007	− 000002	− ·000003	+ 000005	+ ·000006	− 000001
e	Eighths	$f_{e\,1}(h)$	− 000010	− 000003	+ 000004	+ 000002	− 000006	+ 000006	+ 000008	+ 000011	+ 000006	+ 000008
f	Eighths	$f_{s\,1}(h)$	·000000	000000	− 000005	− 000003	− 000005	− 000001	− 000002	+ 000004	+ 000004	− ·000001
g	Mean of *a* and *e*	$f_{e\,1}(h)$	− 000005	− 000003	000000	000000	− 000005	+ 000001	+ 000004	+ 000006	+ 000003	+ 000006
h	Mean of *b* and *f*	$f_{s\,1}(h)$	− ·000007	− 000010	− 000007	− ·000005	− 000005	− 000001	− 000001	+ 000003	+ 000004	+ 000001

Table 16.—Horizontal Force.

	Derived from lunar diurnal variations for what phases.	Typical variation.	Bombay 0	1	2	3	4	5	6	7	8	9
a	New and full moon	$f_{e\,1}(h)$	− 000004	− 000002	+ ·000001	− 000004	− 000007	− ·000004	+ 000001	+ ·000003	+ 000005	− 000001
b	First and last quarters	$f_{s\,1}(h)$	− 000004	000000	+ 000007	+ 000014	+ 000010	+ 000006	+ 000005	+ 000004	000000	000000
c	One and five-eighths	$\frac{1}{\sqrt{2}}\left\{f_{e\,1}(h)+f_{s\,1}(h)\right\}$	+ 000011	− ·000008	− 000019	− 000012	000000	+ 000006	000000	− 000008	− 000008	+ 000003
d	Three and seven-eighths.	$\frac{1}{\sqrt{2}}\left\{-f_{e\,1}(h)+f_{s\,1}(h)\right\}$	+ 000007	+ 000007	+ ·000008	− 000008	− 000012	− 000008	− ·000004	− 000004	+ 000005	− 000010
[illegible]	[illegible]	$f_{e\,1}(h)$	+ ·000008	− 000005	− ·000013	− 000008	000000	+ 000004	·000000	− 000006	− 000005	+ ·000002
[illegible]	[illegible]	[illegible]	+ ·000005	+ 000005	+ 000005	− 000005	− 000008	− 000006	− 000003	− 000003	+ 000003	− 000007
[illegible]	[illegible]	[illegible]	+ 000002	− 000004	− 000006	− ·000006	− 000003	000000	000000	− ·000001	000000	·000000
[illegible]	[illegible]	[illegible]	000000	+ 000003	+ ·000006	+ 000004	+ ·000001	000000	+ 000001	+ 000001	+ ·000001	− 000004

August to October.

Astronomical Hours

10	11	12	13	14	15	16	17	18	19	20	21	22	23
+ 000002	000000	− 000002	000000	− 000001	− 000002	+ 000001	000000	− 000002	000000	000000	− 000005	− 000006	+ 000001
+ 000003	− 000001	− 000001	000000	− 000003	000000	000000	− 000001	000000	+ 000003	+ 000006	+·000007	+ 000009	000000
− 000002	− 000004	·000000	000000	− 000001	− 000002	− 000003	− 000005	000000	− 000002	− 000007	− 000005	− 000002	− 000004
− 000001	+ 000002	− 000001	000000	− 000003	+ 000002	+ 000003	+ 000002	− 000003	− 000002	+ 000001	+ 000002	− 000005	000000
− 000001	− 000003	·000000	000000	− 000001	− 000001	− 000002	− 000003	000000	−·000001	− 000005	− 000004	− 000002	−·000003
000000	+ 000002	− 000001	000000	− 000002	+ 000001	+ 000002	+ 000001	− 000002	− 000001	+ 000001	+ 000002	− 000004	000000
·000000	− 000001	− 000001	000000	− 000001	− 000002	000000	− 000002	− 000001	− 000001	− 000002	− 000004	−·000004	− 000001
+ 000001	000000	− 000001	000000	− 000003	000000	+ 000001	000000	− 000001	+ 000001	+ 000004	+ 000005	+ 000003	000000

November to January.

Astronomical Hours

10	11	12	13	14	15	16	17	18	19	20	21	22	23
000000	− 000004	− 000003	− 000005	000000	+ 000002	+ 000001	+ 000002	+ 000005	+ 000007	+ 000002	+ 000003	− 000001	+ 000004
+ 000009	+ 000010	000000	+ 000005	+ 000003	+ 000005	+ 000008	+ 000013	+ 000007	+ 000001	+ 000001	− 000003	·000000	− 000007
+ 000008	+ 000004	− 000002	+ 000005	+ 000004	− 000006	− 000009	− 000007	+ 000002	− 000003	− 000012	− 000005	− 000009	− 000016
− 000001	000000	+ 000011	−·000003	000000	+·000005	+·000004	+ 000006	+ 000006	+ 000008	− 000003	− 000001	− 000008	+ 000004
+ 000006	+·000003	− 000001	+ 000003	+ 000003	− 000004	− 000006	− 000005	+ 000002	− 000002	− 000008	− 000004	− 000006	− 000011
− 000001	000000	+ 000008	− 000002	000000	+ 000003	+ 000003	+ 000004	+ 000004	+ 000005	− 000002	−·000001	− 000005	+ 000003
+ 000003	000000	− 000002	− 000001	+ 000001	− 000001	− 000003	− 000001	+·000004	+ 000002	−·000003	000000	− 000004	− 000004
+ 000004	+ 000005	+ 000004	+ 000002	+ 000002	+ 000004	+ 000005	+ 000008	+ 000006	+ 000003	000000	−·000002	− 000002	− 000002

February to April.

Astronomical Hours

10	11	12	13	14	15	16	17	18	19	20	21	22	23
+·000003	000000	+ 000006	+ 000005	+ 000003	+ 000003	− 000008	000000	− 000001	− 000001	+ 000001	+ 000001	− 000002	+ 000003
− 000002	−·000001	− 000001	− 000005	− 000007	− 000004	+·000001	+·000004	+ 000002	− 000004	− 000003	−·000014	− 000009	− 000006
+ 000013	+ 000015	+ 000018	+·000003	− 000001	+ 000009	+ 000003	−·000001	·000000	− 000003	− 000009	−·000003	−·000014	+·000001
− 000011	− 000008	−·000004	+ 000003	+ 000002	+ 000007	+ 000003	+ 000008	000000	+·000003	+·000006	+ 000008	+ 000002	+·000005
+ 000009	+ 000011	+ 000013	+·000002	− 000001	+ 000006	+ 000002	000000	·000000	−·000002	− 000006	− 000002	− 000010	000000
− 000008	−·000006	− 000003	+ 000002	+ 000002	+ 000005	+ 000002	+ 000005	·000000	+·000002	+·000004	+ 000005	+ 000001	+ 000003
+ 000006	+ 000006	+ 000009	+ 000004	+ 000001	+ 000005	000000	·000000	−·000001	− 000002	− 000003	000000	− 000006	+ 000002
− 000005	− 000004	− 000002	− 000002	− 000002	000000	+ 000002	+ 000005	+ 000001	− 000001	000000	− 000004	− 000004	−·000001

TABLE 17.—Horizontal Force.

	Derived from lunar diurnal variations for what phases	Typical variation	Bombay									
			0	1	2	3	4	5	6	7	8	9
a	New and full moon	$f_{e\,1}(h)$.	− 000007	− 000006	− 000001	− 000003	− 000003	000000	+ 000003	+ 000002	000000	+ 000003
b	First and last quarters	$f_{s\,1}(h)$.	000000	+ 000005	+ ·000001	+ 000005	+ 000009	·000000	000000	+ 000002	− 000006	− 000004
c	One and five-eighths .	$\frac{1}{\sqrt{2}}\left\{f_{e\,1}(h)+f_{s\,1}(h)\right\}$	+ 000004	+ 000003	+ 000009	+ 000005	+ 000004	+ 000004	− 000004	000000	− 000005	− 000001
d	Three and seven-eighths	$\frac{1}{\sqrt{2}}\left\{-f_{e\,1}(h)+f_{s\,1}(h)\right\}$	+ 000021	+ 000008	+ 000008	+ 000016	+ 000008	+ 000006	+ 000002	000000	·000000	− ·000008
e	Eighths	$f_{e\,1}(h)$	+ 000003	+ 000002	+ 000000	+ 000004	+ 000003	+ 000003	− 000003	000000	− 000003	000000
f	Eighths	$f_{s\,1}(h)$.	+ 000015	+ 000006	+ 000006	+ 000011	+ 000006	+ 000001	+ 000001	000000	000000	− 000005
g	Mean of *a* and *e*	$f_{e\,1}(h)$.	− 000002	− 000002	+ 000003	000000	000000	+ 000002	000000	+ 000001	− 000002	+ 000001
h	Mean of *b* and *f*	$f_{s\,1}(h)$.	+ 000008	+ 000005	+ 000003	+ 000008	+ 000007	+ 000002	+ 000001	+ 000001	− 000003	− ·000005

TABLE 18.—Horizontal Force.

	Derived from lunar diurnal variations for what phases	Typical variation.	Bombay									
			0	1	2	3	4	5	6	7	8	9
a	New and full moon . .	$f_{e\,1}(h)$	− 000002	000000	·000000	− 000004	− 000004	+ 000001	000000	− 000004	− 000002	+ 000003
b	First and last quarters .	$f_{s\,1}(h)$	+ 000012	+ 000012	·000000	+ 000003	+ 000012	+ 000001	− ·000002	+ 000001	− 000008	− 000006
c	One and five-eighths . .	$\frac{1}{\sqrt{2}}\left\{f_{e\,1}(h)+f_{s\,1}(h)\right\}$	− 000001	− 000004	+ 000005	− 000016	− 000015	− 000014	000000	+ 000002	+ 000005	− 000003
d	Three and seven-eighths	$\frac{1}{\sqrt{2}}\left\{-f_{e\,1}(h)+f_{s\,1}(h)\right\}$	− 000005	+ 000001	+ 000013	− 000005	− 000013	− 000006	− 000012	− 000006	− ·000008	− 000008
e	Eighths	$f_{e\,1}(h)$ . .	·000000	− 000003	+ 000008	− 000011	− ·000011	− 000010	000000	+ 000001	+ 000004	− 000002
f	Eighths	$f_{s\,1}(h)$ . . .	− 000003	+ 000001	+ 000009	− 000004	− 000009	− 000004	− 000008	− 000004	− ·000006	− 000005
g	Mean of *a* and *e* . . .	$f_{e\,1}(h)$ . .	− ·000001	− 000001	+ ·000002	− 000008	− ·000008	− 000004	000000	− 000001	+ 000001	000000
h	Mean of *b* and *f* .	$f_{s\,1}(h)$ . .	+ ·000004	+ 000007	+ 000004	000000	+ 000001	− 000001	− 000005	− 000001	− ·000004	− 00000[illegible]

May to July.

Astronomical Hours													
10	11	12	13	14	15	16	17	18	19	20	21	22	23
+ 000002	+ 000001	+ 000001	+ 000003	+ 000004	+ 000004	+ 000002	+ 000005	+ 000005	+ 000003	− 000001	− 000004	− 000005	− 000002
− 000003	− 000001	+ 000004	+ 000002	− 000001	+ 000001	+ 000002	+ 000001	+ 000005	+ 000002	−·000004	− 000013	000000	− 000010
+ 000002	− 000002	+ 000003	+ 000004	+ 000005	000000	+ 000007	+ 000006	000000	000000	− 000007	− 000007	− 000013	− 000006
− 000004	− 000004	− 000006	− 000008	000000	− 000008	− 000007	− 000017	− 000016	− 000008	− 000010	+ 000003	+ 000007	+ 000011
+ 000001	− 000002	+ 000002	+ 000003	+ 000004	000000	+ 000005	+ 000004	000000	000000	− 000005	−·000005	− 000009	−·000004
− 000003	−·000003	− 000004	− 000005	000000	− 000005	−·000005	− 000012	− 000011	− 000008	− 000007	+ 000002	+ 000005	+ 000008
·000000	000000	+ 000002	+ 000003	+ 000004	+ 000002	+ 000004	+ 000005	+ 000003	+ 000001	− 000003	− 000005	−·000007	− 000003
−·000003	− 000002	000009	− 000002	− 000001	− 000002	− 000001	− 000005	− 000003	− 000002	− 000006	− 000005	+ 000002	− 000001

August to October.

Astronomical Hours.													
10	11	12	13	14	15	16	17	18	19	20	21	22	23
000000	− 000004	+ 000005	000000	+ 000006	+ 000003	+ 000001	+ 000001	+ 000005	+ 000002	− 000003	−·000001	− 000005	− 000005
− 000007	−·000008	− 000009	− 000009	+ 000001	− 000003	− 000005	− 000009	− 000002	+ 000001	− 000003	+ 000005	+ 000007	+ 000009
+ 000007	+·000010	+ 000008	+ 000007	+ 000006	+ 000002	000000	+ 000002	+ 000001	000000	− 000007	+ 00000[illegible]	+ 000005	−·000001
+ 000004	000000	+ 000007	+·000001	+ 000001	+ 000007	+ 000005	+ 000007	+ 000008	+ 000008	+·000007	+ 000006	− 000008	− 000007
+ 000005	+ 000007	+ 000006	+ 000005	+ 000004	+ 000001	000000	+ 000001	000000	000000	− 000005	+ 000004	+ 000004	− 000001
+ 000003	000000	+ 000005	000000	+ 000001	+ 000005	+·000004	+ 000005	+ 000005	+ 000006	+ 000005	+ 000004	− 000006	− 000005
+ 000003	+ 000001	+ 000006	+ 000003	+ 000005	+ 000002	000000	+ 000001	+·000003	+ 000001	− 000004	+·000001	000000	− 000003
− 000002	− 000004	− 000002	− 000004	+ 000001	+ 000001	000000	− 000002	+·000001	+·000004	+ 000001	+ 000005	+ 000001	+ 000002

The mean variations $f_{c\,1}(h)$, $f_{s\,1}(h)$ (lines g and h of Tables 11 to 18), are curved in order in figs. 22 to 37.

15. Extending now the process of elimination described in paragraph 11, so as to further clear the observed variations of their elements $f_{c\,1}(h)$, $f_{s\,1}(h)$, &c , as shown on lines a and b of Tables 11 to 18, we subtract $f_{c\,1}(h)$ from the observed lunar diurnal variation at new moon, commencing with 0 hours , subtract $f_{s\,1}(h)$ from the observed variation at first quarter, commencing with 6 hours ; add $f_{c\,1}(h)$ to the observed variation at full moon, commencing with 12 hours , and add $f_{s\,1}(h)$ to the observed variation at last quarter, commencing with 18 hours; thus reducing the residuals formerly obtained to four new residuals, the mean of which may be taken as the residual lunar diurnal variation for the quarters of the moon. A similar process, using lines c and d of Tables 11 to 18, in lieu of lines a and b, leads to the residual lunar diurnal variation for the eighths phases. These residuals, for each quarter of the year and for both the magnetic elements, were found to be practically insignificant, the hourly numbers in all of them lying between the limits $\pm$·000002. From this we conclude that, in each quarter of the year and for each magnetic element separately, our extended formula practically embodies the whole of the observed lunar diurnal variation for the full lunation.

16. Coefficients of the harmonic equivalents of the typical variations, expressed in the form

$$a_1 \cos \frac{2\pi}{24}h + b_1 \sin \frac{2\pi}{24}h + a_2 \cos 2\left(\frac{2\pi}{24}h\right) + b_2 \sin 2\left(\frac{2\pi}{24}h\right) + \&c.,$$

were calculated for each case, the data being taken from Tables 9 and 10, and from lines g and h of Tables 11 to 18, and their values are exhibited in Tables 19 and 20.

TABLE 19.

Declination		a_1	b_1	a_2	b_2	A_1	B_1	A_2	B_2
November to January	$f_{c\,1}(h)$	+ 0000005	− 0000034	+ 0000006	+ 0000004				
	$f_{s\,1}(h)$		..	...	...	+ 0000023	+ ·0000023	+ 0000010	− 0000003
	$f_{c\,2}(h)$	+ ·0000031	+ ·0000026	+ 0000097	+ ·0000040				
	$f_{s\,2}(h)$	..	...	...	...	− 0000030	+ ·0000031	− 0000055	+ ·0000077
February to April	$f_{c\,1}(h)$	− 0000032	− 0000010	+ 0000009	+ ·0000009				
	$f_{s\,1}(h)$	...	...		.	+ ·0000008	− ·0000006	− 0000010	0000000
	$f_{c\,2}(h)$	+ 0000014	− 0000022	+ 0000023	− ·0000009				
	$f_{s\,2}(h)$	...	..	.	.	− ·0000014	− 0000001	− 0000002	+ 0000016
May to July	$f_{c\,1}(h)$	− 0000014	− 0000019	− 0000001	− ·0000011				
	$f_{s\,1}(h)$	..	...	..	..	+ ·0000029	+ 0000008	+ ·0000002	− ·0000010
	$f_{c\,2}(h)$	− ·0000043	·0000000	− 0000078	+ 0000013				
	$f_{s\,2}(h)$	.	...	..	..	− ·0000024	− 0000049	− ·0000017	− ·0000080
August to October	$f_{c\,1}(h)$	− ·0000004	+ 0000020	− 0000007	+ 0000003				
	$f_{s\,1}(h)$	..	...	...	...	− 0000001	− ·0000004	− ·0000017	− ·0000016
	$f_{c\,2}(h)$	− ·0000054	+ ·0000008	− ·0000076	+ 0000047				
	$f_{s\,2}(h)$		...	...	..	− ·0000016	− 0000040	− 0000050	− ·0000048

TABLE 20.

Horizontal Force		a_1	b_1	a_2	b_2	A_1	B_1	A_2	B_2
November to January	$f_c\,_1(h)$	− 0000020	+ 0000015	− 0000021	− 0000013				
	$f_s\,_1(h)$		...			− 0000050	− 0000028	− 0000023	− 0000012
	$f_c\,_2(h)$	+ 0000006	− 0000093	+ 0000001	− 0000126				
	$f_s\,_2(h)$	...	...			+ 0000136	− ·0000017	+ 0000134	+ 0000007
February to April	$f_c\,_1(h)$	− 0000038	− 0000004	+ 0000018	− 0000006				
	$f_s\,_1(h)$					+ 0000015	+ 0000003	− 0000014	+ ·0000025
	$f_c\,_2(h)$	+ ·0000014	− 0000036	+ 0000011	− ·0000067				
	$f_s\,_2(h)$					+ 0000123	+ 0000018	+ 0000071	+ 0000010
May to July	$f_c\,_1(h)$	− 0000023	− 0000003	− 0000010	+ 0000025				
	$f_s\,_1(h)$					+ 0000028	+ 0000026	+ 0000013	+ 0000028
	$f_c\,_2(h)$	− 0000021	− 0000032	− 0000014	− 0000021				
	$f_s\,_2(h)$					+ ·0000053	+ 0000017	+ 0000044	+ 0000007
August to October	$f_c\,_1(h)$	− 0000028	− 0000017	+ 0000018	− 0000006				
	$f_s\,_1(h)$					+ ·0000038	− 0000017	+ 0000008	+ 0000008
	$f_c\,_2(h)$	− 0000004	− 0000037	− 0000003	− 0000033				
	$f_s\,_2(h)$					+ 0000067	− 0000013	+ 0000063	+ 0000010

And from these coefficients were calculated the values of $\frac{1}{2}\sqrt{(a_1+B_1)^2+(b_1-A_1)^2}$; $\frac{1}{2}\sqrt{(a_2+B_2)^2+(b_2-A_2)^2}$; $\frac{1}{2}\sqrt{(a_2-B_2)^2+(b_2+A_2)^2}$; and $\frac{1}{2}\sqrt{(a_1-B_1)^2+(b_1+A_1)^2}$ for each pair of typical variations $f_c(h), f_s(h)$. These quantities are the coefficients A, B, C, D, respectively, of our transformed formula in paragraph 13, and their values are entered in Tables 21 and 22.

TABLE 21.

Declination.	A_1	B_1	C_1	D_1	A_2	B_2	C_2	D_2
November to January	0000031	0000003	0000008	0000010	0000042	0000099	0000012	0000002
February to April	0000021	0000010	0000004	0000013	0000007	0000019	·0000006	0000019
May to July	0000024	0000008	0000006	0000012	0000048	0000080	0000002	0000012
August to October	0000011	0000015	0000008	0000009	0000049	0000078	0000014	0000008

TABLE 22.

Horizontal force	A_1	B_1	C_1	D_1	A_2	B_2	C_2	D_2
November to January . .	0000038	0000017	0000018	·0000017	·0000114	·0000130	0000054	0000024
February to April . .	0000019	0000021	0000010	0000021	0000081	0000070	0000002	·0000043
May to July . .	·0000015	0000011	0000027	0000027	·0000042	0000032	0000015	0000021
August to October . .	0000038	0000012	0000002	0000009	0000053	0000048	0000016	0000016

17. Regarding these numbers as the amplitudes of the eight simple waves whose periods are specified in paragraph 13, we see that generally B_2, and in a less degree A_1—which refer to the lunar half-day and the lunar day respectively—form important elements of the whole luni-solar variation; and scarcely less important are the elements that have the A_2's for their amplitudes. the numbers for the remaining five periods are generally smaller, though for none of them are they so small as to approach general insignificance.

18. Up to this point we have, in comparing our formula with observation, had to adapt to the purpose variations obtained for the period of the lunar day. It would obviously be a more direct procedure to arrange the so-called lunar differences of a given season in solar days, according to the age of the moon, and, although the labour of copying out the numbers so arranged, and of re-computing, is considerable, we have thought the verification of the results obtained by the indirect mode of procedure of sufficient importance to justify the undertaking of the task, at least in respect of the element of declination, and for that season, November to January, when the luni-solar variations at Bombay have the largest range. Accordingly, abstract sheets were prepared for the 1st, 2nd, 3rd, &c., solar days after new moon, embracing all such days in the months November to January of the years 1846 to 1870. The hourly sums and means and the diurnal variations were then taken on each abstract sheet, with the results shown in Table 23.

TABLE 23.—Luni-solar Variations of Declination for each Solar Day after New Moon in the Quarter November to January, expressed in C.G S Units of Force.

Bombay astronomical hours.	Days after new moon														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h. m															
12 12	+ 0000045	+ 0000028	+ 0000091	− 0000072	− 0000036	− 0000007	− 0000054	− 0000019	+ 0000013	+ 0000029	+ 0000010	+ 0000011	+ 0000081	+ 0000113	+ 0000086
13 12	+ 0000010	+ ·0000028	+ 0000082	− 0000002	− 0000023	− 0000071	− 0000042	+ 0000060	− 0000019	− 0000125	+ 0000013	− 0000009	− 0000029	+ 0000076	− 0000005
14 12	− 0000007	+ 0000064	+ 0000079	+ 0000112	− 0000004	− 0000080	− 0000027	− 0000054	+ 0000027	− 0000094	− 0000086	0000000	− 0000124	+ 0000041	0000000
15 12	+ 0000017	− 0000012	+ 0000093	+ 0000130	+ 0000083	− 0000012	− 0000028	− 0000085	− 0000018	− 0000108	− 0000056	− 0000018	− 0000101	− 0000006	− 0000077
16 12	+ 0000051	+ 0000010	+ 0000065	+ 0000144	+ 0000027	+ 0000038	+ 0000013	− 0000075	− 0000035	− ·0000096	− 0000059	− 0000037	− 0000142	− 0000043	− 0000161
17 12	− 0000068	− 0000092	+ 0000067	+ 0000064	+ 0000041	+ 0000015	+ 0000070	+ 0000025	+ 0000028	+ 0000028	− 0000060	− 0000044	− 0000159	− 0000069	− 0000219
18 12	− 0000101	− 0000126	− 0000027	+ 0000056	+ 0000097	+ 0000069	+ 0000039	+ 0000049	+ 0000071	+ 0000070	− 0000014	− 0000033	− 0000157	− 0000163	− 0000229
19 12	− 0000172	− 0000219	+ 0000041	+ 0000067	− 0000002	+ 0000150	+ 0000191	+ 0000040	+ 0000077	+ 0000148	− 0000020	− 0000005	− 0000130	− 0000158	− 0000295
20 12	− 0000029	− 0000101	− 0000017	− 0000087	+ 0000059	+ 0000110	+ 0000128	+ 0000066	− 0000029	+ 0000118	− 0000010	+ 0000062	− 0000041	− 0000252	− 0000244
21 12	− ·0000002	− 0000020	− 0000145	− 0000100	+ 0000007	+ 0000060	+ 0000035	− 0000052	− 0000044	+ 0000023	+ 0000003	+ 0000068	+ 0000063	− 0000170	− 0000206
22 12	+ 0000117	− 0000082	− 0000126	− 0000162	− 0000135	− 0000015	− 0000040	− 0000125	+ 0000016	− 0000031	+ 0000074	+ 0000167	+ 0000234	− 0000176	+ 0000028
23 12	+ 0000091	− 0000019	− 0000263	− 0000050	− 0000157	− 0000148	− 0000093	− 0000060	− 0000032	− 0000004	+ 0000075	+ 0000069	+ 0000211	− 0000081	+ ·0000171
0 12	+ 0000163	− 0000046	− 0000071	− 0000156	− 0000030	− 0000079	− ·0000139	− 0000033	− 0000059	− 0000005	+ 0000114	− 0000016	+ 0000218	+ 0000076	+ 0000183
1 12	+ 0000283	+ 0000191	− 0000016	+ 0000078	+ 0000029	− 0000109	− ·0000164	− 0000148	− 0000104	− 0000132	+ 0000041	+ 0000014	+ 0000209	+ 0000160	+ 0000221
2 12	+ 0000177	+ 0000285	+ 0000062	+ 0000064	− 0000051	− 0000127	− 0000025	− 0000194	− 0000113	− 0000202	+ 0000014	+ 0000070	+ 0000026	+ 0000187	+ 0000276
3 12	+ 0000140	+ 0000290	+ 0000147	+ 0000043	+ 0000118	+ 0000053	+ 0000038	− 0000071	− 0000179	− 0000248	− 0000110	− 0000131	− 0000059	+ 0000071	+ 0000113
4 12	− 0000163	+ 0000079	+ 0000165	+ 0000133	+ 0000148	+ 0000238	+ 0000093	+ 0000117	− 0000094	+ 0000014	− 0000164	− 0000135	− 0000239	− 0000012	− 0000151
5 12	− 0000165	− 0000129	− 0000036	+ 0000099	+ 0000082	+ 0000125	+ 0000134	+ 0000112	+ 0000087	+ 0000131	+ 0000019	− 0000111	− 0000214	− 0000130	− 0000059
6 12	− 0000135	− 0000070	− 0000112	+ 0000033	− 0000024	+ 0000032	− 0000001	+ 0000161	+ 0000114	+ 0000138	+ 0000016	− 0000035	+ 0000004	+ 0000093	+ 0000068
7 12	− 0000100	− 0000071	− 0000043	− 0000027	− 0000064	− 0000011	+ 0000019	+ 0000112	+ 0000065	+ 0000148	+ 0000031	− 0000022	+ 0000031	+ 0000045	+ 0000157
8 12	− 0000129	− 0000059	− 0000046	− 0000126	− 0000022	− 0000041	− 0000064	+ 0000059	+ 0000107	+ 0000088	+ 0000106	+ 0000019	+ 0000135	+ 0000141	+ 0000061
9 12	− 0000081	+ 0000027	− ·0000019	− 0000108	− 0000043	− 0000063	− 0000050	+ 0000023	+ 0000027	+ 0000102	+ 0000010	+ 0000080	+ 0000035	+ 0000087	+ 0000062
10 12	− 0000032	+ 0000019	+ 0000015	− 0000069	− 0000062	− 0000054	− 0000005	+ ·0000083	+ 0000046	+ 0000050	− 0000015	+ 0000047	+ 0000093	+ 0000067	+ 0000100
11 12	+ 0000049	+ 0000033	+ 0000022	− 0000063	− 0000038	− ·0000062	− 0000019	+ 0000005	+ 0000047	− 0000047	+ 0000066	− 0000003	+ 0000063	+ 0000055	+ 0000113

TABLE 23—continued.

Bombay astronomical hours		Days after new moon														
		16	17	18	19	20	21	−9	−8	−7	−6	−5	−4	−3	−2	−1
h.	m.															
12	12	+ ·0000077	− ·0000029	− ·0000001	− ·0000017	− ·0000014	− ·0000056	+ ·0000018	− ·0000060	− ·0000064	+ ·0000014	− ·0000097	+ ·0000070	− ·0000010	+ ·0000016	+ ·0000060
13	12	+ ·0000108	− ·0000052	+ ·0000006	− ·0000021	− ·0000048	+ ·0000051	− ·0000004	− ·0000023	− ·0000095	− ·0000013	− ·0000060	+ ·0000050	− ·0000039	+ ·0000060	+ ·0000111
14	12	+ ·0000072	− ·0000033	+ ·0000052	− ·0000025	− ·0000005	+ ·0000062	+ ·0000013	− ·0000016	− ·0000077	+ ·0000072	− ·0000034	+ ·0000026	− ·0000063	− ·0000012	+ ·0000101
15	12	− ·0000015	− ·0000068	+ ·0000098	+ ·0000071	+ ·0000108	+ ·0000029	+ ·0000086	+ ·0000061	− ·0000056	+ ·0000037	− ·0000076	+ ·0000005	+ ·0000023	− ·0000009	+ ·0000031
16	12	− ·0000100	− ·0000063	+ ·0000118	+ ·0000081	+ ·0000059	+ ·0000067	+ ·0000068	+ ·0000002	+ ·0000010	+ ·0000019	− ·0000013	+ ·0000044	− ·0000074	·0000000	+ ·0000037
17	12	− ·0000081	− ·0000084	+ ·0000055	+ ·0000116	+ ·0000091	+ ·0000006	+ ·0000042	+ ·0000045	+ ·0000037	+ ·0000086	− ·0000005	− ·0000034	− ·0000051	− ·0000035	+ ·0000022
18	12	− ·0000161	− ·0000116	+ ·0000006	+ ·0000053	+ ·0000112	+ ·0000019	+ ·0000021	+ ·0000103	+ ·0000067	+ ·0000123	+ ·0000109	− ·0000015	+ ·0000006	+ ·0000013	− ·0000126
19	12	− ·0000171	− ·0000133	− ·0000146	+ ·0000038	+ ·0000103	+ ·0000177	+ ·0000013	+ ·0000194	+ ·0000082	+ ·0000163	+ ·0000048	+ ·0000035	− ·0000033	− ·0000063	− ·0000289
20	12	− ·0000053	− ·0000085	− ·0000191	− ·0000010	− ·0000083	+ ·0000193	+ ·0000027	+ ·0000080	+ ·0000132	+ ·0000187	+ ·0000055	+ ·0000020	+ ·0000035	− ·0000051	− ·0000196
21	12	− ·0000141	− ·0000133	− ·0000204	− ·0000069	+ ·0000004	+ ·0000095	+ ·0000014	− ·0000019	+ ·0000214	+ ·0000157	+ ·0000155	+ ·0000243	+ ·0000164	− ·0000044	+ ·0000036
22	12	− ·0000003	− ·0000067	− ·0000194	− ·0000164	− ·0000143	− ·0000068	− ·0000004	− ·0000027	+ ·0000202	+ ·0000147	+ ·0000234	+ ·0000151	+ ·0000236	+ ·0000007	− ·0000081
23	12	+ ·0000020	+ ·0000048	− ·0000167	− ·0000192	− ·0000009	− ·0000141	+ ·0000002	− ·0000068	+ ·0000082	− ·0000125	+ ·0000180	+ ·0000081	+ ·0000191	+ ·0000109	+ ·0000117
0	12	+ ·0000004	+ ·0000028	− ·0000108	− ·0000112	− ·0000132	− ·0000215	− ·0000043	− ·0000096	− ·0000061	− ·0000171	+ ·0000070	+ ·0000033	+ ·0000258	+ ·0000176	+ ·0000181
1	12	+ ·0000043	+ ·0000252	− ·0000085	− ·0000106	− ·0000056	− ·0000218	− ·0000291	+ ·0000010	− ·0000116	− ·0000253	+ ·0000014	− ·0000053	+ ·0000247	+ ·0000117	+ ·0000274
2	12	+ ·0000158	+ ·0000387	+ ·0000158	+ ·0000083	+ ·0000017	− ·0000192	− ·0000176	− ·0000216	− ·0000135	− ·0000230	− ·0000128	− ·0000085	+ ·0000023	− ·0000072	+ ·0000211
3	12	+ ·0000146	+ ·0000247	+ ·0000232	+ ·0000198	+ ·0000020	− ·0000086	− ·0000103	− ·0000159	− ·0000138	− ·0000132	− ·0000141	− ·0000273	− ·0000074	− ·0000022	+ ·0000087
4	12	+ ·0000053	+ ·0000075	+ ·0000212	+ ·0000130	+ ·0000142	+ ·0000211	+ ·0000056	+ ·0000129	− ·0000108	− ·0000211	− ·0000063	− ·0000237	− ·0000238	− ·0000104	− ·0000092
5	12	− ·0000130	− ·0000062	+ ·0000094	+ ·0000091	+ ·0000160	+ ·0000190	+ ·0000100	+ ·0000146	+ ·0000022	− ·0000007	− ·0000074	− ·0000098	− ·0000169	− ·0000060	− ·0000060
6	12	− ·0000039	+ ·0000001	+ ·0000021	− ·0000011	− ·0000044	− ·0000013	+ ·0000044	+ ·0000017	− ·0000018	− ·0000019	+ ·0000031	− ·0000047	− ·0000126	+ ·0000005	− ·0000154
7	12	− ·0000009	− ·0000046	+ ·0000065	− ·0000035	− ·0000091	+ ·0000005	+ ·0000005	+ ·0000014	− ·0000007	+ ·0000042	− ·0000058	− ·0000007	− ·0000068	+ ·0000017	− ·0000118
8	12	+ ·0000038	− ·0000062	− ·0000017	− ·0000081	− ·0000090	− ·0000043	+ ·0000027	+ ·0000028	+ ·0000007	+ ·0000011	− ·0000068	+ ·0000056	− ·0000075	− ·0000013	− ·0000069
9	12	+ ·0000043	− ·0000026	− ·0000043	− ·0000029	− ·0000022	+ ·0000018	+ ·0000030	− ·0000021	+ ·0000055	+ ·0000055	+ ·0000012	− ·0000027	− ·0000070	− ·0000001	+ ·0000011
10	12	+ ·0000071	·0000000	+ ·0000036	− ·0000067	− ·0000090	− ·0000028	+ ·0000022	− ·0000037	− ·0000006	+ ·0000001	− ·0000082	− ·0000011	− ·0000070	− ·0000011	− ·0000087
11	12	+ ·0000045	− ·0000016	+ ·0000001	− ·0000063	+ ·0000017	− ·0000066	+ ·0000045	− ·0000085	− ·0000080	+ ·0000038	− ·0000017	+ ·0000067	− ·0000030	− ·0000028	+ ·0000001

These variations are curved in thin lines in fig. 48, being numbered 1 to 20 and −9 to −1 in order.

19. In calculating luni-solar variations for each day of the moon's age by our extended formula, we carried the harmonic equivalents of the typical variations of lines 1 and 5 of Tables 9 and 10, and of lines g and h of Tables 11 and 15, to six pairs of terms, and used these equivalents instead of the typical variations from which they were derived. The coefficients found were, for November to January—

TABLE 24.—Declination.

	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	a_5	b_5	a_6	b_6
$f_{e\cdot 1}(h)$ . .	+ 0000005	− 0000034	+ 0000006	+ 0000004	− 0000002	− 0000009	− 0000002	− 0000006	+ 0000012	+ 0000016	+ 0000011	+ 0000006
$f_{s\cdot 1}(h)$ . .	+ 0000023	+ 0000023	+ 0000010	− 0000003	− 0000004	+ 0000001	+ 0000008	0000000	− 0000004	− 0000005	− 0000002	+ 0000001
$f_{e\cdot 2}(h)$ . .	+ 0000031	+ 0000026	+ 0000097	+ ·0000040	+ 0000033	− 0000031	+ 0000001	+ 0000024	− 0000007	+ 0000003	0000000	− 0000008
$f_{s\cdot 2}(h)$ . .	− 0000030	+ 0000031	− 0000055	+ 0000077	− 0000023	+ 0000023	+ 0000001	− 0000009	+ 0000008	− 0000015	+ 0000005	+ 0000001

TABLE 25.—Horizontal Force.

	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	a_5	b_5	a_6	b_6
$f_{e\cdot 1}(h)$	− 0000020	+ 0000015	− 0000021	− 0000013	+ 0000005	+ 0000007	0000000	+ 0000018	+ 0000003	− ·0000002	− 0000019	− 0000001
$f_{s\cdot 1}(h)$ . . .	− 0000050	− 0000023	− ·0000023	− 0000012	+ 0000004	+ 0000002	+ 0000006	− 0000011	− 0000008	− 0000005	− 0000001	− 0000008
$f_{e\cdot 2}(h)$.	+ 0000006	− 0000093	+ 0000001	− 0000126	− 0000009	− 0000053	− 0000009	− 0000019	− 0000006	+ 0000003	− 0000004	+ 0000015
$f_{s\cdot 2}(h)$	+ 0000136	− 0000017	+ 0000134	+ 0000007	+ 0000038	− 0000028	+ 0000005	− 0000009	− 0000009	− 0000011	+ 0000009	− 0000010

20. And now, using 12 as the hour of occurrence of new moon on the initial solar day, the luni-solar variations found were as follows:—

TABLE 26.—Declination. November to January. Values, for each Solar Hour of the Moon's Age, of

$$f_{c1}(h)\cos\left(2\pi\frac{t}{P}\right)+f_{s1}(h)\sin\left(2\pi\frac{t}{P}\right)+f_{c2}(h)\cos 2\left(2\pi\frac{t}{P}\right)+f_{s2}(h)\sin 2\left(2\pi\frac{t}{P}\right)$$

Bombay astronomical hours.	Days after new moon.														
	−9	−8	−7	−6	−5	−4	−3	−2	−1	+1	+2	+3	+4	+5	+6
12	− ·0000036	− ·0000042	− ·0000040	− ·0000033	− ·0000020	− ·0000002	+ ·0000015	+ ·0000030	+ ·0000039	+ ·0000041	+ ·0000035	+ ·0000024	+ ·0000009	− ·0000012	− ·0000024
13	− ·0000011	− ·0000022	− ·0000029	− ·0000030	− ·0000026	− ·0000015	·0000000	+ ·0000014	+ ·0000026	+ ·0000034	+ ·0000037	+ ·0000031	+ ·0000020	+ ·0000006	− ·0000012
14	+ ·0000023	+ ·0000012	+ ·0000003	− ·0000002	− ·0000002	+ ·0000003	+ ·0000012	+ ·0000024	+ ·0000034	+ ·0000041	+ ·0000044	+ ·0000039	+ ·0000027	+ ·0000010	− ·0000013
15	+ ·0000058	+ ·0000053	+ ·0000047	+ ·0000040	+ ·0000035	+ ·0000029	+ ·0000038	+ ·0000040	+ ·0000044	+ ·0000050	+ ·0000051	+ ·0000048	+ ·0000038	+ ·0000021	− ·0000002
16	+ ·0000093	+ ·0000089	+ ·0000077	+ ·0000056	+ ·0000036	+ ·0000018	+ ·0000007	+ ·0000003	+ ·0000003	+ ·0000019	+ ·0000033	+ ·0000051	+ ·0000054	+ ·0000052	+ ·0000041
17	+ ·0000125	+ ·0000115	+ ·0000088	+ ·0000048	+ ·0000007	− ·0000033	− ·0000062	− ·0000075	− ·0000073	− ·0000054	− ·0000023	+ ·0000012	+ ·0000044	+ ·0000070	+ ·0000080
18	+ ·0000129	+ ·0000120	+ ·0000092	+ ·0000048	− ·0000002	− ·0000051	− ·0000088	− ·0000111	− ·0000114	− ·0000099	− ·0000066	− ·0000023	+ ·0000020	+ ·0000049	+ ·0000085
19	+ ·0000103	+ ·0000120	+ ·0000118	+ ·0000096	+ ·0000064	+ ·0000021	− ·0000021	− ·0000057	− ·0000077	− ·0000082	− ·0000070	− ·0000043	− ·0000009	+ ·0000027	+ ·0000059
20	+ ·0000055	+ ·0000109	+ ·0000146	+ ·0000161	+ ·0000152	+ ·0000122	+ ·0000076	+ ·0000026	− ·0000023	− ·0000059	− ·0000078	− ·0000078	− ·0000046	− ·0000042	+ ·0000018
21	− ·0000006	+ ·0000070	+ ·0000132	+ ·0000172	+ ·0000183	+ ·0000167	+ ·0000125	+ ·0000063	− ·0000003	− ·0000061	− ·0000076	− ·0000124	− ·0000117	− ·0000084	− ·0000035
22	− ·0000079	− ·0000018	+ ·0000053	+ ·0000107	+ ·0000139	+ ·0000146	+ ·0000125	+ ·0000082	+ ·0000024	− ·0000038	− ·0000091	− ·0000128	− ·0000138	− ·0000123	− ·0000082
23	− ·0000155	− ·0000108	− ·0000048	+ ·0000020	+ ·0000081	+ ·0000122	+ ·0000142	+ ·0000135	+ ·0000103	+ ·0000055	+ ·0000004	− ·0000058	− ·0000099	− ·0000118	− ·0000114
0	− ·0000217	− ·0000192	− ·0000139	− ·0000057	+ ·0000029	+ ·0000107	+ ·0000167	+ ·0000199	+ ·0000199	+ ·0000165	+ ·0000107	+ ·0000035	− ·0000089	− ·0000099	− ·0000135
1	− ·0000237	− ·0000245	− ·0000195	− ·0000123	− ·0000035	+ ·0000057	+ ·0000139	+ ·0000195	+ ·0000217	+ ·0000202	+ ·0000153	+ ·0000079	− ·0000005	− ·0000082	− ·0000141
2	− ·0000179	− ·0000211	− ·0000216	− ·0000184	− ·0000125	− ·0000051	+ ·0000028	+ ·0000095	+ ·0000139	+ ·0000154	+ ·0000139	+ ·0000094	+ ·0000032	− ·0000035	− ·0000095
3	− ·0000056	− ·0000120	− ·0000168	− ·0000189	− ·0000184	− ·0000153	− ·0000102	− ·0000039	+ ·0000024	+ ·0000075	+ ·0000111	+ ·0000120	+ ·0000105	+ ·0000071	+ ·0000025
4	+ ·0000074	+ ·0000009	− ·0000063	− ·0000127	− ·0000172	− ·0000189	− ·0000177	− ·0000137	− ·0000085	− ·0000002	+ ·0000069	+ ·0000128	+ ·0000164	+ ·0000169	+ ·0000147
5	+ ·0000128	+ ·0000087	+ ·0000026	− ·0000043	− ·0000106	− ·0000152	− ·0000173	− ·0000165	− ·0000128	− ·0000069	+ ·0000002	+ ·0000075	+ ·0000135	+ ·0000173	+ ·0000181
6	+ ·0000096	+ ·0000078	+ ·0000044	− ·0000002	− ·0000050	− ·0000095	− ·0000125	− ·0000137	− ·0000132	− ·0000104	− ·0000062	− ·0000011	+ ·0000039	+ ·0000082	+ ·0000109
7	+ ·0000044	+ ·0000037	+ ·0000021	− ·0000001	− ·0000027	− ·0000055	− ·0000080	− ·0000097	− ·0000105	− ·0000101	− ·0000087	− ·0000063	− ·0000035	− ·0000002	+ ·0000025
8	+ ·0000021	+ ·0000017	+ ·0000009	− ·0000003	− ·0000016	− ·0000033	− ·0000049	− ·0000058	− ·0000072	− ·0000075	− ·0000073	− ·0000063	− ·0000048	− ·0000027	− ·0000006
9	+ ·0000010	+ ·0000020	+ ·0000018	+ ·0000011	+ ·0000002	− ·0000012	− ·0000025	− ·0000037	− ·0000048	− ·0000052	− ·0000056	− ·0000052	− ·0000043	− ·0000029	− ·0000014
10	− ·0000002	+ ·0000006	+ ·0000013	+ ·0000018	+ ·0000020	+ ·0000018	+ ·0000012	+ ·0000001	− ·0000012	− ·0000024	− ·0000036	− ·0000048	− ·0000043	− ·0000039	− ·0000031
11	− ·0000034	− ·0000025	− ·0000012	+ ·0000005	+ ·0000019	+ ·0000031	+ ·0000037	+ ·0000035	+ ·0000026	+ ·0000012	− ·0000003	− ·0000022	− ·0000036	− ·0000044	− ·0000047

TABLE 26—continued.

Bombay astronomical hours.	Days after new moon.													
	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18	+19	+20
12	− 0000035	− 0000040	− 0000038	− 0000028	− 0000018	+ 0000004	+ 0000021	+ 0000033	+ 0000040	+ 0000041	+ 0000034	+ 0000021	+ 0000003	− 0000015
13	− 0000027	− 0000039	− 0000044	− 0000043	− 0000034	− 0000022	− 0000005	+ 0000012	+ 0000026	+ 0000036	+ 0000039	+ ·0000035	+ 0000026	+ 0000013
14	− 0000035	− 0000054	− 0000068	− 0000074	− ·0000074	− 0000063	− ·0000046	− 0000025	− 0000003	+ 0000017	+ 0000032	+ 0000041	+ 0000043	+ 0000038
15	− 0000026	− 0000054	− 0000077	− 0000096	− 0000106	− 0000107	− 0000097	− 0000080	− 0000057	− 0000029	− 0000002	+ 0000023	+ 0000041	+ 0000052
16	+ 0000020	− 0000011	− 0000045	− 0000079	− 0000109	− 0000127	− 0000134	− 0000125	− 0000108	− ·0000069	− 0000029	+ 0000012	+ 0000048	+ 0000073
17	+ 0000074	+ 0000051	+ 0000013	− 0000015	− ·0000071	− 0000107	− 0000125	− 0000127	− 0000109	− 0000074	− 0000027	+ 0000022	+ 0000069	+ 0000104
18	+ 0000092	+ 0000080	+ 0000052	+ 0000013	− 0000031	− 0000070	− 0000096	− 0000107	− ·0000097	− 0000068	− 0000027	+ 0000021	+ 0000068	+ 0000106
19	+ 0000078	+ 0000081	+ 0000068	+ 0000039	0000000	− 0000043	− 0000082	− 0000109	− 0000120	− 0000112	− 0000088	− 0000047	+ 0000001	+ 0000048
20	+ 0000055	+ 0000084	+ 0000093	+ 0000083	+ 0000053	+ 0000006	− 0000047	− 0000101	− 0000144	− 0000169	− 0000170	− 0000147	− 0000102	− 0000043
21	+ 0000025	+ 0000080	+ 0000120	+ 0000138	+ 0000128	+ 0000092	+ 0000036	− 0000038	− 0000101	− 0000157	− 0000190	− 0000194	− 0000168	− 0000116
22	− 0000024	+ 0000042	+ 0000101	+ 0000145	+ 0000166	+ 0000157	+ 0000123	+ 0000065	− 0000003	− 0000072	− 0000129	− 0000165	− 0000173	− 0000154
23	− 0000086	− 0000036	+ ·0000021	+ 0000080	+ 0000128	+ 0000155	+ 0000155	+ 0000129	+ 0000080	+ 0000016	− 0000053	− 0000114	− 0000159	− 0000177
0	− 0000152	− 0000120	− 0000072	− 0000007	+ 0000059	+ 0000117	+ 0000158	+ ·0000160	+ ·0000137	+ 0000083	+ 0000011	− 0000066	− 0000140	− 0000194
1	− 0000170	− 0000164	− 0000123	− 0000058	+ 0000021	+ 0000099	+ 0000161	+ 0000193	+ 0000191	+ 0000154	+ 0000086	+ 0000001	− 0000091	− 0000169
2	− ·0000136	− 0000149	− 0000130	− 0000084	− 0000015	+ 0000062	+ 0000135	+ 0000187	+ 0000213	+ 0000207	+ 0000167	+ 0000098	+ 0000014	− 0000073
3	− 0000024	− 0000068	− 0000093	− 0000097	− 0000076	− 0000036	+ 0000020	+ 0000076	+ 0000127	+ 0000160	+ 0000172	+ 0000155	+ 0000114	+ 0000052
4	+ 0000102	+ 0000040	− 0000027	− 0000088	− 0000123	− 0000136	− 0000122	− 0000082	− 0000026	+ 0000040	+ 0000097	+ 0000138	+ 0000155	+ 0000143
5	+ 0000160	+ 0000113	+ ·0000049	− 0000021	− 0000087	− 0000133	− 0000154	− ·0000147	− 0000112	− 0000057	+ 0000008	+ 0000071	+ 0000118	+ 0000144
6	+ 0000119	+ 0000109	+ 0000082	+ 0000045	+ 0000001	− 0000038	− 0000065	− 0000076	− 0000070	− 0000049	− 0000013	+ 0000026	+ 0000062	+ 0000088
7	+ 0000049	+ 0000064	+ 0000069	+ 0000067	+ ·0000055	+ 0000041	+ 0000026	+ 0000014	+ 0000007	+ 0000007	+ 0000011	+ 0000020	+ 0000031	+ 0000040
8	+ 0000014	+ 0000032	+ ·0000045	+ 0000052	+ 0000054	+ 0000051	+ 0000046	+ 0000039	+ 0000031	+ 0000024	+ 0000021	+ 0000020	+ 0000020	+ 0000020
9	+ 0000002	+ 0000018	+ 0000028	+ 0000035	+ 0000037	+ 0000036	+ 0000031	+ 0000024	+ 0000018	+ 0000013	+ 0000011	+ 0000010	+ 0000012	+ 0000014
10	− 0000018	− 0000003	+ 0000011	+ 0000025	+ 0000031	+ 0000036	+ 0000035	+ 0000029	+ 0000019	+ 0000009	− 0000001	− 0000008	− 0000012	− ·0000010
11	− 0000040	− 0000027	− 0000010	+ 0000008	+ 0000024	+ 0000039	+ 0000046	+ 0000044	+ 0000036	+ 0000022	+ 0000004	− 0000011	− 0000026	− 0000034

TABLE 27.—Horizontal Force. November to January. Values, for each Solar Hour of the Moon's Age, of

$$f_{c,1}(h)\cos\left(2\pi\frac{t}{P}\right)+f_{s,1}(h)\sin\left(2\pi\frac{t}{P}\right)+f_{c,2}(h)\cos 2\left(2\pi\frac{t}{P}\right)+f_{s,2}(h)\sin 2\left(2\pi\frac{t}{P}\right)$$

Bombay astronomical hours.	Days after new moon.														
	−9	−8	−7	−6	−5	−4	−3	−2	−1	+1	+2	+3	+4	+5	+6
12	−·0000034	−·0000033	−·0000032	[illegible]	−·0000029	−·0000028	−·0000029	−·0000029	−·0000032	−·0000031	−·0000028	−·0000025	−·0000017	−·0000008	+·0000005
13	−·0000019	−·0000012	−·0000007	[illegible]	+·0000002	+·0000002	·0000000	−·0000005	−·0000009	−·0000014	−·0000018	−·0000018	−·0000015	−·0000010	·0000000
14	−·0000014	+·0000005	+·0000022	[illegible]	+·0000037	+·0000035	+·0000025	+·0000011	−·0000002	−·0000014	−·0000021	−·0000022	−·0000016	−·0000005	+·0000010
15	−·0000018	·0000000	+·0000014	+·0000012	+·0000021	+·0000012	−·0000003	−·0000022	−·0000039	−·0000051	−·0000055	−·0000050	−·0000035	−·0000012	+·0000021
16	−·0000067	−·0000048	−·0000027	−·0000008	+·0000003	+·0000005	−·0000001	−·0000017	−·0000036	−·0000053	−·0000065	−·0000069	−·0000059	−·0000032	−·0000008
17	−·0000138	−·0000100	−·0000051	−·0000005	+·0000037	+·0000063	+·0000071	+·0000060	+·0000037	+·0000004	−·0000030	−·0000061	−·0000068	−·0000062	−·0000039
18	−·0000170	−·0000116	−·0000048	+·0000027	+·0000077	+·0000118	+·0000134	+·0000126	+·0000098	+·0000054	+·0000006	−·0000037	−·0000066	−·0000073	−·0000059
19	−·0000163	−·0000125	−·0000049	−·0000001	+·0000061	+·0000113	+·0000143	+·0000150	+·0000130	+·0000090	+·0000036	−·0000020	−·0000068	−·0000099	−·0000109
20	−·0000167	−·0000179	−·0000156	−·0000106	−·0000038	+·0000041	+·0000110	+·0000157	+·0000175	+·0000161	+·0000136	+·0000046	−·0000038	−·0000109	−·0000169
21	−·0000165	−·0000236	−·0000260	[illegible]	−·0000171	−·0000072	+·0000040	+·0000144	+·0000218	+·0000255	+·0000236	+·0000175	+·0000077	−·0000035	−·0000149
22	−·0000076	−·0000201	−·0000295	[illegible]	−·0000295	−·0000219	−·0000102	+·0000032	+·0000158	+·0000250	+·0000298	+·0000287	+·0000221	+·0000118	−·0000020
23	+·0000091	−·0000062	−·0000201	[illegible]	−·0000344	−·0000327	−·0000253	−·0000135	+·0000001	+·0000131	+·0000233	+·0000287	+·0000281	+·0000221	+·0000112
0	+·0000249	+·0000110	−·0000034	−·0000164	−·0000259	−·0000303	−·0000292	−·0000229	−·0000130	−·0000016	+·0000097	+·0000182	+·0000223	+·0000213	+·0000154
1	+·0000312	+·0000233	+·0000125	−·0000005	−·0000104	−·0000185	−·0000226	−·0000223	−·0000180	−·0000106	−·0000017	+·0000070	+·0000134	+·0000161	+·0000149
2	+·0000323	+·0000287	+·0000211	+·0000108	−·0000005	−·0000107	−·0000184	−·0000222	−·0000217	−·0000178	−·0000099	−·0000009	+·0000076	+·0000139	+·0000171
3	+·0000285	+·0000276	+·0000224	+·0000140	+·0000037	−·0000067	−·0000152	−·0000220	−·0000241	−·0000221	−·0000164	−·0000082	+·0000009	+·0000093	+·0000152
4	+·0000192	+·0000210	+·0000197	+·0000152	+·0000085	+·0000003	−·0000076	−·0000146	−·0000177	−·0000204	−·0000178	−·0000142	−·0000076	−·0000006	+·0000058
5	+·0000067	+·0000120	+·0000156	+·0000170	+·0000151	+·0000146	+·0000061	−·0000005	−·0000068	−·0000119	−·0000148	−·0000152	−·0000132	−·0000088	−·0000034
6	−·0000017	+·0000051	+·0000113	+·0000155	+·0000172	+·0000164	+·0000129	+·0000077	+·0000017	−·0000040	−·0000082	−·0000106	−·0000105	−·0000081	−·0000040
7	−·0000050	+·0000010	+·0000064	+·0000104	+·0000126	+·0000127	+·0000108	+·0000074	+·0000032	−·0000012	−·0000042	−·0000059	−·0000058	−·0000040	−·0000008
8	−·0000065	−·0000012	+·0000040	+·0000083	+·0000110	+·0000120	+·0000111	+·0000085	+·0000048	·0000000	−·0000026	−·0000049	−·0000059	−·0000050	−·0000028
9	−·0000093	−·0000033	+·0000029	+·0000087	+·0000129	+·0000152	+·0000150	+·0000127	+·0000086	+·0000036	−·0000012	−·0000052	−·0000076	−·0000080	−·0000064
10	−·0000094	−·0000040	+·0000017	+·0000068	+·0000105	+·0000125	+·0000125	+·0000105	+·0000070	+·0000027	−·0000010	−·0000042	−·0000058	−·0000068	−·0000040
11	−·0000065	−·0000040	−·0000015	+·0000007	+·0000022	+·0000030	+·0000027	+·0000019	+·0000005	−·0000010	−·0000023	−·0000028	−·0000025	−·0000015	+·0000005

TABLE 27—continued.

Bombay astronomical hours.	Days after new moon.													
	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18	+19	+20
12	+ ·0000020	+ 0000034	+ 0000045	− 0000012	+ 0000062	+ 0000062	+ 0000058	+ 0000049	+ 0000036	+ ·0000021	+ 0000007	− 0000007	− 0000019	− 0000027
13	+ 0000009	+ 0000020	+ 0000029	+ 0000034	+ 0000036	+ 0000034	+ 0000029	+ 0000020	+ 0000007	− 0000003	− 0000012	− 0000020	− 0000024	− 0000024
14	+ 0000026	+ ·0000040	+ 0000049	+ 0000050	+ 0000043	+ 0000030	+ 0000015	− 0000012	− 0000033	− 0000050	− ·0000060	− 0000063	− 0000057	− 0000043
15	+ 0000045	+ 0000070	+ 0000085	+ 0000090	+ 0000085	+ 0000067	+ 0000042	+ 0000013	− 0000016	− ·0000041	− 0000055	− 0000063	− 0000060	− 0000046
16	+ ·0000029	+ 0000066	+ 0000099	+ 0000128	+ 0000133	+ 0000128	+ 0000108	+ 0000076	+ 0000039	− 0000003	− 0000037	− 0000066	− 0000082	− 0000085
17	0000000	+ 0000046	+ 0000093	+ 0000133	+ 0000157	+ 0000161	+ 0000143	+ 0000101	+ 0000049	− 0000018	− 0000072	− 0000121	− 0000155	− 0000164
18	− 0000025	+ 0000023	+ 0000078	+ 0000118	+ 0000148	+ 0000156	+ 0000137	+ 0000095	+ 0000033	− 0000039	− 0000107	− 0000164	− 0000200	− 0000189
19	− 0000093	− 0000058	− 0000009	+ 0000045	+ 0000094	+ 0000126	+ 0000136	+ 0000122	+ 0000083	+ 0000023	− 0000038	− 0000098	− 0000145	− 0000173
20	− 0000201	− 0000197	− 0000161	− 0000096	− 0000015	+ ·0000069	+ 0000140	+ 0000189	+ 0000206	+ 0000187	+ 0000139	+ 0000067	− ·0000014	− 0000091
21	− 0000240	− 0000292	− 0000294	− 0000247	− 0000156	− 0000037	+ 0000085	+ 0000197	+ 0000275	+ 0000308	+ 0000289	+ 0000222	+ 0000120	+ ·0000001
22	− 0000154	− 0000255	− 0000326	− 0000333	− 0000283	− 0000181	− 0000019	+ 0000097	+ 0000228	+ 0000321	+ 0000358	+ 0000337	+ 0000259	+ 0000138
23	− 0000022	− 0000157	− 0000266	− 0000333	− 0000341	− 0000286	− 0000180	− 0000040	+ 0000115	+ 0000254	+ 0000353	+ 0000396	+ 0000375	+ 0000294
0	+ 0000054	− 0000066	− 0000181	− 0000274	− 0000324	− 0000318	− 0000255	− 0000148	− 0000008	+ 0000139	+ 0000282	+ 0000362	+ ·0000400	+ 0000381
1	+ 0000097	+ ·0000014	− 0000085	− 0000180	− 0000258	− 0000291	− 0000281	− 0000227	− 0000130	− 0000010	+ 0000118	+ 0000227	+ 0000318	+ 0000351
2	+ 0000158	+ 0000112	+ 0000033	− 0000062	− 0000156	− 0000232	− 0000272	− 0000268	− 0000220	− 0000133	− 0000021	+ 0000097	+ 0000207	+ 0000285
3	+ ·0000177	+ 0000162	+ 0000113	+ 0000039	− 0000051	− 0000134	− 0000197	− ·0000225	− 0000212	− 0000160	− 0000077	+ 0000033	+ 0000128	+ 0000215
4	+ 0000106	+ 0000130	+ 0000125	+ 0000093	+ 0000041	− 0000012	− 0000078	− 0000120	− 0000138	− 0000128	− 0000089	− 0000032	+ ·0000041	+ ·0000112
5	+ 0000022	+ 0000066	+ 0000105	+ 0000116	+ 0000102	+ 0000067	+ 0000017	− 0000037	− 0000086	− 0000120	− ·0000132	− 0000172	− 0000100	− 0000029
6	+ 0000010	+ 0000060	+ 0000099	+ 0000117	+ 0000139	+ 0000088	+ 0000034	− 0000063	− ·0000088	− 0000140	− 0000174	− 0000182	− 0000162	− 0000118
7	+ ·0000080	+ 0000070	+ 0000098	+ 0000108	+ 0000102	+ 0000076	+ 0000031	− 0000023	− 0000087	− 0000127	− 0000161	− 0000172	− 0000160	− 0000129
8	+ 0000005	+ 0000010	+ 0000071	+ 0000091	+ 0000094	+ 0000080	+ 0000048	+ ·0000004	+ ·0000004	− 0000070	− 0000092	− 0000149	− 0000149	− ·0000129
9	− 0000032	+ 0000009	+ 0000050	+ 0000082	+ 0000099	+ ·0000095	+ 0000071	+ 0000029	− 0000023	− 0000079	− 0000128	− 0000160	− 0000170	− 0000158
10	− 0000010	+ 0000028	+ 0000063	+ 0000089	+ 0000101	+ ·0000094	+ 0000068	+ 0000026	− 0000025	− 0000078	− 0000128	− ·0000153	− 0000159	− 0000148
11	+ 0000028	+ 0000053	+ 0000074	+ 0000088	+ 0000091	+ 0000082	+ 0000062	+ 0000033	− 0000001	− 0000034	− 0000063	− 0000083	− ·0000089	− 0000089

21. The numbers of Table 26 are curved in dotted lines in fig. 48. The general correspondence of fact, as indicated by the thin-line curve, with our formula, as indicated by the dotted curves, cannot fail to arrest the attention of the reader; and, whatever be the physical conditions lying behind the phenomena, it cannot but be helpful towards their discovery that a compact mental grasp of the phenomena, such as the formula affords, should take the place of such notions of disconnected variations as are the first outcome of a study of the observations

22. The numbers of Table 27 are curved in dotted lines in fig. 49, and on the same form are curved, in thin lines, the observed lunar diurnal variations of horizontal force, taken from Column 2 of Tables 5 to 8, for each of the eight phases of the moon. Here again the comparison of observation, as indicated by the thin-line curves, with the formula, through its representative dotted curves, shows remarkably close agreement.

23 That in their more prominent inflexions the dotted curves of fig. 48 are generally of smaller range than the thin-line curves is sufficiently explained by the fact that the typical variations for each phase are averages for three or four consecutive days, whilst the thin-line curves represent variations for first days, second days, third days, &c., after new moon—all single days of the moon's age. Indeed, it is easy to show that the typical variations $f_{c.1}(h)$, $f_{s.1}(h)$ should have been multiplied by

$$\frac{\frac{4\pi}{29\cdot53}}{\sin\frac{4\pi}{29\cdot53}} = 1\cdot031$$ when derived from the observed variations at the quarters, and by

$$\frac{\frac{3\cdot39\pi}{29\cdot53}}{\sin\frac{3\cdot39\pi}{29\cdot53}} = 1\cdot022$$ when derived from the observed variations at the eighths phases;

and similarly that $f_{c.2}(h)$, $f_{s.2}(h)$ should have been multiplied by $\frac{\frac{8\pi}{29\cdot53}}{\sin\frac{8\pi}{29\cdot53}} = 1\cdot132$ and

by $\frac{\frac{6\cdot78\pi}{29\cdot53}}{\sin\frac{6\cdot78\pi}{29\cdot53}} = 1\cdot092$ in the same two cases respectively. The inadvertent omission to apply these factors is most influential in respect of the larger of them (1·132 and 1·092), since the range of the $f_{c.2}(h)$, $f_{s.2}(h)$ variations is much greater than that of the $f_{c.1}(h)$ and $f_{s.1}(h)$.

24. In the case of the horizontal force curves (fig. 49) the factors would, if applied, affect both the thick-line and the thin-line curves alike, the latter representing the data from which the former were constructed; here, accordingly, the two sets of curves approach more nearly to identity of range.

25. The thick lines of figs. 48 and 49 have been drawn to represent the two last terms alone of our formula, viz., of

$$f_{c2}(h) \cos 2\left(\frac{2\pi}{P}t\right) + f_{s2} \sin 2\left(\frac{2\pi}{P}t\right).$$

Noticing that they deviate but little from the dotted lines, it may be inferred that the bulk of the systematic part of the variations is embraced by the restricted formula; and accordingly it is to that part of the variation that goes through a cycle of change in half a lunation that attention should first be directed in seeking for efficient physical causes.

TREVANDRUM.

26. Having thus found that order and system of a simple kind pervade the manifestations of minute lunar variations of magnetic force at Bombay, we turn with interest to records of a similar nature obtained at the near, but more nearly magnetically equatorial, station of Trevandrum. In a volume entitled 'Observations of Magnetic Declination made at Trevandrum and Agustia Malley in the Observatories of H. H. the MAHARAJAH of Travancore,' and published by H. S KING and Co, of London, in 1874, the late JOHN ALLAN BROUN discussed the declination observations made at Trevandrum during the ten years 1854 to 1864, and—by a method of his own, which is, in essential points, similar to that described in the preceding pages—obtained values of the lunar diurnal variations for each month of the year and at each of the four quarters of the moon. Curves representing these variations appear as Plate 6 of the volume, and a glance at those for the months December and January, which—as at Bombay—have the largest range, is sufficient to show that our formula would correctly represent their principal features at each of the four quarters of the moon; indeed, those curves have a close resemblance to the corresponding winter curves for Bombay. An inspection of the curves for all the months leads to a suspicion that a period of ten years is scarcely sufficient to secure a counterbalancing, in all months, of casual irregularities; but, by combining together the variations of those consecutive months, the curves of which are similar in character, greater regularity is obtained, and the nature of the change of the typical variations with change of season is more distinctly brought out. The grouping of months adopted with these objects in view was as follows, viz.:—(1) December and January; (2) February and March; (3) April to June; (4) July to September, and (5) October and November. The average lunar diurnal variations for these several groups of months were calculated from the variations for individual months given in Tables XXXVII. to XL. (pages 125 and 126) of the book referred to. These average variations, which were expressed in minutes of arc, were then converted into force by multiplying by an adopted value of the horizontal force at Trevandrum (0·37141 C.G.S. unit), and by sine 1′, with results that are exhibited in the following Tables.

Table 28.—Declination. Trevandrum.

Twenty-fifths of Trevandrum lunar day	0	1	2	3	4	5	6	7	8	9	10
New Moon . .	+ ·000054	+ ·000030	+ ·000009	− ·000015	− ·000024	− ·000026	− ·000022	− ·000011	− ·000006	− ·000002	− ·000001
First Quarter	+ ·000014	+ ·000011	+ ·000002	+ ·000005	·000000	− ·000004	− ·000007	− ·000006	− ·000007	− ·000005	+ ·000002
Full Moon	+ ·000003	+ ·000005	+ ·000005	·000000	− ·000007	− ·000017	− ·000025	− ·000026	− ·000021	− ·000009	+ ·000002
Last Quarter	+ ·000023	+ ·000026	+ ·000016	− ·000002	− ·000020	− ·000027	− ·000037	− ·000032	− ·000019	− ·000001	+ ·000019

Table 29.—Declination. Trevandrum.

Twenty-fifths of Trevandrum lunar day	0	1	2	3	4	5	6	7	8	9	10
New Moon . . .	+ ·000015	+ ·000012	+ ·000001	− ·000010	− ·000019	− ·000022	− ·000016	− ·000014	− ·000009	− ·000008	− ·000002
First Quarter . .	+ ·000014	+ ·000005	− ·000001	− ·000002	− ·000004	− ·000005	− ·000008	− ·000007	− ·000006	− ·000004	·000000
Full Moon	+ ·000002	− ·000002	− ·000005	− ·000010	− ·000013	− ·000015	− ·000017	− ·000015	− ·000008	+ ·000008	+ ·000019
Last Quarter	+ ·000007	+ ·000008	− ·000009	− ·000017	− ·000023	− ·000022	− ·000011	− ·000001	+ ·000006	+ ·000013	+ ·000018

Table 30.—Declination. Trevandrum.

Twenty-fifths of Trevandrum lunar day	0	1	2	3	4	5	6	7	8	9	10
New Moon .	− ·000015	− ·000018	− ·000013	− ·000006	·000000	+ ·000005	+ ·000005	+ ·000003	+ ·000001	+ ·000002	+ ·000003
First Quarter	− ·000008	− ·000003	·000000	·000000	− ·000003	− ·000006	− ·000001	− ·000006	− ·000004	− ·000004	− ·000002
Full Moon	+ ·000008	+ ·000005	+ ·000004	·000000	·000000	− ·000001	− ·000001	+ ·000003	+ ·000006	+ ·000006	+ ·000002
Last Quarter .	− ·000001	− ·000005	− ·000011	− ·000009	− ·000010	− ·000006	+ ·000001	+ ·000006	+ ·000009	+ ·000010	+ ·000001

Table 31.—Declination. Trevandrum.

Twenty-fifths of Trevandrum lunar day.	0	1	2	3	4	5	6	7	8	9	10
New Moon	− ·000022	− ·000017	− ·000004	+ ·000003	+ ·000007	+ ·000010	+ ·000006	+ ·000002	+ ·000001	− ·000004	− ·000002
[illegible]	+ ·000001	+ ·000004	+ ·000003	+ ·000002	+ ·000004	+ ·000004	+ ·000008	+ ·000001	+ ·000001	+ ·000001	− ·000003
[illegible]	·000000	+ ·000004	+ ·000008	+ ·000008	+ ·000011	+ ·000013	+ ·000019	+ ·000018	+ ·000014	+ ·000001	− ·000007
[illegible]	− ·000006	− ·000004	+ ·000006	+ ·000018	+ ·000008	+ ·000008	+ ·000009	+ ·000005	·000000	− ·000007	− ·000002

December and January.

11	12	13	14	15	16	17	18	19	20	21	22	23	24*
000000	+ 000003	+ 000008	+ 000006	+·000003	− 000005	− 000015	− 000029	− 000033	− 000020	− 000004	+ 000014	+ 000037	+ 000051
+ 000009	+ 000022	+ 000032	+ 000032	+ 000013	− 000012	− 000034	− 000040	− 000037	− 000027	− 000006	+ 000008	+ 000019	+ 000015
+ 000024	+ 000040	+ 000033	+ 000025	+·000008	− 000006	− 000010	− 000010	− 000009	− 000005	− 000002	− 000002	·000000	+·000004
+ 000022	+ 000015	+ 000009	+ 000003	+ 000001	000000	− 000003	− 000008	− 000008	− 000007	− 000004	+ 000002	+ 000010	+ 000024

February and March.

11	12	13	14	15	16	17	18	19	20	21	22	23	24
+·000001	+ 000004	+ 000008	+ 000010	+ 000012	+ 000008	+ 000002	− 000007	−·000007	− 000006	+·000001	+ 000012	+ 000016	+ 000021
+ 000008	+ 000011	+ 000012	+ 000006	+ 000002	− 000013	− 000023	− 000020	− 000016	− 000005	+ 000008	+·000017	+ 000015	+·000017
+ 000026	+ 000019	+ 000012	+ 000005	+ 000001	− 000005	− 000005	− 000002	+ 000001	+ 000002	− 000001	+ 000003	+ 000003	+ 000004
+ 000016	+ 000010	+ 000005	+ 000002	+ 000003	+ 000004	000000	− 000003	− 000006	− 000004	− 000002	000000	+ 000004	+·000005

April to June.

11	12	13	14	15	16	17	18	19	20	21	22	23	24
+·000005	+ 000004	+·000003	+ 000004	+·000003	− 000003	− 000003	− 000004	+·000001	+ 000006	+·000010	+ 000011	+ 000003	− 000007
−·000002	−·000006	− 000007	−·000008	−·000005	− 000002	+ 000004	+·000008	+·000012	+ 000013	+ 000014	+ 000003	+ 000006	000000
− 000004	−·000007	−·000006	−·000008	−·000007	− 000006	·000000	+·000001	−·000002	−·000001	−·000001	+ 000001	+ 000004	+ 000005
−·000001	−·000002	−·000001	·000000	−·000003	−·000005	− 000003	−·000003	− 000001	+·000002	+·000006	+ 000011	+ 000009	+ 000004

July to September.

11	12	13	14	15	16	17	18	19	20	21	22	23	24
−·000001	− 000001	+·000001	+·000001	+ 000003	+ 000006	+·000011	+·000015	+·000013	+·000009	+ 000008	−·000007	−·000012	−·000018
− 000009	− 000013	−·000010	− 000007	+·000001	+ 000006	+·000009	+ 000009	+·000006	+ 000004	·000000	− 000008	− 000008	− 000002
−·000010	− 000008	− 000014	− 000012	− 000007	+·000001	+ 000008	+·000002	−·000010	−·000013	− 000009	−·000008	− 000002	−·000001
− 000006	− 000004	− 000003	+ 000003	+ 000003	+ 000001	+ 000004	−·000001	−·000003	− 000004	−·000003	− 000003	−·000006	− 000010

* In Broun's mode of reduction the lunar day is divided into 25 parts.

TABLE 32 —Declination. Trevandrum.

Twenty-fifths of Trevandrum lunar day	0	1	2	3	4	5	6	7	8	9	10
New Moon	+ 000019	+ 000014	+ 000001	− 000007	− 000011	− 000015	− 000013	− 000010	− 000007	− 000001	− 000002
First Quarter	+ 000006	000000	− 000003	− 000005	− 000007	− 000008	− 000008	− 000009	− 000008	− 000012	− 000015
Full Moon	+ 000007	+ 000007	+ 000009	+ 000005	+ 000002	− 000002	− 000008	− 000012	− 000014	− 000011	− 000008
Last Quarter	+ 000003	+ 000001	+ 000003	− 000006	− 000008	− 000009	− 000008	− 000008	− 000006	+ 000002	+ 000007

27. By treating each of these sets of numbers in the manner described in paragraphs 8

TABLE

Twenty fifths of a solar day after noon.		0	1	2	3	4	5	6	7	8	9	10
December and January	$f_{o\cdot 1}(h)$	+ 000009	000000	− 000004	− ·000008	− 000008	− ·000008	− 000006	− 000002	− ·000002	000000	000000
	$f_{e\cdot 1}(h)$	− 000001	− 000001	+ ·000002	− ·000001	·000000	− 000002	+ 000001	+ ·000003	·000000	+ ·000002	+ ·000001
	$f_{e\cdot 2}(h)$	+ 000041	+ 000028	+ 000010	− 000008	− 000018	− 000017	− 000014	− 000009	− 000003	− ·000002	·000000
February and March	$f_{o\cdot 1}(h)$	000000	+ ·000002	− 000001	− 000004	− ·000007	− 000009	− 000008	− ·000009	− 000005	− 000005	− ·000003
	$f_{e\cdot 1}(h)$	− 000005	− 000004	− 000001	+ ·000001	− 000001	+ 000002	+ ·000003	+ ·000001	− 000002	− 000003	− 000003
	$f_{e\cdot 2}(h)$	+ 000013	+ 000006	− 000004	− 000011	− ·000014	− 000014	− 000010	− 000005	− 000002	− ·000002	+ 000001
April to June	$f_{o\cdot 1}(h)$	− 000004	− 000006	− 000003	000000	+ 000001	+ 000002	+ ·000003	+ ·000002	+ 000001	+ ·000001	000000
	$f_{e\cdot 1}(h)$	+ ·000004	+ ·000003	+ 000002	+ 000001	+ 000003	+ 000001	− 000001	− 000001	+ 000001	+ 000002	+ ·000001
	$f_{e\cdot 2}(h)$	− 000009	− 000012	− ·000011	− ·000007	− 000002	+ 000002	+ 000002	+ ·000001	000000	+ 000002	+ 000003
July to September . .	$f_{o\cdot 1}(h)$	− ·000006	− 000002	+ 000003	+ 000003	+ 000002	+ 000003	+ ·000005	+ ·000007	+ 000006	+ 000002	+ ·000001
	$f_{e\cdot 1}(h)$	·000000	+ ·000001	+ 000002	− ·000002	− 000002	+ ·000002	+ ·000002	+ 000002	·000000	·000000	+ 000001
	$f_{e\cdot 2}(h)$	− ·000012	− ·000009	− ·000002	+ ·000003	+ ·000005	+ ·000005	+ ·000001	− 000003	− 000004	− ·000004	− ·000003
October and November	$f_{o\cdot 1}(h)$	+ 000008	+ ·000004	− ·000008	− 000004	− ·000004	− 000006	− ·000006	− ·000005	− 000005	+ ·000004	− ·000003
	$f_{e\cdot 1}(h)$	·000000	+ ·000001	+ ·000002	+ ·000002	000000	·000000	− ·000002	− 000003	− 000004	− 000004	− 000004
	$f_{e\cdot 2}(h)$	+ ·000010	+ ·000008	+ ·000002	− ·000005	− 000008	− 000010	− ·000007	− ·000004	− ·000002	·000000	+ ·000002

These variations are curved in figs. 17 to 21 and 38 to 47.

October and November.

11	12	13	14	15	16	17	18	19	20	21	22	23	24
+ 000002	+ 000004	+ 000004	+ 000006	+ 000006	+ 000003	− 000002	− 000008	− 000008	− 000005	000000	+ 000005	+ 000010	+ 000016
− 000009	− 000001	+ 000013	+ 000016	+ 000018	+ 000010	+ 000006	− 000003	− 000008	− 000004	+ 000001	+ 000008	+ 000009	+ 000012
− 000004	+ 000002	+ 000006	+ 000007	+ 000005	− 000003	− 000003	− 000003	000000	+ 000002	+ 000003	+ 000005	+ 000002	+ 000008
+ 000012	+ 000011	+ 000008	+ 000006	+ 000006	+ 000002	+ 000001	− 000001	− 000004	− 000004	− 000006	− 000003	000000	+ 000001

and 14, the several values of the typical variations $f(h)$ were obtained as follows :-

33.

11	12	13	14	15	16	17	18	19	20	21	22	23	24
− ·000001	·000000	+ 000002	000000	000000	− 000001	− 000002	− 000004	− 000004	+ 000002	+ 000006	+ 000009	+ 000012	+ ·000009
+ 000001	+ 000001	000000	− 000001	− 000002	− 000002	− 000004	− 000001	+ 000004	+ 000006	+ 000003	000000	− 000005	− 000004
+ 000003	+ 000005	+ 000006	+ 000006	+ 000003	− 000004	− 000013	− 000024	− 000029	− 000024	− 000010	+ 000008	+ 000027	+ 000039
− 000001	+ 000001	+ 000004	+ 000007	+ 000010	+ 000010	+ 000008	+ 000004	+ 000004	+ 000003	+ 000002	000000	− 000003	000000
− 000002	− ·000002	− 000001	− 000002	− 000002	− 000001	+ 000002	+ 000002	+ 000003	+ 000004	+ 000007	+ 000004	000000	− 000002
+ 000003	+ 000005	+ 000005	+ 000004	+ 000002	− 000001	− 000006	− 000010	− 000010	− 000005	+ 000002	+ 000014	+ 000021	+ 000020
000000	− 000001	− 000002	000000	000000	− 000001	− 000001	− 000001	000000	+ 000001	+ 000002	+ 000004	+ 000002	− 000001
− 000002	− 000001	− 000003	− 000004	− 000006	− 000006	− 000005	− ·000004	− 000002	000000	+ ·000002	+ ·000004	+ 000006	+ 000005
+ 000005	+ 000004	+ 000004	+ 000002	·000000	− 000003	− 000002	000000	+ 000003	+ 000006	+ 000008	+ 000007	+ 000001	− 000005
000000	000000	− 000001	− 000003	− 000002	− 000002	000000	− 000001	− 000003	− 000003	− 000001	− 000002	− 000002	− 000005
+ 000001	+ 000002	+ ·000002	+ 000002	+ 000002	+ 000001	000000	− 000003	− 000003	− 000004	− 000004	− 000002	000000	000000
− 000002	− 000001	+ 000001	+ 000002	+ 000003	+ 000006	+ 000010	+ 000013	+ 000012	+ 000008	000000	− 000007	− 000009	− 000011
− 000002	− 000002	− 000002	− 000001	000000	000000	− ·000001	− 000001	+ 000001	+ 000004	+ 000007	+ 000008	+ 000008	+ 000008
− 000004	− 000003	− 000002	− 000002	− 000004	− 000007	− 000005	− 000001	+ 000006	+ 000007	+ 000010	+ 000008	+ 000007	+ 000003
+ 000004	+ 000006	+ 000006	+ 000007	+ 000007	+ 000006	+ 000002	− 000004	− 000008	− 000009	− 000007	− 000002	+ ·000002	+ 000006

28. On these curves we first remark that, as at Bombay, the variations of the $f_{c1}(h)$ and $f_{s1}(h)$ types are quite subordinate in magnitude to those of the $f_{c2}(h)$ type; that the $f_{c2}(h)$ curve is twice reversed in the course of the year, and that, whilst for December and January (fig. 17) it has a greater range than the corresponding winter curve (fig. 1) for Bombay, the reverse curve for July to September (fig. 20) has a smaller range than the corresponding autumn curve (fig. 4) for Bombay. The relation of the $f_{c2}(h)$ curves of the two stations generally is such that the superposition of a constant variation of the character of the winter $f_{c2}(h)$ curve, but of smaller range, upon the Bombay curves for each season would convert these into curves having the same principal characteristics as those of the same season for Trevandrum, and, in agreement with this, the reversal in the first half of the year occurs later at Trevandrum than at Bombay, and the reversal in the last half-year occurs earlier than at Bombay.

And here again we must insist on the reality of the variations now brought to light as true physical phenomena, which exhibit themselves not alone at a single point on the earth's surface, but are well marked in the observations of two stations that differ by nearly ten degrees in latitude, and the observatories at which were necessarily furnished with independent instruments, and had independent directors.

POSTSCRIPT.

(Added September 23, 1886.)

The late JOHN ALLAN BROUN, in discussing the Trevandrum observations for the ten years 1854 to 1864, came to the conclusion that in the lunar diurnal variations of declination in each month of the year the amount of movement is, as in the solar diurnal variation, greater during the day than during the night; and the writer of this paper has shown in it that in each of four seasons of the year the same holds at Bombay with respect both to the declination and horizontal force. Whilst admitting, however, that BROUN has priority of publication of this fact as regards the declination at Trevandrum, the writer avers that he has in no way been guided by BROUN's previous investigation, nor was he aware of the existence of BROUN's paper * when he made the discovery with respect to the declination at Bombay. Moreover, the writer's line of investigation was that of SABINE, modified by himself, and not that of BROUN; and an account of the first results of its employment was read before the Royal Society on the 1st February, 1872, a date antecedent to the reading of BROUN's paper before the Royal Society of Edinburgh, which did not take place till the 6th May, 1872. The writer also readily acknowledges that in the paper already referred to BROUN enunciated some relations of the movements near sunrise in the lunar diurnal variations of declination in different months, which relations fit in well with the idea of a luni-solar variation when once that idea is conceived; but he finds

* 'Transactions of the Royal Society of Edinburgh,' vol. 26, page 735.

no trace of that idea in any of BROUN's writings. Neither does he find in BROUN's writings any of the following results that are described in the present paper, viz. —

(1.) The discovery of relations subsisting between the lunar diurnal variation of declination *as a whole* at one phase of a lunation with the same at other phases in each season of the year [this as distinguished from the movements that occur *near sunrise*, and the nature of which was described by BROUN].

(2.) The discovery that a part, being the bulk of the whole, of the lunar diurnal variation runs through a cycle of change in a lunation; to this part the name of the luni-solar variation is given.

(3.) A hypothesis is advanced—a hypothesis of phenomenal relations and not of physical causation—that the luni-solar variation of a given season is a combination of solar diurnal variations of constant types that go through cycles of wave-like change of amplitude in the periods of a lunation, half a lunation, &c., &c., and a formula is found to give expression to this hypothesis.

(4.) Following up this hypothesis, the observations of declination for the quarter November to January of the years 1846 to 1870 are divided into categories of solar days according to the age of the moon; and the excess solar diurnal variation—the excess over the mean solar diurnal variation for the approximate full lunation—is entered in the Tables for each day, and the mean value is calculated for each category. Curves representing these mean excess solar diurnal variations for each day of the moon's age are found to agree fairly well with the formula expressing the luni-solar variation of the quarter November to January, a result which tends to confirm the hypothesis.

(5.) The mode of change, from season to season, of the character of the elements of the luni-solar variations is described.

(6.) All the results, without exception, of the investigation with reference to the lunar variations of horizontal force must be placed on this list.

(7.) That the luni-solar variations of declination are related to those of the horizontal force in the following manner, viz. :—(*a*) In the winter season the luni-solar variation of declination due to any phase of the moon is similar to the luni-solar variation of horizontal force due to a phase later by one-eighth of a lunation; and (*b*) in the summer and autumn seasons it is, on the other hand, the luni-solar variations of horizontal force that precede by an eighth of a lunation the similar variations of declination.

(8.) It is shown that, when the Bombay observations have pointed the way, BROUN's determinations of the lunar diurnal variations of declination at Trevandrum for the four quarters of the moon in each month of the year, when properly treated, support the hypothesis of result (3).

II *On the Properties of Matter in the Gaseous and Liquid States under various Conditions of Temperature and Pressure* *

By the late THOMAS ANDREWS, *M.D., LL.D., F.R.S.*

Communicated by Professor STOKES, *D.C.L., P.R.S.*

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ACCORDING to DALTON, the particles of one gas possess no repulsive or attractive power with regard to the particles of another gas; and accordingly, if m measures of a gas A be mixed with n measures of another gas B, each will occupy $m+n$ measures of space. The density of A in such a mixture will be $\frac{m}{m+n}$, and of B, $\frac{n}{m+n}$, the pressure upon any one particle of such a gaseous mixture arising solely from particles of its own kind. "It is scarcely necessary," DALTON remarks, "to insist upon the application of this hypothesis to the solution of all our difficulties respecting the constitution of mixed gases where no chemical union ensues. The moment we admit it every difficulty vanishes. The atmosphere, or, to speak more properly, the compound of atmospheres, may exist together in the most intimate mixture without any regard to their specific gravities, and without any pressure upon one another. Oxygen gas, azotic gas, hydrogenous gas, carbonic acid gas, aqueous vapour, and probably several other elastic fluids, may exist in company under any pressure, and at any temperature, without any regard to their specific gravities, and without any pressure upon one another, while each of them, however paradoxical it may appear, occupies the whole space allotted to them all."†

In conformity with this law, GAY LUSSAC found that the vapours of alcohol and water mix like two gases which have no action upon one another. The density of the

* The manuscript of this memoir was found among the author's papers, and was sent to me by Professor TAIT, who thought that, as the former papers on the same subject were published in the 'Philosophical Transactions,' the manuscript ought to be submitted to the Royal Society. It has every appearance of being complete, though probably intended to be, in part at least, fair-copied before presentation. It was without title, but the title has been supplied from an earlier draft This title shows that the paper must have been written a great many years ago, while Dr. ANDREWS was still connected with Queen's College, Belfast. For many years before his death the state of Dr ANDREWS' health prevented him from continuing his scientific labours, even to the extent of preparing for publication an account of the results at which he had arrived in an investigation which was doubtless carried on with that conscientious accuracy which characterised all his work —G. G S.

† 'Manchester, Phil Soc. Mem.,' vol. 5, 1802, p. 543

mixed vapours agreed closely with the density calculated according to DALTON'S law.* In 1836 MAGNUS published an important memoir on the same subject. He found that, if two liquids which do not mix with one another are introduced into a barometer tube, the tension of the mixed vapours at any temperature is equal to the sum of the tensions of the vapours of the two liquids. But when the liquids have the property of mixing with one another the behaviour of their vapours he found to be altogether different. The tension of the mixed vapours was no longer equal to the sum of the tensions of each vapour separately. This statement appears at first view to contradict the experiments of GAY LUSSAC, but, as MAGNUS himself has pointed out, the conditions under which the observations of the two eminent physicists were made were essentially different. In the experiments of GAY LUSSAC the mixed liquids were wholly converted into vapour, and therefore the mixed vapours formed were not in contact with any liquid, while in those of MAGNUS an excess of the mixed liquids was always present, and in contact with the vapour.†

The same subject was afterwards investigated with great care by REGNAULT in his elaborate work on the elastic force of vapours. His experiments were made by a similar method to that adopted by MAGNUS, but they embrace a larger number of bodies, and the results are given in fuller detail. The conclusions at which REGNAULT arrived are the same as those previously stated by MAGNUS, viz., that two volatile liquids, which are not capable of dissolving each other, give a vapour tension equal to the sum of the tensions which the same liquids give separately, but that two volatile liquids, which are capable of dissolving each other, give a complex vapour whose tension is always less than the sum of the tensions of the vapours of the two liquids, and often less than the tension of the vapour of the more volatile liquid alone.‡

The case of the elastic force or tension of a mixture of gas and vapour has also been investigated by REGNAULT. In his earlier researches on this subject he found the elastic force of aqueous vapour, at the point of saturation, in presence of air or nitrogen, to be always a little feebler than its elastic force *in vacuo*. The difference was, however, small, rarely exceeding one-fiftieth of the whole tension, and REGNAULT in his earlier investigations was inclined to attribute it to some constant error in his method.§ In his later researches he returned to the same subject, and in order to aid in the solution of the question he made a number of direct determinations of the density of the vapour of water within the limits of temperature at which he had formerly worked. He also extended his experiments to mixtures of air with vapours of other liquids more volatile than water. The results he obtained were in accordance with DALTON'S law, provided the mixed gas and vapour were compressed so as to

* 'Annales de Chimie,' vol. 95, 1815, p. 314; Biot, 'Traité de Physique,' vol. 1, p. 298, where GAY LUSSAC'S experiments are first described

† 'POGGENDORFF, Annalen,' vol. 38, 1836, p. 488.

‡ 'Mémoires de l'Académie des Sciences,' vol. 26, 1862, pp. 722 and 729.

§ 'Annales de Chimie,' vol. 15, 1845, p. 137.

cause an abundant deposition of liquid, and the observations were made immediately after the application of the pressure. The slight deviations from that law under other conditions of the experiment he attributed to the hygroscopic affinity of the sides of the containing vessel, which condense a portion of the vapour and lower its tension below that due to saturation As the result of all his observations, REGNAULT concludes that the law of DALTON may be considered to be theoretically true in the case of mixtures of gases and vapours, and that it would probably be in all cases verified rigorously by experiment if the mixed gas and vapour could be enclosed in a vessel whose sides were formed of the volatile liquid itself *

The only experiments, so far as I know, on the effects of pressure upon mixtures of the ordinary gases are a few recorded by REGNAULT on mixtures of atmospheric air and carbonic acid, and of hydrogen and sulphurous acid. The observations were made within limits of pressure extending from two-thirds of an atmosphere to two atmospheres, and the results indicated that within these limits the compressibility is intermediate between that which each gas, if isolated, would exhibit for the same variations of pressure. †

The result of these experimental investigations is to confirm, with one exception, the law of DALTON for all cases of mixtures of gases or of vapours, or of gases and vapours which have no chemical action upon one another. The exception referred to is that of a mixture of vapours derived from liquids capable of dissolving one another, and in presence of the compound liquid. To such a case the law of DALTON, as originally enunciated, is clearly inapplicable, since chemical affinities come into play which disturb the result. The diminution of volume and disengagement of heat which occur when water and alcohol in the liquid state are mixed prove, as GAY LUSSAC long ago pointed out, that there is a well-marked affinity between those liquids, resulting in the formation of a chemical compound, and that it is the tension of the vapour of this compound which is actually observed.

It would, however, be a hasty inference to conclude that the law of DALTON has been fully established by experiment, the more so as in none of the investigations to which I have referred was the pressure carried beyond two atmospheres. As the apparatus described in my former communications was well adapted to this inquiry,‡ I

* 'Mémoires de l'Académie des Sciences,' vol 26, 1862, pp 680–696

† *Ibid*, p. 258.

‡ The following foot-note occurs in the earlier draft —

"I wish here to supply an omission in my former paper, and to explain, for the information of future experimenters, the mode of packing the screws and of connecting the glass tubes with the metallic flanges In fig. 3 of the plate accompanying this paper a section of the steel screw is shown, from which it will be seen that the screw enters a female screw in the flange (fig 7) for about half an inch, and afterwards a leather packing of more than one inch in length This packing is formed of a number of circular disks of leather punched through in the centre and saturated *in vacuo* with melted lard After each disk is introduced it is pressed firmly into its place in the flange by hammering lightly with a wooden mallet upon an iron bolt, which fitted loosely the cylindrical cavity. The flange, when

have examined with care the behaviour of a mixture of nitrogen and carbonic acid gases under varied conditions of pressure and temperature. According to the definition I ventured formerly to give of a vapour, viz., that it is a gas at any temperature below its critical point, that is to say, at any temperature at which it can be changed by the application of external pressure to the liquid state, carbonic acid is a vapour at temperatures below 31° C., and a gas proper at higher temperatures. Accordingly the properties of the mixture of nitrogen and carbonic acid have been examined at temperatures both above and below the critical point of pure carbonic acid.

The gaseous mixture, carefully dried, after flowing through the tube in which it was to be compressed, was collected over mercury and analysed. After all reductions were made its composition was found to be

$$3 \text{ vol. } CO_2 \text{ and } 4{\cdot}05 \text{ vol. N.}$$

At the conclusion of the long series of experiments now to be described, which occupied several months, during which this mixture was exposed from day to day to

filled with its packing, was strongly clamped upon the edge of a table, the flat surface resting on the table, and the steel screw afterwards turned till it passed through the packing and entered the table for a short way. On removing the flange from the table, the leather was always found to project beyond the surface of the brass; it was cut away till it formed a thin cylinder round the screw, which cylinder was afterwards strengthened by tying it with a silk thread. I have been thus minute in describing this operation, as a screw successfully packed in this way will resist for many months a pressure of several hundred atmospheres. The upper flange which carried the glass tube was traversed by a cylindrical cavity terminating in a hollow cone, and the glass was fashioned into a form which, when covered with its packing, fitted into this cavity. The packing was effected by rolling round the glass cone and cylinder several strands of a fine hempen thread covered with shoemaker's wax, which was applied slightly heated The flange was then warmed, and the glass tube very firmly pressed down into its place, and the whole allowed to cool. A junction of this kind will also bear a pressure of several hundred atmospheres without yielding."

The following is given in the text of the same draft:—

"So perfectly did the apparatus work that in a long investigation, extending over several months, during which the apparatus was never dismounted, I was able without any difficulty to make a continuous series of experiments at pressures varying from 50 to 300 atmospheres, and to read the changes of volume of the gases of these pressures in glass tubes with almost as great ease as an ordinary barometer. One precaution only requires to be mentioned. If the pressure is either largely augmented or diminished in the course of an experiment, the readings should not be made for some time after, as the apparatus, without the slightest leakage, takes time to adjust itself to the new condition. I presume it is the slow change of volume in the leather packings to which this is due. Indeed, the effect is equally marked on taking off the pressure suddenly as on augmenting it. The mercury in the manometer will rise, in the former case, visibly to the eye for some time. When working at very high pressures, I was formerly in the habit of making the observations as quickly as possible after each augmentation of pressure, but since I discovered the true action of the apparatus I have always given time for all the parts to adjust themselves to the new condition, when the readings can be made with great ease and accuracy."

pressures varying from 40 to 300 atmospheres, it was again analysed as it existed in the tube, and found to consist of

3 vol. CO_2 and 4 02 vol. N

In the following Tables p is the pressure as indicated by a hydrogen-manometer, t the temperature of the manometer, ϵ a fraction representing the ratio of the volume of the mixture after compression to its volume at the temperature t' under one atmosphere, t' the temperature of the mixed gases, and θ the volume which 1 volume of the mixture measured at 0° and 760 millimetres would occupy, at the temperature at which the observation was made, and under the pressure indicated by the hydrogen-manometer.

TABLE I.—Compressibility of 3 v. CO_2+4 v. N at 2°·2.

p	t	ϵ	t'	θ
	°		°	
42 05	7 30	$\frac{1}{47\ 18}$	2 32	0 02138
48 68	7 30	$\frac{1}{55\ 90}$	2 34	0 01805
51·64	7 22	$\frac{1}{59\ 97}$	2 08	0 01681
54 73	7·26	$\frac{1}{64\ 05}$	2 38	0 01575
58·43	7 21	$\frac{1}{69\ 28}$	2·06	0 01455
62·63	7 21	$\frac{1}{75\ 48}$	2 10	0 01338
67 20	7 20	$\frac{1}{82\ 04}$	2 16	0 01229
79 47	7 18	$\frac{1}{101\ 0}$	2 21	0 00998
87 73	7·17	$\frac{1}{114\ 4}$	2·21	0 00881
97 21	7 17	$\frac{1}{130\ 7}$	2 17	0·00772
108·60	7 17	$\frac{1}{151\ 0}$	2 21	0 00668
123·40	7 17	$\frac{1}{177\ 6}$	2 25	0 00568

[In the following Table the denominator of ϵ in the second line is given in both manuscripts as 49·06, which must have been an error of copying. 46·09 corresponds with θ.—G. G. S.]

TABLE II.—Compressibility of 3 v. CO_2 and 4 v. N at 7°·5.

p	t	ε	t'	θ
39·68	7°·47	$\frac{1}{43\cdot71}$	7°·50	0·02350
41·56	7·48	$\frac{1}{46\cdot00}$	7·50	0·02231
43·66	7·42	$\frac{1}{48\cdot75}$	7·50	0·02109
46·00	7·46	$\frac{1}{51\cdot67}$	7·53	0·01987
51·48	7·48	$\frac{1}{58\cdot05}$	7·51	0·01744
54·75	7·42	$\frac{1}{63\cdot24}$	7·59	0·01625
58·24	7·54	$\frac{1}{68\cdot00}$	7·50	0·01512
62·46	7·50	$\frac{1}{78\cdot00}$	7·50	0·01391
67·07	7·48	$\frac{1}{80\cdot60}$	7·50	0·01274
73·00	7·59	$\frac{1}{89\cdot17}$	7·50	0·01149
79·22	7·65	$\frac{1}{99\cdot18}$	7·51	0·01054
87·20	7·55	$\frac{1}{111\cdot0}$	7·50	0·00926
96·50	7·80	$\frac{1}{126\cdot0}$	7·08	0·00808
108·70	7·54	$\frac{1}{148\cdot8}$	7·48	0·00691
144·00	7·58	$\frac{1}{207\cdot5}$	7·54	0·00495
166·40	7·79	$\frac{1}{243\cdot5}$	7·50	0·00422
173·40	7·63	$\frac{1}{264\cdot7}$	7·49	0·00404
215·70	7·67	$\frac{1}{309\cdot5}$	7·50	0·00332
283·90	7·58	$\frac{1}{377\cdot9}$	7·49	0·00272

TABLE III.—Compressibility of 3 v. CO_2 and 4 v. N at 31°·3.

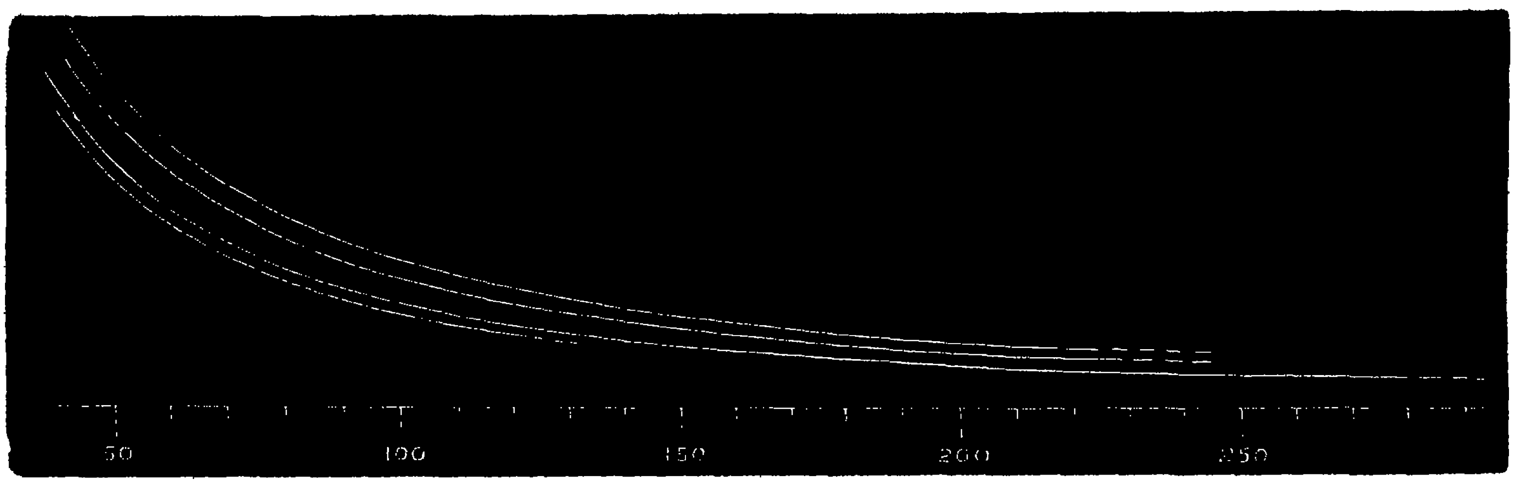
p	t	ε	t'	θ
42·26	11°·61	$\frac{1}{45\cdot60}$	31°·35	0·02445
49·43	11·94	$\frac{1}{54\cdot80}$	31·31	0·02055
55·82	11·86	$\frac{1}{62\cdot18}$	31·21	0·01795
63·87	12·38	$\frac{1}{72\cdot35}$	31·40	0·01542
72·34	12·38	$\frac{1}{88\cdot40}$	31·14	0·01337
89·26	12·38	$\frac{1}{106\cdot6}$	31·06	0·01045
110·00	11·96	$\frac{1}{136\cdot7}$	31·36	0·00816
145·60	11·63	$\frac{1}{190\cdot2}$	31·35	0·00586
221·70	11·70	$\frac{1}{295\cdot9}$	31·30	0·00377

TABLE IV.—Compressibility of 3 v. CO_2 and 4 v. N. at 48°·4.

p	t	ϵ	t'	θ
41 90	° 8 39	$\frac{1}{44\ 66}$	° 48 22	0 02635
48·80	8 42	$\frac{1}{52\ 50}$	48 11	0 02239
55 86	12 06	$\frac{1}{61\ 10}$	48 48	0 01929
64 18	12·08	$\frac{1}{70\ 92}$	48 43	0 01662
72 54	12 18	$\frac{1}{81\ 56}$	48 66	0 01448
110 50	12·36	$\frac{1}{131\ 4}$	48 38	0 00896
147·10	12 36	$\frac{1}{182\ 3}$	48 49	0 00646
223·60	12 40	$\frac{1}{276\ 5}$	48 47	0 00430

No liquid carbonic acid was formed in any of these experiments, although the experiments in the first series were made at a temperature 29° below the critical point of pure carbonic acid. The results of the foregoing experiments are represented graphically in the following figure, in which the curves for the different temperatures present a remarkable similarity of form, giving no clear indications of a difference of character for temperatures above and below 31°.

FIG. 1

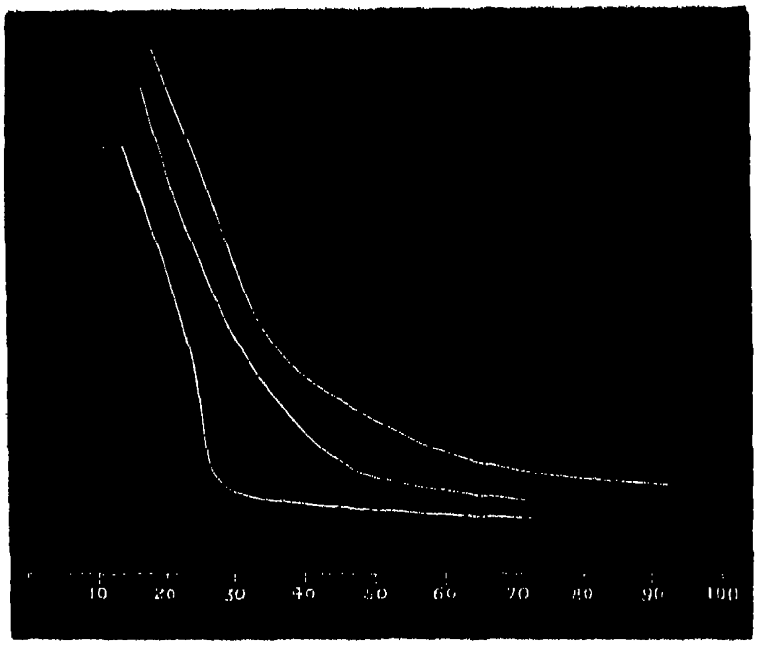


If we assume DALTON'S law to be true, and BOYLE'S law to be true in the case of nitrogen for the pressures employed, the following curves will represent the changes of volume of the carbonic acid in the mixture at the indicated pressures and temperatures. [See fig. 2, which is taken from a provisional figure in the MS.]

I have endeavoured to ascertain approximately the form of the carbonic acid curve in this mixture, assuming as sufficiently accurate for my purpose that nitrogen in contracting obeys the law of BOYLE, but the curves so obtained, although they give indications of a fall to the liquid volume at the two lower temperatures, are wholly

different from those for the unmixed gas. It is manifest, therefore, that the law of DALTON does not hold under the conditions of these experiments. Like the other laws of the gaseous condition of matter, it appears to be strictly true only in the (ideal) perfect state. The small differences in the elastic force of aqueous vapour when alone and when mixed with air, as given in the original experiments of REGNAULT, are, it is highly probable, due, not to accidental or extraneous causes, but to the law of DALTON being only approximately true in the case of such a mixture of gas and vapour.

Fig. 2.

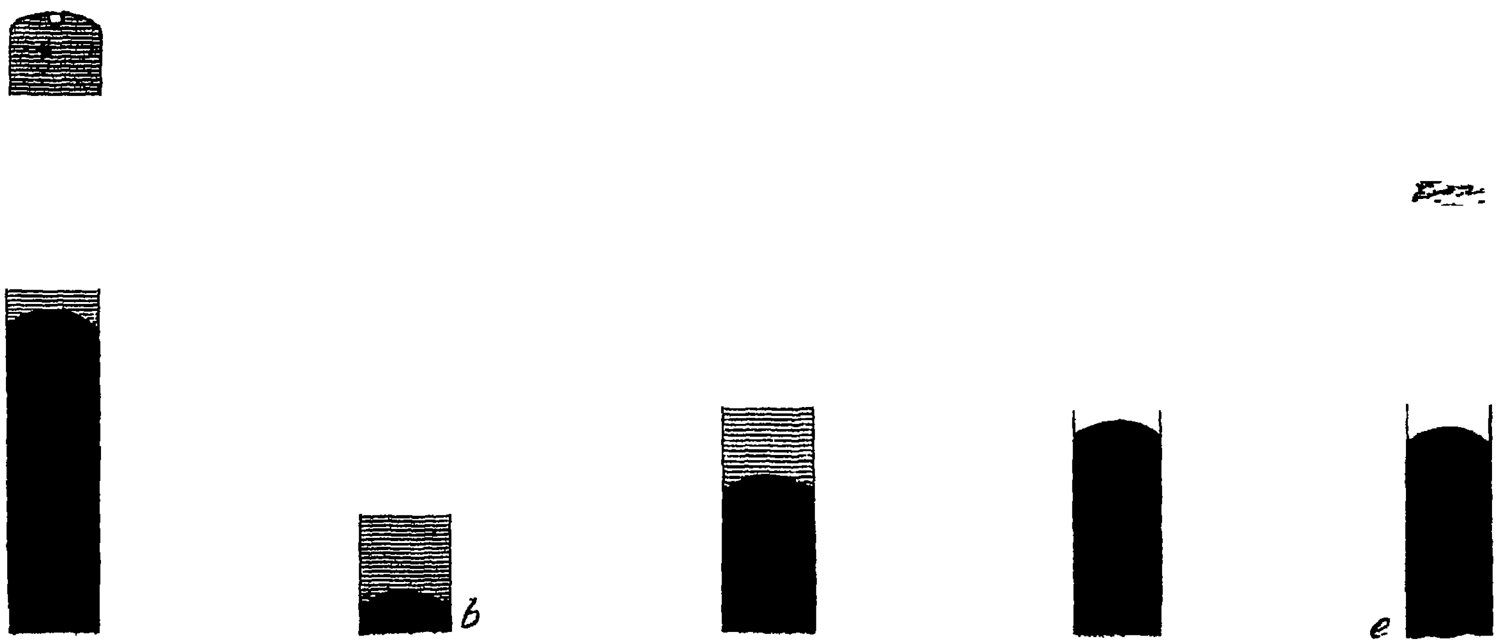


As no liquid was formed in any of these experiments, it follows that the admixture of carbonic acid with a permanent gas, such as nitrogen, has the effect of lowering the critical point or critical temperature. It will be remembered that for pure carbonic acid the critical temperature is 30·9°. In order to examine the phenomena with more precision, mixtures of carbonic acid and nitrogen were taken containing a smaller proportion of nitrogen than the foregoing.

With a mixture of 6·2 vol. carbonic acid and 1 vol. nitrogen, and which therefore contained 13·9 per cent. of nitrogen, liquid first appeared at the temperature of 3°·5, when the pressure was raised to 48·3 atmospheres. As the pressure was increased the volume of the liquid augmented, and after each increase of pressure the liquid continued for some time slowly to augment. Thus at 82 atmospheres the relative volumes of gas and liquid in the above mixture were at first 8·5 and 5·8, but on leaving the apparatus to itself there was a small increase in the volume of the liquid. When the pressure was raised to 102 atmospheres the volume of the gas, at first 1·7, gradually diminished till it became reduced to a minute globule, which also at last disappeared, the nitrogen dissolving in the liquid carbonic acid. These are the ordinary phenomena of the solution of a gas in a liquid, the gas retaining the form of a small globule, as is shown in figure 3*a*, till it finally disappeared. But on repeating the experiment at higher temperatures the phenomena exhibited were very different.

The temperature being kept steady, the liquid carbonic acid appeared at first with the usual concave surface, and on augmenting the pressure its volume for some time steadily increased without any marked change in its appearance (fig. 3, *b*). By further increase of pressure, the surface of separation became a faint line without curvature (fig 3, *c*), and on continuing the pressure it finally disappeared, the whole mass assuming the homogeneous state (fig. 3, *d*).

Fig 3



The position of the surface of separation in the tube before it disappeared depended upon the temperature at which the observations were made. At 14° the liquid occupied about two-thirds of the entire space just before the surface was effaced.

It is difficult to fix with precision the exact pressure at which for a given temperature the last trace of the surface of separation disappears, but a definite point can be obtained if, after having completely effaced the surface by pressure, we diminish the pressure till a cloud appears, when the liquid surface in a faint form is immediately restored (fig. 3, *e*). The appearance of this cloud is remarkable. It occupies several millimetres of the tube, and as it subsides the plane surface of separation appears, not in the middle of the space occupied before by the cloud, but about one-third from the lower end.

For a mixture composed of 1 volume of nitrogen and 3·43 volumes carbonic acid, or containing 22·5 per cent. nitrogen, the critical point of temperature was found to be 14°·0, and the corresponding pressure 98 atmospheres. This point was determined by gradually lowering the temperature till it was just possible to obtain a small trace of liquid by the application of pressure. With this mixture a number of experiments were made at lower temperatures than the critical point (14°), in order to fix the pressures at which, for the same temperature, the liquid first appeared, and was afterwards effaced. As it was difficult to fix the latter point directly with precision, I found it better in the first place completely to efface the liquid, and afterwards to

remove the pressure till the cloud appeared. In this way precise and concordant results were obtained.

At the temperature of 6°·3, liquid first appeared when the pressure was raised to 68·7 atmospheres, and, after being effaced by pressure, it reappeared at 113·2 atmospheres.

At 9°·9, liquid appeared at a pressure of 77·6 atmospheres, and, after being effaced by pressure, it reappeared at 107·8 atmospheres.

At 13°·2, liquid appeared at 91 6 atmospheres, and, after being effaced by pressure, it reappeared at 103·2 atmospheres.

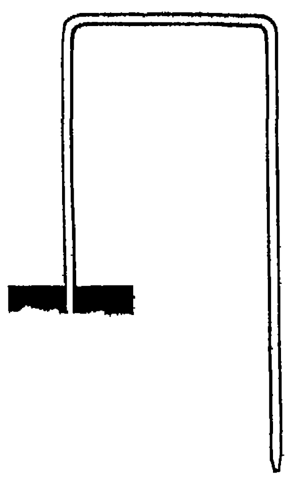
It is evident that, as we approach the critical point, the pressures at which the liquid first appears, and after being effaced reappears, will come almost exactly to coincide. This pressure, as already stated, for the mixture under examination was 98 atmospheres. If we take the mean of the pressures at which the liquid appears and reappears after being effaced for the above temperatures, we shall have the following results :—

°	Atmospheres.
6·3	90·9
9·9	92·7
13·2	97·4
14·0	98·0

The last numbers are the temperature and pressure for the critical state of the mixture.

In making these experiments, it was found convenient to modify the form of the apparatus by bending the carbonic acid tube before introducing it into the apparatus, as shown in the annexed figure. On compressing the mixture below the critical

Fig. 4



point, the liquid carbonic acid accumulated in the lower end of the tube, although a part of it was at first formed in contact with the mercury. But the latter portion soon fell down, and the whole of the liquid collected below.

I was long perplexed with anomalous results, the carbonic acid sometimes liquefying by the application of pressure at temperatures above 20°, at other times refusing to liquefy at temperatures several degrees below that point. After many trials these irregularities were at last traced to the gaseous mixture having separated into two portions, one rich and the other poor in carbonic acid, when the pressure was reduced after liquefaction so as to convert the whole mixture into the gaseous state In this case the lower end of the tube, which had been before occupied by the liquid, contained a large excess of carbonic acid, particularly if the tube had been previously exposed to $-10°$ so as to liquefy nearly the whole of the carbonic acid in the mixture. On reducing the pressure so as to restore the carbonic acid to the gaseous state, and raising the temperature to 26°, I have succeeded in liquefying the carbonic acid by the application of pressure alone at this temperature, provided the experiment was performed without loss of time. But, on allowing the mixture to remain for some time in the gaseous state, diffusion gradually occurred, and the temperature at which liquefaction was possible underwent a corresponding reduction. The diffusion was not complete for several hours, after which no liquid was formed by the application of pressure till the temperature was reduced to 14°. The true critical point for this mixture was therefore just below 14°, and when liquid was formed by pressure at higher temperatures it arose from carbonic acid gas being present in excess in the lower end of the tube.

Advantage was taken of this mode of separating a mixture of nitrogen and carbonic acid to ascertain whether any or what change of volume occurs in the diffusion of these gases at high pressures. For this purpose the gases were left for about twenty hours under nearly the same pressure as that at which the observations were subsequently made. The volume of the mixture was then carefully read at a fixed temperature and pressure. The mixed gases were now exposed to high pressure at the temperature of $-12°$ so as to liquefy the carbonic acid. They were again restored to the original pressure and temperature, and the volume was read. It was always found to have increased, usually about one-sixtieth part. On allowing the whole to remain at rest, and preserving the temperature and pressure steadily the same, the original volume was gradually restored by the expansion which occurred as the gases slowly diffused into one another. In an hour and a half the mixture had recovered four-fifths of the first contraction. On the other hand, the application of strong pressure at temperatures at which no liquefaction occurred did not produce any change in the volume of the mixture when the original temperature and pressure were restored. The following are the details of three sets of observations:—

At 8°·5, and under a pressure of 46·4 atmospheres, the volume of the mixed gases, after complete diffusion for many hours, was found to be 162·2. After liquefying the carbonic acid by strong pressure at $-12°$, and again restoring the temperature to 8°·5 and the pressure to 46·4, the volume of the mixed gas was reduced to 159·5, a contraction of 2·7 volumes having taken place from the separation of the diffused gases.

In an hour and a half diffusion had partially occurred and the volume had increased to 161·5.

At 16°·0, and under a pressure of 47·9 atmospheres, the volume of the mixture was 164·6; after liquefaction of the carbonic acid and restoration of the original temperature and pressure the volume was 161·9, or a contraction of 2·7 volumes. In an hour and a half the volume had increased to 164·1.

Similar experiments at 20°, and under a pressure of 46·4 atmospheres, gave a reduction of volume from 175·8 to 173·5 after separation of the gases.

It hence appears that when carbonic acid and nitrogen diffuse into one another at high pressures an increase of volume takes place; and, on the other hand, when they are separated from one another there is a diminution of volume. A similar change of volume no doubt occurs at ordinary pressures, but its amount would be so small as to render the observation difficult. The result now described is the reverse of what commonly happens when liquids diffuse into one another, but it is in accordance with what might be expected in the case of the diffusion of two bodies such as carbonic acid and nitrogen of unequal compressibility, and which do not combine with one another.

The following are the general conclusions to which this inquiry has led:—

1. The law of gaseous mixtures, as enunciated by DALTON, is largely deviated from in the case of mixtures of nitrogen and carbonic acid at high pressures, and is probably only strictly true when applied to mixtures of gases in the so-called perfect state.

2. The critical point of temperature is lowered by admixture with a permanent gas.

3. When carbonic acid gas and nitrogen diffuse into each other at high pressures the volume of the mixture is increased.

4. In a mixture of liquid carbonic acid and nitrogen at temperatures not greatly below the critical point, the liquid surface loses its curvature and is effaced by the application of pressure alone, while at lower temperatures the nitrogen is absorbed in the ordinary way, and the curvature of the liquid surface is preserved so long as any portion of the gas is visible.

III On Evaporation and Dissociation.—Part III. *A Study of the Thermal Properties of Ethyl Oxide*

By William Ramsay, *Ph.D.*, *and* Sydney Young, *D.Sc.*

Communicated by Professor G. G. Stokes, *D.C.L.*, *P.R S.*

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[Plates 6–10]

In a memoir published in the Royal Society's 'Philosophical Transactions,' 1886 (Part I.), p. 123, " On the Thermal Properties of Ethyl Alcohol," we gave the results of a research on the vapour-pressures of alcohol, the densities of its vapour—both unsaturated and saturated—and the expansion and compressibility of liquid alcohol at various temperatures; and from these data were deduced the amounts of heat required to vaporize alcohol at those temperatures. Our object in these researches has been to compare carefully the behaviour of stable with that of unstable bodies, and, if possible, to acquire some clear ideas of the nature of chemical combination But, as the properties of stable bodies are still to a great extent unknown, we have deemed it advisable to extend our research with the view of investigating this relationship; and for that purpose we have made a similar series of measurements of the thermal constants of ethyl oxide $(C_2H_5)_2O$. The data, and the deductions from the data, are the subject of the following memoir.

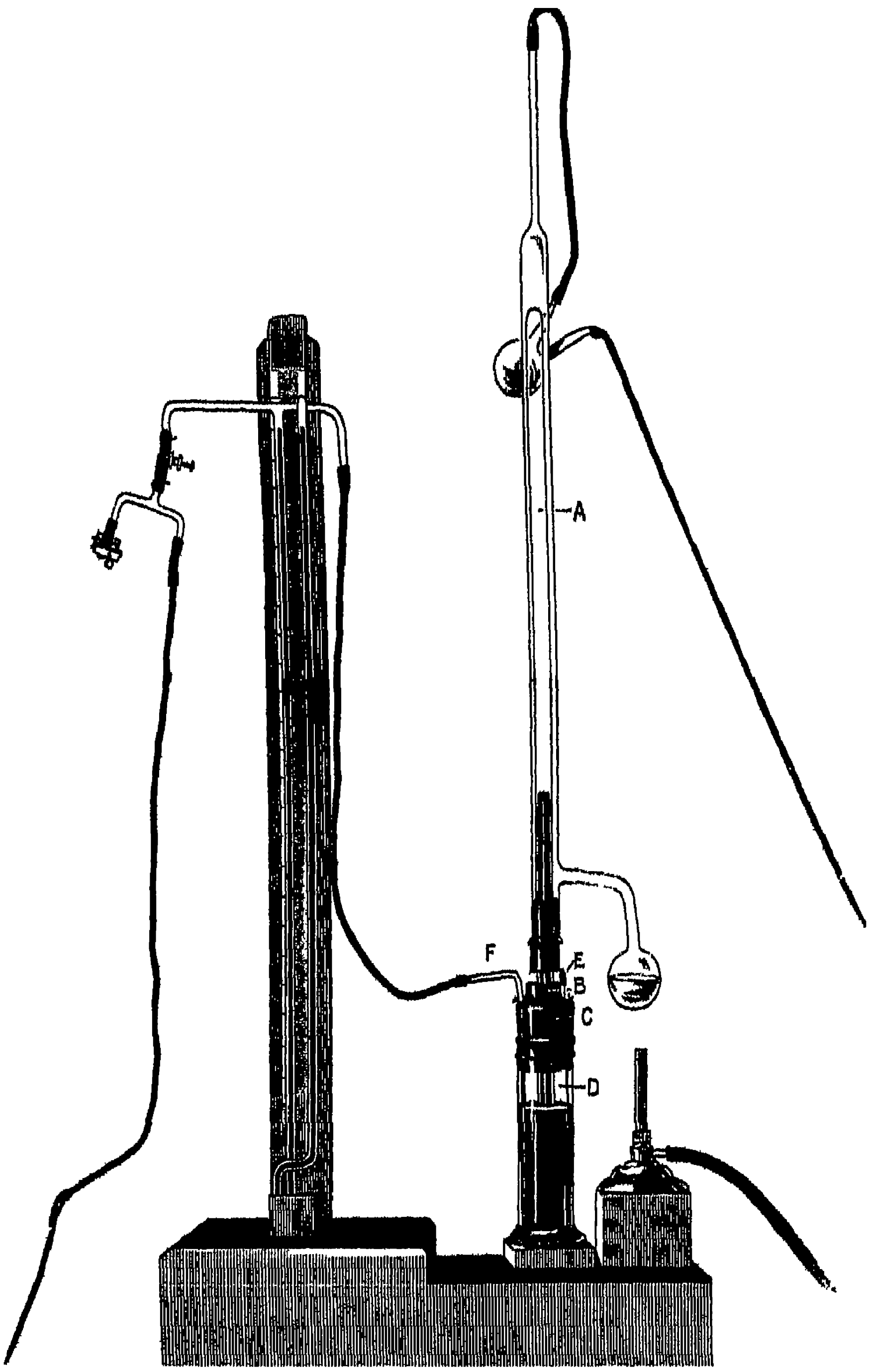
Experiments on the vapour-pressure, vapour-density, expansion, and other properties of ether have been made by Regnault, Kopp, Pierre, Mendelejeff, Avenarius, and others, and their results shall be quoted when necessary.

Preparation of Pure Ether.

A quantity of absolute alcohol was converted into ether by means of sulphuric acid in the usual way. The distillate was first shaken up with caustic soda, to remove sulphurous anhydride, and was then redistilled. In order to remove a great part of the alcohol in the distillate, it was allowed to stand over calcium chloride, and again distilled. It was then repeatedly shaken with water to remove the last traces of alcohol, and it was then again dried with calcium chloride and distilled. The distillate was cohobated with metallic sodium until gas ceased to be evolved; it was then distilled from the sodium, and left in contact with clean, fresh sodium for many months. It was again distilled, and was found to boil with absolute constancy at 34·72° at a pressure of 763·1 millims. The thermometer used was graduated in

tenths of a degree, and had been frequently tested and indirectly compared with an air thermometer. In order to exclude water, the ether was preserved in a stoppered tube with a mercury joint above the stopper.

Fig. 1.



Apparatus employed.

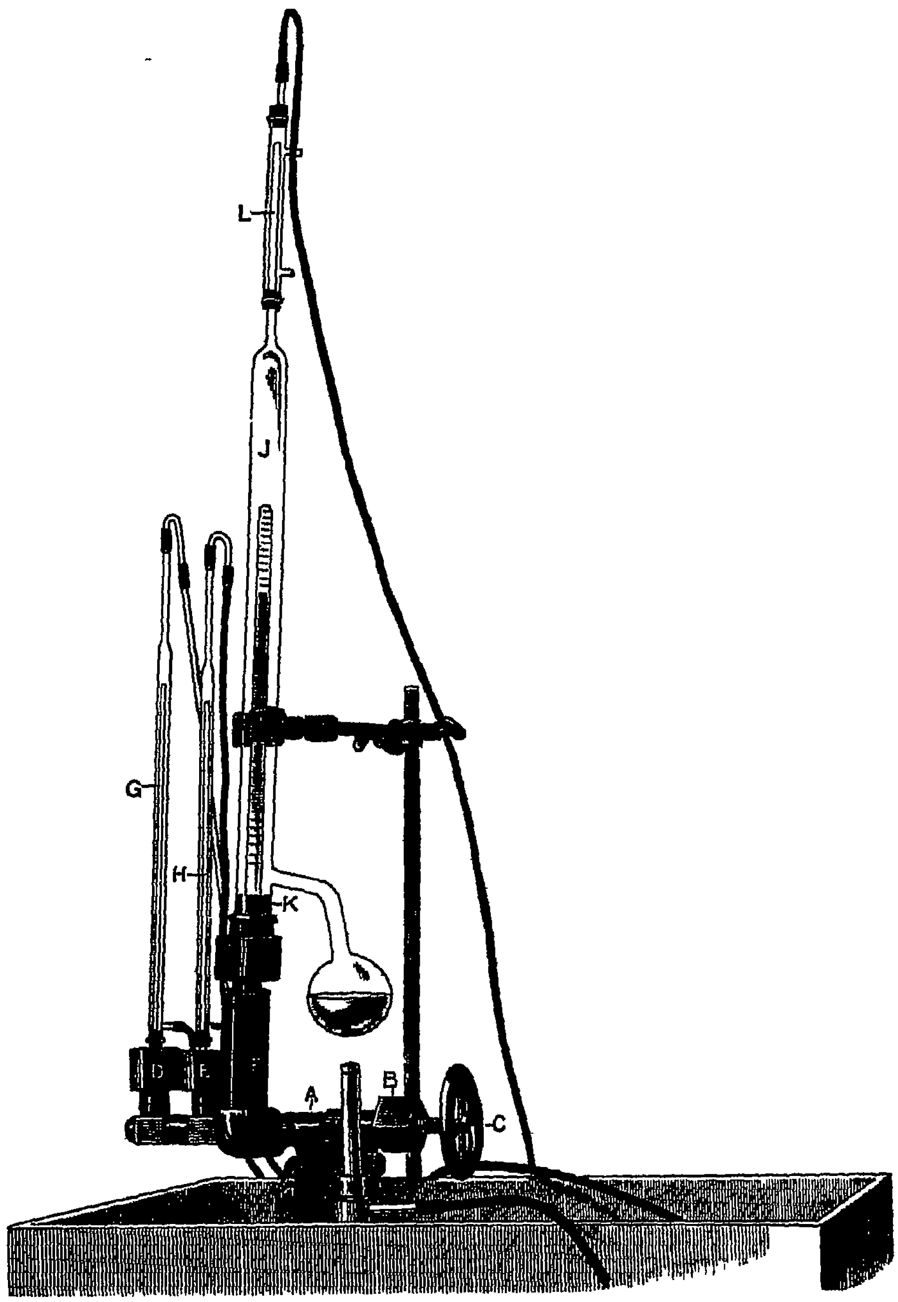
Three different pieces of apparatus were employed in this research.

One for the determination of vapour-pressures at low temperatures. The apparatus has already been described in the 'Philosophical Transactions' for 1884, p. 37, and an improved form in the 'Journal of the Chemical Society' for 1885, p. 42. As it was impossible to use an india-rubber joint in presence of ether, a tight glass stopcock, smeared with slightly deliquesced phosphoric anhydride, was substituted.

The densities of the saturated and unsaturated vapour at low temperatures were

determined by an apparatus modified from that devised by Professor HOFMANN. The form adopted was simpler than that employed in the research on alcohol. The graduated tube A (fig. 1) was completely filled with warm dry distilled mercury; the ether, contained in a small light bulb, was introduced; the tube was then inverted into a temporary mercury-trough B, on the top of a large india-rubber cork C, which closed the top of a large glass jar D, full of mercury, and communicating with the reservoir

Fig. 2



by means of a hole through the cork. Through this hole the tube was inserted, and pushed down, until its extremity was distant from the bottom of the jar about 2 centims. A quantity of mercury was then forced out of the jar through the tube E, which did not dip so deeply into the mercury as the graduated tube. The tube E was then permanently closed. The tube F, which just passed through the cork, was connected with a pump and gauge, by means of which the pressure on the surface of the mercury could be altered and read.

To find the weight of ether employed, the graduated tube was jacketed with the vapour of alcohol, boiling under atmospheric pressure. For lower temperatures the tube was surrounded with flowing water.

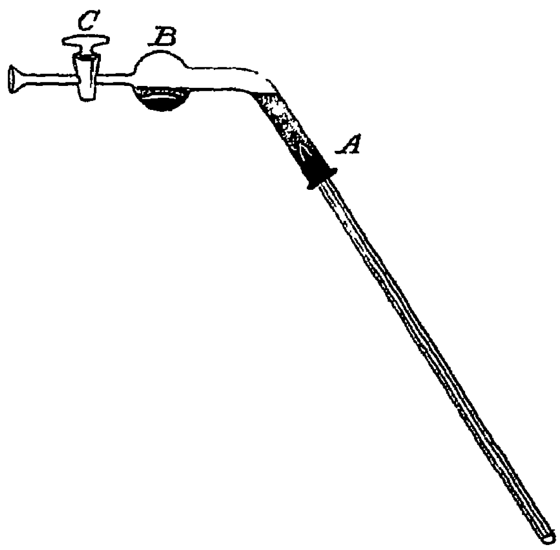
The constants at high temperatures were ascertained by help of the apparatus employed in our research on alcohol, which will now be described

The body of the apparatus consists of a wrought-iron tube A (fig. 2), firmly fixed in a horizontal position by being clamped in a vice. As in ANDREWS's apparatus, one end is closed by a cap B, through which an iron screw passes, the joint being made tight by a packing of greased leather in which the screw C works, passing through the interior of an india-rubber cork, which closely fits the cap. On screwing on the cap, the india-rubber is compressed, so that a very high pressure can be withstood without leakage. The iron tube has no opening at the other end, but is provided with three vertical branches, D, E, and F, closed in a similar manner by iron caps, through which the gauges and the experimental tube pass. The gauge G is intended for registering high pressures, and H for low pressures. To the open end of G is sealed a glass reservoir, of known capacity, while H is a plain tube, constricted at one end. These, and also the experimental tube, which is also constricted at its open end, dip into clean, distilled mercury, filling the iron tube completely. The gauges are jacketed by narrow glass tubes, through which water flows; the temperature of the water was registered by a small thermometer, placed in the stream, immediately after passage through the gauges. It was found by experiment that, with a rapid current, the temperature did not rise sensibly during its passage. The experimental tube is fitted with a jacket J, in the bulb of which a liquid boiled under known pressure ('Chem. Soc. Journ.,' vol. 47, p. 640). The experimental tube passes through a perforated india-rubber cork, closing the jacket, and protected from the action of the condensed hot liquid by a layer of mercury K. The top of the jacket is furnished with a small condenser L, to prevent escape of vapour. The pressure under which the pure liquid was boiling was read by means of a gauge and barometer.

The experimental tube was filled with ether by fitting it, with help of a ring cut from india-rubber tube, into the end of the tube A (fig. 3). This tube is bent to an obtuse angle, and widened into a bulb at B. On the further side of B there is a tight stopcock C. Into the bulb is introduced some pure mercury and a quantity of the pure ether, much more than sufficient to fill the experimental tube. By warming the experimental tube some air is expelled; the ether in the bulb is then boiled on the surface of the mercury, the stopcock being open. When vapour freely escapes the stopcock is closed, and the tube is held in such a manner that ether covers the open end of the experimental tube. The latter is again warmed, and the ether, which enters on cooling, is boiled. The experimental tube, being at a higher temperature than the boiling-point of ether under the reduced pressure, liquid ether, trickling down into it, is at once gasified and carries with it all air, and a series of bubbles rapidly rises through the ether in the bulb. When it is judged that all air is expelled

the experimental tube is cooled, and ether rushes in to fill it. It is easy to make sure, by the absence of a bubble, of complete expulsion of air. The tube is then tilted, so that mercury covers the end of the experimental tube, and a portion of the latter is warmed. The ether boils off through the mercury, and, on cooling, its place is occupied by mercury. By tapping, the column of mercury may be made to descend to any desired point. When quite cold, the experimental tube is disconnected, placed in the iron cap, and gently warmed, so as to cause a globule of mercury to hang to its constricted open end It is then plunged under the surface of the mercury in the branch of the iron tube, and the cap is screwed tight. From this description it will be noticed that all possibility of the presence of air in the liquid to be experimented on is completely excluded; and our results prove that this was the case, for the readings of vapour-pressure at different volumes of gas and liquid give, for the same temperature, absolutely identical results.

Fig 3



It was thus possible to alter volume by means of the screw; to read pressures accurately by the use of the high and low pressure gauges, the readings of which were compared when possible; and to secure constant known temperatures by means of the vapour-jacket.

Experimental Results

1. *Vapour-pressures at Low Temperatures.*

Calibration of Thermometer.—The thermometer employed was a new one by Negretti and Zambra, divided into tenths of a degree, and registering from $-20°$ to $+50°$. The zero-point at the atmospheric pressure was $+0·19°$. The bulb was immersed in mercury, contained in a vessel from which air could be exhausted by means of a Carré's pump. Unless the bulb is dipped in mercury, the temperature it registers is altered by the cooling of the air by exhaustion, or heating by compression; the mercury serves to keep the temperature constant. It was found that the tempera-

ture was apparently lowered 0·25° for a fall of pressure of 700 millims. The thermometer was next tested by a few determinations of the vapour-pressure of water. The water on the cotton-wool encasing the bulb having been frozen, pressure was raised to 6 millims., and the melting-point of ice was observed The mercury stood constant for a long time at −0·11. An apparent fall of temperature, due to reduction of pressure, of +0·19−(−0·11)=0 3°, had thus occurred for a fall of pressure of 754 millims., which agrees with sufficient accuracy with the former observation, 0·25°, for 700 millims. This change of zero-point was considered to be proportional to the pressure, and corrections introduced accordingly. A comparison of the vapour-pressures of water by our method gave results coincident with those of REGNAULT up to 33°, and it was assumed that the graduation of the thermometer was equally regular below 0°.

Vapour-pressures at Low Temperatures.

SERIES I.

Pressure.	Temperature.	Pressure.	Temperature.
millims.	°	millims.	°
73 65	−17·73	141 0	−5·46
74 15	−17·68	152 7	−3·90
84·50	−15·30	153·85	−3·73
96·55	−12 85	168 9	−1·81
110 65	−10 33	184·05	−0·11
124 80	− 7 94	184·10	−0·07
139·85	− 5·65	186 05	+0·13
140 65	− 5 55	197·35	+1·42

SERIES II.

Pressure.	Temperature	Pressure	Temperature.
millims.	°	millims.	°
79 95	−16·24	216·25	+ 3 37
105·45	−11 23	258 25	7·33
133 05	− 6 74	316·40	11·96
162·60	− 2·71	392 90	17·19
181·80	− 0·33	593·05	27·64

These results were plotted and a curve drawn through them. The pressures for each 5° between −15° and +5°, read from the curve, are as follows :—

Temperature.	Pressure.
°	millims.
−15	86·0
−10	112·8
− 5	144·8
0	184 9
+ 5	233 0

The pressure at 0° agrees very well with that given by REGNAULT, 184·39 millims ('Mémoires de l'Académie,' vol. 26, 1862, p 393); but at the other temperatures the agreement is not nearly so close. These results will be considered subsequently.

2. *Vapour-densities at Low Temperatures.*

Weight of Ether taken.—The quantity of ether taken was not determined directly by weighing, but was calculated from vapour-density determinations at the boiling-point of alcohol under atmospheric pressure.

Series I.—Barometer, 763·1 millims. (reduced to 0°)
Boiling-point of alcohol, 78·4°.

In each set of observations the volume was altered very slightly; in the first, for instance, it varied from 154·3 to 155·3 cub. centims, while the pressure varied from 75·45 to 74 75 millims. The number of readings and the mean values are given in every case.

Number of readings.	Pressure reduced to 0°, Mean.	Volume, Mean.	Pressure × Volume	Vapour-density
	mms	c cs		
9	75 06	154·77	11617	Taken as normal, =37
6	85 55	135 15	11562	
5	105 56	110 01	11613	
2	160 85	71 85	11557	37·13
3	438 82	25 87	11352	37 80

The mean value of $p.v.$ calculated from the first three sets of readings is 11599·5. Taking the normal vapour-density of ethyl oxide as 37·0, the weight is 0·0393 gram. The pressure of the saturated vapour at this temperature is nearly 3,000 mms.; and it will be noticed that the value of $p.v.$ is constant, although the volume has been reduced from 3 to 2.

Vapour-density at 12·9°.—The densities of the unsaturated and saturated vapour were then determined at 12·9°, the temperature being maintained constant by means of running water. The data follow in order.

Number of readings.	Pressure reduced to 0°, Mean.	Volume, Mean	Pressure × Volume	Vapour density
	mms	c cs		
2	54 07	175·25	9476	36 84
1	72 90	129 95	9473	36 85
2	118 20	79 92	9447	36·95
1	138 25	67·80	9374	37 24
2	138 70	67 40	9348	37 34
2	170 30	55 00	9355	37·30
2	207 18	44 90	9302	37 53
2	263 45	35 02	9227	37 83
2	309 68	29 60	9166	38 08
2	325·00	28·05	9116	38 30
Condensed	330 00	25·10	—	--

Vapour-pressure calculated, 330·48 mms

3. *Constants at High Temperatures.*

For these experiments four different amounts of ether were employed. In the first case a large amount, A, was taken, and its weight calculated from its volume at known temperatures. During the early experiments a leakage took place, and, some ether being lost, the gauges had to be refilled. Measurements were again made to ascertain the weight of the remaining ether; this portion is alluded to as A′. The third amount, B, was too small to be accurately determined in this way, and was estimated by comparison of the volumes occupied by the two quantities under similar conditions of temperature and pressure. The fourth quantity, C, was still smaller, and its weight was deduced from comparison of its volumes with those of B at the same temperatures and pressures.

A. *Results of experiments on large quantity of ether. Determination of weight.*

The mean of the determinations of the specific gravity of ether at 0° by Kopp, Pierre, Mendelejeff, and Perkin was taken. Their numbers were as follows :—

Kopp ('Liebig's Annalen der Chemie und Pharmacie,' vol. 64, p. 214)	0·73658
Pierre ('Annales de Chimie et de Physique,' vol. 15, p. 325) . .	0·73581
Mendelejeff ('Liebig's Annalen der Chemie und Pharmacie,' vol. 119, p. 9)	0·73644
Perkin ('Chem. Soc. Journ.,' vol. 45, p. 474)	0·7371
" " "	0·7352
Mean	0·7362

Dr. Perkin's results were from comparisons of ether and water at 15° and at 25°; the former is 0·72088, and the latter 0·70991. They were reduced to 0° by means of Kopp's formula with the above results

The portion of ether A gave the following measurements .—

Temperature	Volume	Specific gravity.	Weight
17°·95	0·19589	0·71627	0·14031
15·50	0·19554	0·71902	0·14060
16·20	0·19571	0·71805	0·14053
	Mean weight	. . . . .	0·14048

The volume tube was a new one, and was carefully calibrated by weighing with mercury. A low pressure and a high pressure gauge were employed. Where possible, readings on both were taken. The manometers were calibrated by weighing with mercury, and contained air dried over phosphorus pentoxide. The following corrections for volume, pressure, and temperature were applied :—

For volume.—Meniscus of mercury and of liquid.

Expansion of glass by heat.

The expansion of the tube owing to internal pressure was not allowed for, as it would have been much within the errors of reading.

For pressure.—Meniscus of mercury.

Levels of mercury in volume tube and in pressure gauges.

Difference of temperature in water-jacket at time of filling and time of reading.

Deviation of air from BOYLE'S Law, as determined by AMAGAT ('Compt. Rend.,' vol. 99, 1884, p. 1153).

For temperature.—Reduction of the pressure under which the liquid boiled, as read on the gauge, to 0° (see 'Chem. Soc. Journ.,' vol. 47, 1885, p. 640).

The temperatures are those of an air-thermometer.

	Pressure of alcohol	Temperature of alcohol	Volume of liquid.	Volume of 1 gramme.	Specific gravity.	Vapour-pressure.*	Mean
	mms	°C	c cs	c.cs.		mms.	mms.
A′ (1)	133·7	40	0·20220	1·4505	0·68943	921 920 922 922	921
(2)	172·2	45	0·20398	1·4632	0·68342	1085 1085 1086 1086	1085·5
(3)	220·0	50	0·20611	1·4785	0·67636	1276 1276 1278 1278	1277

* In this, and in all other cases, the vapour-pressures were determined at widely different volumes.

Compressibilities of Liquid.

Volume	Volume of 1 gramme	Pressure.
c cs	c cs	mms
0·20576	1 4760	2,635
0 20472	1·4685	5,878
0 20401	1·4635	16,539
0 20365	1 4609	24,986
0 20296	1·4559	33,929

	Pressure of alcohol.	Temperature of alcohol.	Volume of liquid	Volume of 1 gramme	Specific gravity	Vapour pressure.		Mean.
						L P G *	H. P. G.*	
	mms.	°C	c cs	c cs		mms.	mms.	mms.
(4)	278 6	55	0 20753	1 4887	0·67172	1490 1488 1491 1494	..	1491
(5)	350·3	60	0·20999	1·5063	0·66387	1732 1738 1732 1736	..	1734
(6)	437·0	65	0·21141	1·5165	0·65940	2002 2003 2004 2007	..	2004
(7)	541 2	70	0 21355	1·5319	0 65280	2299 2306 2305 2307	..	2304
(8)	665 55	75	0·21556	1 5463	0·64671	2637 2640 2639 2639	..	2639
	(Chlorobenzene)							
(9)	144 8	80	0·21848	1·5673	0 63806	2976 2976 2978	..	2977
(10)	174·25	85	0·21984	1·5770	0·63412	3393 3388 3392 3385	..	3389
(11)	208 35	90	0·22378	1 6053	0·62293	3829 3835 3829 3832	..	3831
(12)	247·7	95	0 22590	1·6205	0·61709	4326	4322 4325 4327 4330	4326
(13)	292·75	100	0 22840	1·6384	0·61034	4849 4855 4871 4852	4855 4853 4860 4852	4857

* Low-pressure gauge and high-pressure gauge.

Compressibilities of Liquid.

Volume.	Volume of 1 gramme	Pressure
c cs.	c cs	mms.
0 22770	1 6334	7,208
0 22700	1 6284	12,955
0 22595	1 6208	19,515
0 22420	1 6083	27,072
0 22246	1 5958	44,154

	Pressure of chlorobenzene	Temperature of chloro-benzene.	Volume of liquid.	Volume of 1 gramme.	Specific gravity	Vapour-pressure *		Mean.
						L P G	H. P G	
	mms.	° C	c cs	c.cs		mms	mms	mms.
(14)	344 15	105	0 23121	1 6586	0 60292	5441		
						5445	5443	
						5438	5440	
						5430	5439	5441
						5458	5439	
						5441	5437	
(15)	402 55	110	0 23456	1 6826	0 59431	6100		
						6078	6083	
						6088	6088	6082
						6063	6063	
						6086	6086	
(16)	468 5	115	0 23825	1·7091	0 58511	..	6773	
							6778	6775
							6772	
							6779	
(17)	542 8	120	0 24215	1 7370	0 57569	..	7496	
							7501	7513
							7520	
							7535	
(18)	626 15	125	0 24566	1·7623	0 56745	..	8274	
							8307	8313
							8327	
							8344	
(19)	718 95	130	0 24987	1 7925	0 55792	..	9156	
							9176	9188
							9216	
							9203	
A (20)	144 8	80	0 22022	1 5676	0 63790	2965		
						2976	.	2971
						2973	.	
	(Bromobenzene)							
(21)	372 65	130	0 25213	1·7948	0 55715	..	9162	
							9135	
							9131	9131
							9134	
							9125	

* Under this heading are occasionally given the pressures under which the liquid was measured. It will be seen that in no case does this pressure differ so much from the vapour-pressure as to sensibly reduce the volume of the liquid

	Pressure of bromobenzene	Temperature of bromo-benzene	Volume of liquid.	Volume of 1 gramme	Specific gravity.	Vapour-pressure. L. P. G	Vapour-pressure. H. P. G	Mean.
	mms	° C	c.cs	c.cs.		mms.	mms	mms.
(22)	430 75	135	0 25585	1 8213	0 54906		10,222	
							10,056 10,085 10,086 10,081	10,077
(23)	495 8	140	0 26077	1·8563	0 53870		11,087	
							11,048 11,058 11,039 11,060	11,051
(24)	568 35	145	0 26651	1 8971	0·52712	.	12,260	
							12,113 12,128 12,115 12,133	12,122
(25)	649 05	150	0·27133	1·9314	0·51774	..	13,348	
							13,260 13,228 13,234 13,242	13,241

Compressibilities of Liquid,

Volume.	Volume of 1 gramme.	Pressure,
c.cs,	c.cs.	mms
0·26993	1·9215	13,818
0 26818	1 9091	18,258
0·26644	1·8967	20,228
0·26574	1·8917	22,705

The density of the saturated vapour was determined at this temperature.

Volume of vapour.	Volume of liquid.	Weight of liquid.	Weight of vapour	Weight of 1 c.c. vapour	Mean.	Volume of 1 gramme vapour.	Vapour-density.
0 37322	0 23042	0 11930	0·02118	0 05675			
0·51918	0 21594	0 11180	0·02868	0 05224	0·05541	18·047	56·22
0·75893	0 19121	0·09900	0·04148	0·05466			
0·91823	0 17377	0·08997	0·05051	0 05501			

	Pressure of aniline	Temperature of aniline	Volume of liquid.	Volume of 1 gramme	Specific gravity	Vapour-pressure	Mean.
	mms.	° C	c cs	c cs		mms	mms
(26)	283 7	150	0 27133	1 9315	0 51774	13,491	
						13,347 13,291 13,290 13,259 13,272	13,292
(27)	331 7	155	0 27766	1·9766	0 50593	14,818	
						14,515 14,532 14,521 14,488	14,514
(28)	386 0	160	0·28433	2 0240	0 49406	16,038	
						15,776 15,768 15,769 15,799	15,778
(29)	447·1	165	0 29170	2 0765	0 48158	17,335	
						17,265 17,209 17,204 17,124	17,201
(30)	515 6	170	0·30186	2 1488	0 46538	18,860	
						18,743 18,666 18,668 18,597	18,671
(31)	592 05	175	0·31443	2 2383	0 44677	20,238	
						20,178 20,181 20,210 20,228	20,199

Compressibilities of Liquid.

Volume	Volume of 1 gramme.	Pressure.
c cs.	c cs.	mms.
0 31199	2 2209	21,802
0 30850	2 1961	22,556
0 30501	2 1712	23,767
0 30152	2 1464	25,496
0·29454	2 0967	30,420
0 29105	2 0719	32,780
0 28756	2 0470	36,362
0 28407	2 0222	40,177

The density of the vapour was here determined.

Volume of vapour	Volume of liquid.	Weight of liquid.	Weight of vapour	Weight of 1 c c. vapour.	Mean.	Volume of 1 gramme vapour	Vapour density
						c cs.	
0·31360	0 24414	0 10907	0 03141	0·10016			
0 58620	0 18492	0 08261	0 05787	0 09872	0 09899	10 103	68·23
0 86187	0 12462	0 05568	0 08480	0 09839			
1 04445	0 08340	0 03726	0 10322	0 09883			

	Pressure of aniline.	Temperature of aniline	Volume of liquid	Volume of 1 gramme.	Specific gravity.	Vapour-pressure	Mean.
	mms.	° C	c cs.	c cs.		mms.	mms.
(32)	677·15	180	0 32943	2 3451	0 42642	22,131	
						21,804	
						21,745	
						21,776	21,793
						21,821	
						21,807	
	(Methyl salicylate)						
(33)	249 35	180	0·32943	2·3451	0·42642	21,942	
						21,820	
						21,804	21,756
						21,645	
(34)	287·8	185	0·34936	2·4869	0·40211	23,746	
						23,691	
						23,688	23,691
						23,695	

Compressibilities of Liquid.

Volume.	Volume of 1 gramme	Pressure.
c cs.	c.cs.	mms.
0 33648	2·3952	25,243
0·33299	2·3704	26,091
0·32950	2 3456	26,888
0·32602	2 3208	27,746
0·32253	2·2959	28,720
0·31905	2·2712	29,924
0·31208	2·2216	32,640
0 30860	2·1968	34,348
0 30510	2·1719	36,355
0·30161	2·1470	38,842
0 29811	2·1221	41,820

Vapour-density Determinations.

Saturated.

Volume of vapour	Volume of liquid	Weight of liquid	Weight of vapour	Weight of 1 c c vapour	Mean	Volume of 1 gramme vapour	Vapour-density
						c cs.	
0 31308	0 24482	0 09844	0 04204	0 13428			
0 62739	0 14396	0 05788	0 08260	0 13166	0 13290	7·5245	79 85
0 89820	0 05282	0 02124	0 11924	0 13276			

Unsaturated.

Volume of vapour	Pressure.	Weight of 1 c.c vapour	Volume of 1 gramme vapour	Vapour-density
1 1106	23,408	0 12649	7 9059	76 91
1 1990	22,947	0 11716	8 5352	72 67

	Pressure of methyl salicylate	Temperature of methyl salicylate	Volume of liquid	Volume of 1 gramme	Specific gravity	Vapour-pressure.	Mean
	mms	° C	c cs.	c cs		mms.	mms.
(35)	330 85	190	0 38325	2 7282	0 36654	25,645	
						25,514	
						25,518	25,530
						25,558	

Compressibilities of Liquid

Volume	Volume of 1 gramme	Pressure
c cs	c.cs.	mms
0 37836	2 6934	25,781
0 37138	2 6437	26,034
0 36440	2 5940	26,448
0 35743	2 5444	27,173
0 34696	2 4699	28,535
0 32953	2 3458	32,351
0 31908	2 2714	35,522
0 31211	2 2218	38,705

Vapour-density Determinations.

Saturated.

Volume of vapour	Volume of liquid	Weight of liquid	Weight of vapour	Weight of 1 c.c vapour	Mean	Volume of 1 gramme vapour.	Vapour-density
0·24674 0·44219 0·69501	0·27581 0 18704 0 07641	0 10110 0 06856 0 02801	0·03938 0 07192 0·11247	0·15960 0 16264 0 16183	0·16136	6 1973	91·44

Unsaturated.

Volume of vapour	Pressure	Weight of 1 c c vapour	Volume of 1 gramme vapour.	Vapour density.
0 91519	25,331	0 15350	6 5148	87 19
0 98687	25,107	0·14235	7·0251	81·58
1 0578	24,705	0·13280	7·5302	77·35
1 1991	23,848	0·11715	8 5359	70 69

	Pressure of methyl salicylate.	Temperature of methyl salicylate.	Volume of liquid	Volume of 1 gramme.	Specific gravity.	Vapour-pressure.	Mean.
	mms	° C.	c cs.	c.cs.		mms.	mms.
(36)	349 45	192	0 40637	2·8928	0·34566	26,482 26,304 26,324 26,354	26,327

Compressibilities of Liquid.

Volume.	Volume of 1 gramme.	Pressure
c.cs.	c.cs.	mms
0·40287	2·8678	26,482
0·39589	2·8181	26,515
0·38888	2·7683	26,786
0·38189	2·7185	27,066
0·37142	2 6440	27,580
0 36444	2·5943	28,236
0 35397	2·5198	29,441
0 34700	2·4701	30,400
0 33655	2·3957	32,518
0·32957	2 3461	34,227
0 31912	2 2716	38,474
0 31215	2·2220	41,819

Vapour-density Determinations.

Saturated

Volume of vapour	Volume of liquid	Weight of liquid.	Weight of vapour.	Weight of 1 c c. vapour	Mean	Volume of 1 gramme.	Vapour-density.
						c cs	
0 32029	0 23773	0 08218	0 05830	0 18203			
0 47047	0 15523	0 05366	0·08682	0·18314	0 18254	5 4782	100 20
0 69794	0 03801	0 01314	0 12734	0 18245			

Unsaturated.

Volume of vapour	Pressure	Weight of 1 c c vapour	Volume of 1 gramme	Vapour-density.
			c cs	
0 91529	25,842	0 15348	6 5155	85 83
1 05800	25,172	0 13278	7 5311	76 23
1 19930	24,219	0 11714	8 5369	69 89

	Pressure of methyl salicylate.	Temperature of methyl salicylate.	Volume of liquid	Volume of 1 gramme.	Specific gravity.	Vapour-pressure.	Mean
	mms.	° C	c.cs.	c cs.		mms	mms.
(37)	359 05	193	0·42563	3 0298	0 33006	26,851	..
						26,787	26,792
						26,797	

Vapour-density Determinations.

Saturated.

Volume of vapour	Volume of liquid	Weight of liquid	Weight of vapour	Weight of 1 c c. vapour.	Mean	Volume of 1 gramme.	Vapour-density.
						c.cs.	
0 43540	0 15824	0 05223	0 08825	0 20269	0 20112	4·9722	108 72
0 51645	0 11285	0 03725	0 10323	0 19956			

(38.) APPARENT Critical Point. T = 193·8° (Methyl salicylate).

Volume	Volume of 1 gramme.	Pressure	Vapour-density *	Volume	Volume of 1 gramme.	Pressure.	Vapour-density.
c.cs.	c.cs.	mms.		c.cs.	c.cs.	mms.	
1·1886	8·4612	24,466	70·08	0·48711	3·4675	27,125	..
1·1320	8·0582	24,913	72·26	0·43616	3·1047	27,279	..
1·0580	7·5315	25,372	75·92	0·41475	2·9524	27,273	..
0·98082	6·9820	25,827	80·45	0·38714	2·7559	27,842	.
0·89363	6·3614	26,316	86·66	0·37841	2·6937	28,443	
0·75374	5·3655	26,809	100·85	0·35711	2·5421	30,407	.
0·70044	4·9976	27,044	107·34	0·33235	2·3659	34,384	..
0·62930	4·4797	27,050	119·72	0·32103	2·2853	39,579	..
0·55802	3·9723	27,079	.				

(39.) T = 195° (Methyl salicylate).

Volume.	Volume of 1 gramme.	Pressure.	Vapour-density.	Volume.	Volume of 1 gramme.	Pressure.	Vapour-density.
c.cs.	c.cs.	mms.		c.cs.	c.cs.	mms.	
1·1992	8·5373	24,461	69·08	0·48711	3·4675	27,708	..
1·0580	7·5315	25,662	75·25	0·45192	3·2170	27,816	..
0·95122	6·7713	26,386	81·41	0·41686	2·9674	28,112	..
0·84318	6·0022	26,947	89·92	0·38189	2·7185	29,097	..
0·73595	5·2389	27,342	101·54	0·36585	2·6042	30,372	..
0·69230	4·4797	27,569	117·77	0·34142	2·4304	34,346	..
0·55802	3·9723	27,605	..	0·32504	2·3138	39,535	..

(40.) T = 197° (Methyl salicylate).

Volume	Volume of 1 gramme.	Pressure.	Vapour-density.	Volume	Volume of 1 gramme	Pressure.	Vapour-density.
c.cs.	c.cs.	mms.		c.cs.	c.cs.	mms.	
1·1994	8·5379	25,116	68·12	0·48716	3·4679	28,538	..
1·0581	7·5322	26,264	73·84	0·45196	3·2178	28,724	..
0·95130	6·7719	27,027	79·81	0·41690	2·9677	29,289	..
0·84326	6·0027	27,732	87·75	0·39941	2·8432	29,679	..
0·73601	5·2393	28,059	99·36	0·38543	2·7311	30,363	..
0·62935	4·4800	28,459	114·57	0·34947	2·4877	34,327	..
0·63807	3·9727	28,495	..	0·33065	2·3538	39,510	..

* As it is uncertain whether the substance is a vapour, this term must be accepted with an extended meaning.

(41) T = 200° (Methyl salicylate).

Volume.	Volume of 1 gramme	Pressure	Vapour-density.	Volume	Volume of 1 gramme	Pressure.	Vapour-density.
c cs.	c cs.	mms.		c cs.	c cs.	mms	
1 1995	8 5387	25,545	67 39	0 55811	3 9729	29,847	123 97
1 0582	7 5329	26,740	72 98	0 48720	3 4682	30,381	.
0 95137	6 7723	27,643	78 52	0 45199	3 2175	30,725	..
0 84332	6 0032	28,511	85 89	0 41693	2 9679	31,307	..
0 73607	5·2398	29,108	96 38	0 37254	2·6519	34,318	..
0 62941	4 4804	29,663	110 60	0 34008	2·4209	39,500	..

(42.) T = 205° (Methyl salicylate).

Volume.	Volume of 1 gramme	Pressure	Vapour-density.	Volume.	Volume of 1 gramme.	Pressure.	Vapour-density
c cs	c cs.	mms		c cs.	c cs.	mms.	
1 1996	8 5394	26,354	66 01	0 55817	3 9734	31,873	117 30
1 0583	7 5336	27,624	71 38	0 48725	3 4685	32,623	..
0 95148	6 7732	28,664	76 52	0 45204	3 2179	33,133	..
0 84341	6 0028	29,655	83 44	0 41698	2 9682	34,334	..
0 73616	5 2403	30,484	92 99	0 38200	2 7193	36,634	..
0 62948	4 4809	31,366	105 69	0 34710	2 4708	42,731	..

(43.) T = 210° (Methyl salicylate).

Volume.	Volume of 1 gramme.	Pressure	Vapour-density.	Volume	Volume of 1 gramme	Pressure	Vapour-density
c.cs.	c.cs.	mms.		c.cs.	c.cs	mms.	
1 1997	8·5400	27,161	64 71	0·55824	3 9738	34,022	108 50
1 0584	7 5343	28,579	69 71	0 48731	3 4689	35,230	..
0 95159	6·7739	29,784	74 40	0 45210	3 2183	36,242	..
0 84351	6 0045	30,833	81 08	0 41702	2 9686	37,696	..
0 73624	5 2409	32,000	89·50	0 38204	2·7196	40,743	..
0 62955	4 4815	33,182	100 94				

(44) T = 223° (Methyl salicylate).

Volume.	Volume of 1 gramme	Pressure.	Vapour-density	Volume	Volume of 1 gramme.	Pressure	Vapour-density.
c cs.	c.cs.	mms.		c.cs.	c cs.	mms.	
1 2002	8·5436	29,231	61 72	0 55846	3 9754	39,040	99·32
1 0588	7·5371	30,954	66 07	0 52302	3 7231	40,100	103 25
0·95196	6 7766	32,426	70 15	0·48749	3·4702	41,090	108 10
0 84304	6 0069	34,067	75 33	0 45226	3 2194	42,837	111 77
0 73653	5·2430	35,591	82 61	0 41719	2 9698	45,819	113 28
0·62980	4 4832	37,493	91·71				

B. *Weight.*—The weight of portion B was ascertained by comparisons of its volume with that of A at the same temperatures and pressures. As the real pressures were in no cases the same for both, it was necessary to construct curves showing the relation of the pressures of A to its volume at constant temperatures, and to read off the volumes at the required pressures. The mean of twelve observations at various temperatures and pressures gave the number

0·01227 gramme.

(45.) T = 50° (Alcohol).

Vapour-pressure	Mean.
mms.	mms.
1273	
1275	1275
1277	
1275	

(46.) T = 75° (Alcohol).

Vapour-pressure.	Mean.
mms	mms.
2638	2637·5
2637	

(47.) T = 100° (Chlorobenzene).

Volume.	Volume of 1 gramme.	Pressure.	P. V.	Vapour-density	Volume.	Volume of 1 gramme.	Pressure.		P. V.	Vapour-density.
c.cs.	c.cs.	mms.			c cs.	c.cs.	mms.			
1·1610	94·63	2978	3458	41 14	0·80529	65·63	4104		3305	43·04
1·0906	88 89	3150	3435	41·40	0·76967	62 73	4265		3283	43·33
1 0201	83·14	3334	3401	41 82	0 73419	59·84	4436		3257	43·67
0·94895	77·34	3546	3365	42 27	0 69876	56·95	4618		3227	44·08
0·87710	71 48	3818	3350	42·46	0·68101	55·50	4720		3214	44·25
0·80529	65·63	4099	3301	43·09	0·66325	54·06	4820		3197	44·49
0·73419	59·84	4432	3254	43·71			L.P.G.	H.P.G.		
0·69876	56·95	4636	3239	43·91	..	..	4865	4857		
0·66325	54 06	4815	3194	44·54	..	..	4868	4861		
1·1964	97·51	2893	3461	41·09	..	..	4841	4848	Mean	
1·0554	86·02	3243	3423	41·55	..	..	4851	4853	4853	..
0·91310	74·42	3668	3349	42·47	..	..	4852	4844		
0 84116	68·56	3946	3319	42 88	..	..	4855	4847		

(48.) $T = 130°$ (Chlorobenzene)

Volume	Volume of 1 gramme	Pressure H P G	P V.	Vapour-density	Vapour-pressure. H P G	Mean.
c.cs.	c.cs.	mms.			mms.	
0 34641	28 23	8975	3109	·49 43	9114	9133
0 34293	27 95	9038	3099	49 58	9127	
0 33945	27 66	9097	3088	49 77	9136	
.	..	..	..	.	9136	
..	..	.	..	..	9136	
..	..	..	..		9151	

(49.) $T = 150°$ (Aniline).

Volume	Volume of 1 gramme	Pressure.	P. V.	Vapour-density	Volume.	Volume of 1 gramme	Pressure.	P V.	Vapour density.
c cs	c.cs.	mms.			c cs.	c.cs.	mms		
1 1979	97 63	3375	4043	39 90	0 38146	31 09	9,066	3458	46 64
1 1272	91 87	3564	4017	40 15	0 34661	28 25	9,809	3400	47 44
1 0568	86 13	3783	3998	40 35	0 31180	25 41	10,581	3299	48 89
0 98590	80 35	4027	3970	40 63	0 29436	23 99	11,019	3244	49 73
0 91430	74 51	4304	3935	40 99	0·27692	22 57	11,500	3185	50 65
0 84223	68 64	4561	3917	41 18	0 25946	21 15	11,998	3113	51 82
0 77064	62 81	5034	3879	41 58	0 24199	19 72	12,570	3042	53·03
0 69965	57 02	5481	3835	42 06	0 22449	18 30	13,204	2964	54·42
0 62860	51 23	6024	3787	42·60	..	..	13,265		..
0·55739	45 43	6674	3720	43 36	..	.	13,289	Mean	..
0 48657	39 66	7484	3641	44 30	..	..	13,284	13,283	.
0 41640	33 94	8497	3538	45 59	..	..	13,294		..

(50.) $T = 175°$ (Aniline).

Volume.	Volume of 1 gramme.	Pressure	P. V.	Vapour-density	Volume	Volume of 1 gramme	Pressure	P. V.	Vapour-density.
c cs	c cs	mms.			c cs.	c.cs.	mms.		
1 19860	97 69	3589	4302	39 71	0·38170	31 11	9,906	3781	45 18
1 12790	91 93	3803	4290	39 83	0 34683	28 27	10,715	3716	45 97
1·05740	86 18	4035	4267	40 04	0 31199	25 43	11,673	3642	46 91
0 98649	80 40	4307	4249	40 21	0 27709	22 58	12,730	3527	48·43
0·91483	74 56	4626	4232	40 37	0·24214	19 73	14,021	3395	50·32
0 84276	68 69	4987	4203	40 65	0 22463	18 31	14,713	3305	51 69
0·77111	62 85	5396	4161	41 06	0 20712	16·88	15,531	3217	53 11
0 70008	57 06	5903	4133	41 34	0 18958	15 45	16,379	3105	55 02
0 62900	51 26	6485	4079	41 88	0·17204	14·02	17,304	2977	57·38
0·55774	45 46	7208	4020	42 49	0 15447	12·59	18,388	2840	60 14
0 48687	39 68	8065	3927	43 51	0 13690	11·16	19,354	2650	64 47
0·41665	33·96	9248	3853	44 34					

Vapour-pressure = 20,150.

(51.) T = 185° (Methyl salicylate).

Volume	Volume of 1 gramme	Pressure.	P. V	Vapour-density	Volume	Volume of 1 gramme	Pressure	P. V	Vapour-density
c.cs.	c.cs.	mms.			c.cs.	c.cs.	mms.		
1·1990	97·72	3,692	4427	39·45	0·31208	25·48	12,106	3778	46·23
1·1283	91·95	3,915	4417	39·54	0·24221	19·74	14,641	3546	49·25
1·0577	86·21	4,159	4399	39·70	0·20718	16·89	16,290	3375	51·75
0·91510	74·58	4,755	4351	40·14	0·17209	14·03	18,324	3153	55·38
0·73580	59·97	5,811	4276	40·85	0·13694	11·16	20,667	2830	61·71
0·55790	45·47	7,434	4147	42·11	0·11934	9·726	21,893	2613	66·85
0·38181	31·12	10,284	3927	44·48	0·10174	8·292	22,984	2338	94·69

$$\text{Vapour-pressure} = \left.\begin{matrix}23{,}522\\23{,}518\end{matrix}\right\} 23{,}520$$

(52.) T = 190° (Methyl salicylate).

Volume.	Volume of 1 gramme	Pressure.	P. V.	Vapour-density.	Volume	Volume of 1 gramme.	Pressure	P. V	Vapour-density.
c.cs.	c.cs.	mms.			c.cs.	c.cs.	mms.		
1·0578	86·21	4,210	4453	39·64	0·20720	16·89	16,650	3450	51·18
0·91520	74·59	4,828	4419	39·96	0·17211	14·03	18,772	3231	54·65
0·73590	59·97	5,902	4343	40·65	0·13696	11·16	21,288	2916	60·56
0·55790	45·47	7,549	4212	41·92	0·11935	9·727	22,663	2705	65·27
0·38185	31·12	10,455	3992	44·23	0·10174	8·292	24,033	2445	72·20
0·31211	25·44	12,335	3850	45·86	0·08411	6·855	25,126	2113	83·54
0·24223	19·74	14,930	3616	48·82					

Vapour-pressure = 25,462

(53.) T = 192° (Methyl salicylate)

Volume.	Volume of 1 gramme	Pressure	P. V.	Vapour-density	Volume.	Volume of 1 gramme.	Pressure.	P. V	Vapour-density
c.cs.	c.cs.	mms.			c.cs.	c.cs.	mms.		
1·0580	86·42	4,230	4475	39·62	0·20723	16·89	16,774	3476	51·01
0·91530	74·60	4,847	4436	39·97	0·17213	14·03	18,909	3255	54·48
0·73600	59·98	5,930	4364	40·63	0·13697	11·16	21,448	2938	60·36
0·55800	45·48	7,590	4235	41·86	0·11937	9·729	22,848	2727	65·02
0·38189	31·12	10,544	4027	44·03	0·10176	8·293	24,319	2475	71·65
0·31215	25·44	12,428	3879	45·71	0·08412	6·856	25,641	2157	82·21
0·24226	19·74	15,032	3642	48·69					

Vapour-pressure = 26,342.

(54) T = 193° (Methyl salicylate).

Volume	Volume of 1 gramme	Pressure	P V	Vapour-density	Volume.	Volume of 1 gramme	Pressure	P V	Vapour-density
c cs.	c.cs.	mms.			c cs.	c.cs.	mms.		
1 0580	86 42	4,244	4490	39 58	0 17213	14 03	19,035	3276	54 24
0 91530	74 60	4,859	4447	39 96	0 13697	11·16	21,660	2967	59 89
0 73600	59 98	5,941	4372	40 64	0 11937	9 729	23,091	2756	64 47
0 55800	45 48	7,618	4251	41 80	0 10176	8 293	24,498	2493	71 28
0·38189	31 12	10,577	4239	43 99	0 08412	6 856	25,878	2177	81 63
0 31215	25 44	12,462	3890	45 68	0 06648	5 418	26,659	1772	100 26
0 24226	19 74	15,115	3662	48·53	0 06294	5·130	26,764	1685	105 48
0 20723	16 89	16,883	3499	50 79					

Vapour-pressure = 26,816.

(55.) Apparent Critical Point. T = 193·8° (Methyl salicylate).

Volume	Volume of 1 gramme	Pressure	P V	Vapour-density	Volume	Volume of 1 gramme	Pressure	P. V.	Vapour-density
c.cs	c cs.	mms			c cs.	c.cs.	mms.		
1 0580	86·42	4,252	4498	39 57	0·17213	14 03	19,100	3288	54 14
0 91530	74 60	4,867	4455	39 96	0 13697	11·16	21,731	2977	59 80
0 73600	59 98	5,951	4380	40 64	0 11937	9 729	23,211	2771	64 25
0 55800	45·48	7,627	4256	41 82	0 10176	8 293	24,682	2512	70 87
0 38189	31 12	10,587	4043	44 03	0 08412	6 856	25,980	2185	81 45
0 31215	25 44	12,499	3902	45 62	0 06648	5 418	26,984	1794	99 22
0 24226	19 74	15,152	3671	48 49	0 05941	4 842	27,039	1607	110 80
0 20723	16 89	16,944	3511	50 69	0 05235	4·267	27,091	1418	125 51

(56.) T = 195° (Methyl salicylate).

Volume	Volume of 1 gramme	Pressure	P V	Vapour-density	Volume.	Volume of 1 gramme	Pressure.	P V	Vapour-density
c.cs.	c cs.	mms.			c.cs	c.cs.	mms.		
1 1993	97 74	3,797	4554	39 19	0 20723	16 89	17,105	3545	50 35
1·1258	91 97	4,007	4522	39 47	0 17213	14 03	19,213	3307	53·96
1 0580	86 42	4,260	4507	39 60	0 13697	11 16	21,906	3001	59 48
0 9153	74 60	4,884	4470	39 92	0 11937	9 729	23,335	2785	64 07
0 7360	59 98	5,969	4393	40 62	0 10176	8 293	24,956	2539	70 28
0 5580	45 48	7,651	4269	41 80	0·08412	6 856	26,400	2221	80 36
0 38189	31 12	10,631	4060	43 96	0 06648	5 418	27,435	1824	97·85
0 31215	25 54	12,560	3921	45·52	0·05235	4·267	27,601	1445	123 51
0 24226	19·74	15,241	3692	48 33					

(57.) T = 223·25° (Methyl salicylate).

Volume	Volume of 1 gramme	Pressure	P V	Vapour-density	Volume	Volume of 1 gramme	Pressure	P V.	Vapour-density
c.cs	c cs.	mms.			c cs.	c.cs.	mms.		
1 2001	97 81	4,044	4853	38 99	0 24244	19 76	16,784	4069	46 50
1 1293	92 04	4,289	4844	39·07	0 20738	16 90	18,918	3923	48 26
1 0587	86 29	4,556	4824	39 23	0 17225	14 04	21,669	3733	50 70
0 98771	80 50	4,870	4810	39 34	0 13707	11 17	24,991	3426	55·24
0 91597	74 65	5,232	4792	39 49	0 11945	9 736	27,111	3239	58·43
0 73650	60 02	6,412	4722	40 07	0·10183	8 299	29,680	3022	62 61
0 55843	45 51	8,260	4613	41 02	0 08418	6 861	32,289	2718	69 62
0 38217	31 15	11,567	4421	42 81	0 06653	5 422	35,623	2370	79·84
0 31238	25 46	13,698	4279	44 22					

C. The weight of C was deduced by comparison of the volumes with those of B at the same temperatures and pressures. The mean value was found to be

0·0035982 gramme.

(58.) T = 50° (Alcohol).

Volume	Volume of 1 gramme.	Pressure	P V	Vapour-density	Volume.	Volume of 1 gramme	Pressure	P V	Vapour-density.
c cs.	c.cs.	mms.			c cs.	c cs.	mms		
1 0887	302 6	855	930 8	38·80	0 73288	203·7	1240	909·5	39·75
1 0183	283 0	909	925 6	39·02	0 71519	198·7	1263	904 0	39·99
0 98965	263 3	974	922·6	39 15	0 70458	195·8	1272	896 9	40 30
0·87533	243 3	1051	921 1	39 25	0 69751	193 8	1273	888·6	40·68
0·80384	223·4	1139	916·4	39 45					

(59.) T = 195° (Methyl salicylate).

Volume	Volume of 1 gramme.	Pressure.	P. V.	Vapour-density.	Volume.	Volume of 1 gramme.	Pressure.	P. V.	Vapour-density.
c.cs.	c cs.	mms.			c.cs.	c.cs.	mms.		
1 1285	313·6	1224	1381	37·89	0·27723	77·05	4,718	1308	40·01
1 0580	294 0	1301	1376	38 02	0 24226	67 33	5,356	1298	40 33
0·98700	274 3	1391	1373	38·12	0 20723	57 59	6,166	1278	40 96
0.91530	254·4	1503	1376	38·04	0 17213	47 84	7,312	1269	41·58
0·73595	204·5	1858	1367	38 27	0 13697	38·07	8,972	1229	42·59
0·55800	155·1	2431	1357	38 58	0·11937	33·17	10,108	1207	43 38
0·38189	106 13	3480	1329	39 38	0·10176	28 28	11,584	1179	44·40
0·34700	96·44	3820	1319	39·68	0·08412	23·38	13,450	1131	46·26
0·31215	86·75	4219	1317	39·74	0·06648	18 48	16,096	1070	48·91

(60.) T = 222·85° (Methyl salicylate).

Volume.	Volume of 1 gramme.	Pressure.	P V.	Vapour-density.	Volume	Volume of 1 gramme.	Pressure.	P V.	Vapour-density
c.cs	c cs	mms			c cs.	c.cs.	mms.		
1 2001	333 5	1231	1477 3	37 53	1·0905	303 1	1354	1476 6	37 55
1 1627	323·7	1268	1476 9	37 55	1 0587	294·2	1393	1474 8	37 60
1 1293	313 8	1307	1476 0	37 57					

Reduction and Arrangement of Results.

I. *Vapour-pressures* —The vapour-pressures experimentally observed and calculated are given in the annexed Table, as well as those calculated by REGNAULT from his observations.

Temperature	Vapour-pressures. Observed	Calculated	REGNAULT	Temperature	Vapour-pressures. Observed	Calculated	REGNAULT
° −20		62 99	68 90	° 95	4326	4326 69	4401 81
−15	86 00	85 22	89·31	100	4855	4859 01	4953 30
−10	112 3	111 81	114 72	105	5441	5439 35	5556 23
− 5	144 8	144 69	146 08	110	6082	6070 38	6214·63
0	184 9	184 9	184 39	115	6775	6754 93	6933 26
5	233 0	233 52	230 89	120	7513	7495 73	7719 20
10	290 8	291 78	286 83	125	8313	8295 62	
15	360 0	360 93	353 62	130	9155	9157 42	
20	439 8	442 36	432 78	135	10077	10084 0	
25	534·8	537 51	525 93	140	11051	11078 2	
30*	..	647·93	634 80	145	12122	12142·9	
35†	771 8	775·25	761 20	150	13262	13281·0	
40	921 0	921 18	907 04	155	14514	14495 1	
45	1085 5	1087·53	1074·15	160	15778	15788 1	
50	1276	1276 11	1264 83	165	17201	17162 9	
55	1491	1488 97	1481 06	170	18671	18622 2	
60	1734	1728 13	1725 01	175	20189	20168 4	
65	2004	1995 71	1998 87	180	21775	21804 3	
70	2304	2293 91	2304 90	185	23623	23532 4	
75	2638	2625·04	2645 41	190	25513	25355·1	
80	2974	2991 40	3022·79	192	26331	26111 2	
85	3389	3395 46	3439·53	193	26800	26495 0	
90	3831	3839 71	3898 26				

* The results given up to 30°, as observed, were read from the curve mentioned on p 62.

† This result was calculated from the boiling-point under atmospheric pressure. The remainder are the means of actual observations.

The formula which REGNAULT employed in his extensive research on vapour-pressures was suggested by BIOT. It is

$$\log p = a + b\alpha^t + c\beta^t.$$

The concordance between the found and calculated pressures through a range of temperatures so great as from $-20°$ to $180°$ shows how well this formula interprets the facts. The constants employed were calculated from the observations at 0°, 45°, 90°, 135°, and 180°. The numbers directly read were, however, not taken; but small portions of the curve about these points were previously smoothed by means of the simpler formula

$$\log p = a + b\alpha^t.$$

The constants for the larger formula are

$$a = 5\cdot9834771, \qquad \log \alpha = \bar{1}\cdot99827459,$$
$$\log b = 0\cdot5240258, \qquad \log \beta = \bar{1}\cdot99130336;$$
$$\log c = \bar{1}\cdot5733238,$$

b and c are both negative.

The greatest difference, calculated as temperature between the found and calculated results between $-15°$ and 180°, is 0·2°; but above 180° the difference increases gradually, amounting to 0·8° at 193°; but, indeed, it is doubtful whether any formula can be expected to hold in the immediate neighbourhood of the critical point.

The vapour-pressures of ether were measured between $-20°$ and 120° by REGNAULT; though his results agree with ours at certain temperatures, yet there is, on the whole, considerable discrepancy; and, in our opinion, he himself furnishes the explanation. In only one case was the specimen of ether used by him purified from alcohol by repeated shaking with water; and this specimen appears to have been used only in determining the specific heat of the vapour. He points out that after standing much lower vapour-pressures were obtained than with freshly distilled ether; for instance, at 0° his calculated number is 184·39 mms. In the first series, in which the ether had been distilled from calcium chloride, the pressure at $-0\cdot08°$ was 181·7 mms., corresponding to 182·5 mms. at 0°. In the third series the same ether had stood for a year, and was redistilled over lime before experiment. The vapour-pressure at 0° was 181·65 mms. The same quantity of ether was again allowed to stand in sealed flasks, and on redetermining the vapour-pressure after six months it had fallen to 174·9 mms., and after three months more to 171·93 mms. REGNAULT states that the chemical composition was unaltered, and that the alteration was of a physical nature; but LIEBEN ('Deutsch. Chem. Gesell. Ber.', *Jahrg.* 4, p. 758) states that pure ether, either alone, or in contact with potash, lime, or sodium, does not change on standing; but that the presence of water, fused sodium chloride, calcium chloride, or anhydrous

copper sulphate induces a change, the liquid exhibiting the iodoform reaction. We have little doubt, therefore, that the specimen of ether used by REGNAULT contained alcohol, and it is known from Professor GUTHRIE's researches that the presence of a minute amount of an impurity has a great influence on vapour-pressures. A further argument in support of this view will be given when we consider the heats of vaporisation. It should here be pointed out, moreover, that REGNAULT's observations below 0° agree nearly as well with our formula as with his own, whereas our observations are in very close accordance with our calculated results.

II. *Compressibilities.*—The isothermals of the liquid state, showing decrease in volume with increase of pressure, were, for the purpose of smoothing, plotted on a sheet of curve-paper, and isobars were drawn representing the relations of volume to temperature. The discrepancies of individual observations were thus eliminated, and from the isobars the numbers were retransferred to the isotherms. It will be seen on inspection of the curves (Plate 6) that the smoothed curves agree well with the observations. The Table which follows shows these results, and includes some of the isotherms corresponding to the gaseous state. The volumes are those of 1 gramme.

Pressure.	Temperature.															
mms.	50°	100°	130°	150°	175°	185°	190°	192°	193°	193 8°	195°	197°	200°	205°	210°	223°
1,000	256·4															
2,000	1·4770	..	..	..	..	..	..	..	.	..	189·6		..	..	.	203 4
4,000	1·4705	67 58	75 72	81·05	87 35	89 86		..	..	.	92·17	..	..	.	.	99 45
6,000	1·4690	1·636	47·55	51 49	56·12	57 93	59 07		.	.	59 86		.			64 55
8,000	1 4675	1·632	33·06	36 52	40 47	41 91	42 72	..	.	..	43 39	.		..		47 23
10,000	1·4660	1 6305	..	27·42	30 94	32 31	32 93	..	..	33·35	33·51		..			36 75
12,000	1 4645	1·629	..	21 17	24 53	25 76	26·33	26 50	..	26 72	26 90		.	..	..	29 84
14,000	1·4635	1·627	..	1 9265	19 87	20 96	21·57	21·73	21 82	21 93	22 08		..	..		24 77
16,000	1·4625	1 6235	..	1·912	16 21	17·39	17 93	18 08	18 21	18 30	18 40		..	.	19 88	21 01
18,000	1·4615	1 621	..	1·9045	13 11	14·47	15 03	15·21	15 33	15·38	15 54	15 76	16 01	16 51	16 95	18 05
20,000	1 4610	1 620	..	1 8995	10 32	11 94	12 58	12·70	12 93	13 01	13 15	13 38	13 62	14 13	14 57	15 68
22,000	1·4605	1 618	..	1·893	2 202	9·59	10·40	10 68	10 79	10 90	11 04	11 26	11 62	12 11	12 58	13 69
24,000	1 4600	1 616	..	..	2·169	2·459	8 327	8 664	8 854	8 962	9 175	9 443	9 808	10 35	10 85	11 96
26,000	1·4600	1 613	..	..	2 140	2 371	2 652	6 252	6·647	6 889	7 223	7 638	8 135	8·804	9 327	10 49
28,000	1·4595	1 6105	..	..	2·119	2 317	2·498	2 611	..	2 750	3·020	5 350	6 367	7 283	7 954	9 198
30,000	1·4595	1 6095	..	..	2·099	2 270	2 411	2 490	..	2·570	2 632	2 790	3 830	5 698	6 627	8 060
32,000	1·4590	1 608	..	..	2 080	2 233	2·350	2 410	..	2 469	2·512	2 604	2·848	3 860	5 241	7 000
34,000	1 4580	1 605	..	..	2 062	2 201	2 300	2 350	..	2 400	2 440	2 509	2 659	3 040	4 073	5 990
36,000	1·4575	1·602	..	..	2 049	2 179	2·264	2 308	.	2·349	2 381	2 440	2 549	2 769	3 295	5 070
38,000	1·4570	1 600	..	..	2 036	2·157	2 237	2 276	..	2 310	2 340	2 390	2 470	2 639	2 929	4 300
40,000	1·4565	1 599	..	..	2·024	2·139	2 211	2·248	..	2·279	2 302	2 343	2 409	2 550	2 760	2 720

Above the critical point, 193·8°, the compressibility of the substance has been given, where its condition may be assumed to approximate to that of a liquid.

Isothermals at even pressures for 50° *and* 100°.—As the limits of pressure are so small, it has been thought advisable to give these isothermals separately, in order to save room.

Temperature	Pressure.	Volume of 1 gramme	Temperature.	Pressure	Volume of 1 gramme	Temperature	Pressure.	Volume of 1 gramme
°	mms.	c.cs.	°	mms.	c cs.	°	mms.	c cs.
50	900	286 3	100	2800	101 1	100	4000	67·58
	1000	256 4		3000	93 67		4200	63 83
	1100	231 7		3200	87 18		4400	60 41
	1200	211 0		3400	81 45		4600	57 26
	1276	196 9		3600	76 32		4800	54 32
				3800	71 74		4859	53 50

III. From these results the curve, which we propose to call the orthobaric curve, was constructed. It represents the relations between the volumes of a gramme of liquid and temperatures, at pressures equal to the vapour-pressures. These were obtained by direct reading, sometimes, however, at pressures slightly higher than the vapour-pressures; and also from the points of intersection of the curves representing compressibility, with the horizontal lines, indicating vapour-pressure. The numbers read from the smoothed curve, representing the latter, and also the corresponding specific gravities, are given in the following Table :—

Temperature	Volume	Specific gravity	Temperature	Volume	Specific gravity	Temperature	Volume.	Specific gravity.
°			°			°		
0	1 3583	0 7362	95	1·617	0 6184	155	1·976	0 5061
40	1 4505	0 6894	100	1 638	0 6105	160	2 021	0 4947
45	1 4650	0·6826	105	1 660	0 6024	165	2 027	0 4817
50	1 4785	0 6764	110	1 684	0 5942	170	2·147	0 4658
55	1·4900	0 6711	115	1 708	0 5855	175	2 238	0·4468
60	1 5020	0 6658	120	1 735	0·5764	180	2 343	0 4268
65	1 5175	0 6590	125	1·763	0 5672	185	2 489	0 4018
70	1·531	0·6532	130	1 792	0 5580	190	2 730	0 3663
75	1 550	0 6452	135	1 823	0 5485	192	2 900	0 3448
80	1·562	0 6402	140	1 857	0 5385	193	3 030	0 3300
85	1 580	0 6329	145	1 893	0 5283			
90	1 600	0 6250	150	1 931	0 5179			

Orthobaric Volumes of 1 *gramme of Vapour*.—The following Table gives the volumes of 1 gramme of the saturated vapour at even temperatures, with the corresponding specific gravities and vapour-densities.

Temperature	Volume of 1 gramme	Specific gravity.	Vapour-density.	Temperature.	Volume of 1 gramme.	Specific gravity.	Vapour-density.
°	c cs			°	c cs.		
0	1209 1	0 000827	37 95	105	47 62	0 02100	45·35
5	973 6	0 001027	38 00	110	42 57	0 02349	46 05
10	791 1	0 001264	38 10	115	38 02	0·02630	46·95
15	646 6	0 001547	38 35	120	34 09	0 02934	47 8
20	534 7	0 001870	38 5	125	31 30	0·03195	48 75
25	436 2	0 002293	38 6	130	27 49	0 03638	49 75
30	373 6	0 002677	38 9	135	24 73	0 04044	50 85
35	316 2	0 003163	39 05	140	22 28	0 04488	52 0
40	268 0	0 003731	39 4	145	20·03	0 04992	53 4
45	229 5	0 004358	39 6	150	18 01	0·05551	54 95
50	196 9	0·005079	39·95	155	16 18	0 06179	56 7
55	170 3	0·005886	40·2	160	14 47	0·06911	58 9
60	147 7	0·006771	40 5	165	12·90	0·07754	61·5
65	128 4	0·007790	41 0	170	11·45	0·08731	64 55
70	112 1	0 008920	41·45	175	10·12	0 09879	68·2
75	98 33	0 01017	41 9	180	8·815	0·1135	73·25
80	86 60	0·01155	42 35	185	7·579	0 1320	79·75
85	76·56	0·01306	42·8	190	6·172	0·1620	91·45
90	67 70	0 01477	43·4	192	5·476	0·1826	100 2
95	60·14	0·01663	43 95	193	4·970	0 2012	108·7
100	53 55	0 01867	44 55				

The curves showing the relations given in the preceding Tables between specific gravity and temperature of liquid and vapour are represented in Plate 10, and the volumes of 1 gramme of liquid and vapour are shown in Plate 9, but mapped against pressure.

DENSITIES of Unsaturated Vapour at even Pressures.—(H = 1.)

Pressures.	Temperatures.															
	50°	100°	130°	150°	175°	185°	190°	192°	193°	193° 8	195°	197°	200°	205°	210°	223°
900	38 96															
1,000	39·15															
1,100	39 39															
1,200	39 65															
2,000											38 35					37 9
2,800		40 95														
3,000		41 25														
3,200		41 55														
3,400		41 86														
3,600		42 19														
3,800		42·52														
4,000		42 88	41 35	40 55	39 85	39 6					39 45					38 75
4,200		43 24														
4,400		43 61														
4,600		44 01														
4,800		44 46														
6,000			43 9	42 55	41·35	40 95	40 6				40 50					39 8
8,000			47 35	45·00	43·0	42 45	42 1				41 9					40 8
10,000				47 95	45 0	44 05	43·7			43 5	43 4					41·95
12,000				51 75	47 3	46·05	45 55	45 45		45 25	45 05					43 05
14,000					50 05	48 5	47 65	47·5	47 4	47 25	47 05					44 45
16,000					53·7	51 15	50 15	49 95	49 7	49·55	49 4				47 2	45 85
18,000					59 0	54·65	53 2	52 8	52 5	52 40	52 0	51 5	51 0	50 0	49·2	47 45
20,000					67 45	59·6	57 2	56 9	56 0	55 75	55 3	54 6	53 95	52·55	51 5	49 15
22,000						67 1	62 9	61 5	61 0	60 50	59 9	58 95	57·5	55 75	54 25	51 2
24,000							72 0	69 5	68 15	67 45	66 05	64 45	62 45	59·8	57 6	53 6
26,000								88 9	83 8	81 00	77 45	73 55	69 5	64 9	61 9	56·5
28,000												97 5	82 45	72·85	67 4	59 85
30,000														86 9	75 5	63 9
32,000															89 5	68 65
34,000															108·4	74 55
36,000																82 5
38,000																94 4
40,000																104 2
42,000																110 5

These results are graphically shown on Plate 7. The curves were smoothed by constructing others showing the relations between temperature and pressure at equal vapour-densities, and then transferring back to the original sheet of curve paper. It can be judged by the position of the circles how nearly the observations agree with the smoothed curves.

IV. *Heats of Vaporisation.*—From the thermodynamic equation

$$\frac{L}{s_1-s_2}=\frac{dp}{dt}\frac{t}{J},$$

the heats of vaporisation at definite intervals of temperature were calculated. The values of the expression dp/dt were calculated in the following manner. By means of the formula $\log p = a + b\alpha^t + c\beta^t$, the vapour-pressures at one-tenth of a degree above and below the definite temperature were calculated, and the difference was multiplied by 5 to obtain the values for 1°. This method gives results probably as nearly correct as it is possible to obtain. The pressures were reduced to grammes per square centimetre, and the value of J was taken as 42,500.

Temperature.		$\frac{dp}{dt}$	$\frac{dp}{dt}$	$\frac{dp}{dt}\frac{t}{J}$	s_1-s_2	L
C.	Abs.	in mms.	in grammes.			
0°	273°	8·843	12·023	0·07723	1207·7	93·27
10	283	12·695	17·26	0·11493	789·8	90·77
20	293	17·585	23·91	0·16483	533·3	87·90
30	303	23·720	32·25	0·22992	372·2	85·60
40	313	31·160	42·37	0·31200	266·6	83·18
50	323	40·095	54·51	0·41430	195·4	80·95
60	333	50·620	68·82	0·53924	146·2	78·84
70	343	62·840	85·44	0·68932	110·58	76·42
80	353	77·005	104·70	0·86958	85·04	73·95
90	363	93·010	123·46	1·0801	66·10	71·39
100	373	110·48	150·21	1·3183	51·85	68·35
110	383	131·48	178·76	1·6113	40·95	65·98
120	393	153·95	209·3	1·9355	32·36	62·63
130	403	178·61	242·8	2·3027	25·67	59·11
140	413	205·8	279·8	2·7191	20·42	55·52
150	423	234·9	319·4	3·1786	16·10	51·18
160	433	266·65	362·5	3·6936	12·45	45·99
170	443	300·4	408·4	4·2572	9·324	39·69
180	453	336·4	457·4	4·8751	6·478	31·58
185	458	363·0	493·5	5·3186	5·035	26·78
190	463	407·0	553·4	6·0282	3·467	20·90
192	465	446·0	606·4	6·6344	2·578	17·10
193	466	472·0	641·7	7·0864	1·942	13·67

The heats of vaporisation have been determined by other observers; BRIX ('LIEBIG'S Annalen,' vol. 44, 1842, p. 169) gives determinations of the heats of vaporisation of water, alcohol, and ether. Translating Réaumur into Centigrade degrees, that of water becomes 539·6 calories; of alcohol, 214·25 calories; and of ether, 89·96 calories. Determinations were next made by ANDREWS ('Chem. Soc. Journ.,' vol. I, 1849, p. 27), who found 90·5 calories. The sample of ether he employed

boiled at 34·9° at 752 mms. pressure FAVRE and SILBERMANN found 91 11 calories ('Annales de Chimie,' vol 37, 1853, p. 465). REGNAULT ('Mémoires de l'Académie,' vol 26, p. 881) gives a formula for calculating the total heat of vaporisation from 0°; it is

$$\lambda = a + bt + ct^2,$$

where $a = 94$, $b = 0{\cdot}45$, and $c = -0\ 00055556$.

From this formula the heat of vaporisation at 0° = 94 calories, but for higher temperatures the specific heat of ether is required, for calculating which he gives the formula

$$Q = at + bt^2,$$

where Q is the total quantity of heat required to raise 1 grm. of ether from 0° to t; $\log a = \bar{1}{\cdot}7234538$; and $\log b = \bar{4}{\cdot}4711026$. Four experiments were made, of which the results of only three were employed in calculating the constants, although the fourth experiment was moderately concordant with the others, the range was only from −30° to +32°. Taking into consideration the small number of experiments, and the not very close agreement between the result of the second experiment and the value calculated from the formula (calculated, Q = 15 821; observed, Q = 15 930), it is doubtful whether these constants would hold good for temperatures much higher than 35°. REGNAULT made two series of experiments, of which there were seven observations in the first and four in the second, on the heats of vaporisation of ether. The ether employed in the first series was purified by the ordinary methods, and distilled from time to time with lime to remove acids and water. In the second series, including experiments at very low pressures, the ether was purified "with the greatest care," and kept in a stoppered flask. Nevertheless, on distillation, a quantity of less volatile liquid remained behind, having, as he states, the percentage composition of ether This modification, according to him, is absent from ether recently distilled, but forms after some months (see remarks on p. 82). In the second series the weight of the residual liquid in the calorimeter was always subtracted from the total weight, the liquid having been distilled from the calorimeter. The following Table gives the individual results of REGNAULT's experiments, and also the total heats at the same temperatures, calculated by means of his formula. The first four experiments were made with the carefully purified ether.

Temperature	Total heat.		Temperature.	Total heat.	
	Observed.	Calculated.		Observed	Calculated.
− 3°7	92 235	92 343	34°83	109 117	109·0
+ 7 51	95 370	97 35	90 05	128 900	..
12·9	97 282	99·72	93 85	130 880	..
15 5	98·801	100 84	108 80	138 196	136 38
17·15	101·278	101 56	120 90	140 781	.
21 95	104 366	..			

From some of these the heats of vaporisation were calculated by help of the formula given for calculating specific heats.

HEATS of Vaporisation.

Temperature	Observed.	Calculated	R and Y	Temperature	Observed	Calculated.	R. and Y.
− 3°7	94 188	94 296	94 4	17°15	92·12	92 40	88 8
+ 7·51	91 38	93 36	91 3	34 83	90 333	90·21	84 5
12 9	90 41	92 85	89·9	120 9	72 49	..	62·5
15 5	90 53	92 57	89 25				

It is noticeable that the agreement between REGNAULT's observed and calculated numbers is much less good at low than at high temperatures, with the exception of the first at −3·7°; and, as the four first determinations were made with the purest sample, more stress has been laid by REGNAULT on his observations with the less pure than with the purer ether.

The heats of vaporisation thus calculated are widely different from the results obtained by us, with the single exception of that at the lowest temperature; but it is also remarkable that the individual experiments with the purer substance exhibit much closer concordance with our results.

As the results about the temperature 35° by all observers exhibit fairly close agreement with the observations of REGNAULT, but differ widely from our calculated values, it appeared desirable to submit them to proof by translating REGNAULT's results into vapour-densities; and for this purpose the values of dp/dt were calculated by means of BIOT's formula, using REGNAULT's constants. By thus doing, the work is entirely REGNAULT's. The results are given in the following Table:—

Temperature.	$\frac{dp}{dt}$	$\frac{dp}{dt}$	$\frac{dp}{dt}\frac{t}{J}$	L	s_1-s_2	s_1	Vapour-density.
°	mms.	grammes					
0	8·44	11 47	0 07388	94 0	1272·4	1273·8	36 12
10	12·22	16·615	0·11063	93·12	841·7	843 1	36·37
20	17 175	23 35	0 16099	92·08	572 0	573 4	36 69
30	23 47	31·91	0·22749	90 86	399 4	400 8	37·01
35	27 155	36 92	0 26756	90·18	377·05	338 5	37·15

As the minimum value of the vapour-density of ether is 37, the first three results are impossible; and, on consulting the Table on p. 64, giving the results of our measurements at 12·9°, and Table 58, p. 80, it will be seen that the density of the saturated vapour rises to 38·25 at 12·9°, and 39·95 at 50°. At 35° the vapour-density read from the curve is 39·05.

Our results also receive confirmation from a number of experiments by HORSTMANN ('LIEBIG's Annalen,' Suppl. 6, 1868, p 63), which, although not very concordant with each other, yet amply suffice to prove that the vapour-density is not constant, and that, therefore, $p\,v.$ is variable.

It follows from the Table already given that, if REGNAULT's results are correct, the thermo-dynamical formula does not always hold; the same discordance was noticed in his observations with alcohol.

It is possible, assuming the impurity in REGNAULT's ether to have been alcohol, which is not unlikely, inasmuch as no mention is made of the sample of ether having been purified by washing with water, to calculate the percentage which must have been present in order to raise the heat of vaporisation from 84·5, calculated by us, to 90·2, calculated from REGNAULT's formulæ; it is 4·4 per cent, and an analysis of such a mixture would give 64·3 per cent. of carbon, instead of 64 86 per cent. contained in pure ether. This does not, however, account for the composition of the high-boiling residue, which, if alcohol, should have contained 52·17 per cent of carbon.

From these experiments it is noticeable that with ether, although the density of the saturated vapour is very abnormal, even more so than with alcohol, yet there is no tendency towards a rise with decrease of temperature It therefore seems probable that, for the same reasons which were stated in the memoir on alcohol, combination of gaseous molecules to form complex molecules does not take place.

It is impossible to state accurately the temperature, pressure, and volume of any substance at the critical point; but the following numbers may be regarded as closely approximate for ether.—

Temperature . .	194°.
Pressure . . .	27,060 mms. =35·61 atmospheres.
Volume . .	probably 4 06 cub. centims. for 1 gramme.

APPENDIX

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Since the foregoing memoir was read, it appeared to us of importance, in consequence of some theoretical deductions, a short account of which has been communicated to the Society by Professor STOKES, to make fresh determinations of the relations between volume, temperature, and pressure of ether at higher temperatures and greater pressures than we had formerly employed.

For these experiments a fresh stock of ether was prepared; the volume tube was new; and the air-gauges were refilled. The weight was not determined directly, but was ascertained by comparison with our previous results at 175°, 185°, and 195°. It was 0·055406 gramme.

The temperature 175° was maintained by jacketing the experimental tube with aniline; methyl salicylate was used for the temperatures 185°, 195°, and 220°; and bromonaphthalene for 250° and 280°. It should be mentioned that fresh samples of aniline and methyl salicylate, carefully fractionated from impurities, were employed.

TEMPERATURE, 175°.

Volume of 1 gramme	Pressure.	Volume of 1 gramme	Pressure	Volume of 1 gramme	Pressure.
c cs.	mms.	c cs.	mms.	c cs.	mms
22 608	12,729	17 199	15,392	13 154	17,994
21 254	13,304	16 524	15,822	11 821	18,943
19 901	13,960	15·847	16,207	10 493	19,930
18 552	14,673	14 496	17,163		

Vapour-pressure. P = 20,180 ; 20,271 ; 20,284 ; 20,277 ; 20,321. Mean, 20,271.

TEMPERATURE, 185°.

Volume of 1 gramme	Pressure	Volume of 1 gramme.	Pressure.	Volume of 1 gramme	Pressure
c cs	mms.	c.cs.	mms	c cs.	mms.
22 615	13,248	13 158	18,988	8·530	22,956
19·907	14,549	11 825	20,115	7·880	23,451
17 204	16,061	10 496	21,346		
14 500	17,948	9 181	22,437		

Vapour-pressure. P = 23,750 ; 23,760 ; 23,774 ; 23,770. Mean, 23,763.

The mean of previous determinations of vapour-pressure at 175° is 20,189 ; and at 185° 23,623. Considering that the samples of ether, aniline, and methyl salicylate were different, and that the gauges were refilled, the agreement is satisfactory.

TEMPERATURE, 195°.

Volume of 1 gramme.	Pressure.	Volume of 1 gramme	Pressure.	Volume of 1 gramme.	Pressure.
c.cs.	mms.	c cs.	mms.	c cs.	mms.
22·620	13,723	10 498	22,503	5·308	27,373
19·911	15,102	9·183	23,904	4·665	27,599
17·207	16,752	7 881	25,325	4·025	27,704
14 503	18,793	7·237	25,988	3·386	27,735
13·161	19,931	6 593	26,529	2 748	28,846
11·827	21,169	5 950	27,039		

TEMPERATURE, 220°.

Volume of 1 gramme	Pressure.	Volume of 1 gramme	Pressure.	Volume of 1 gramme	Pressure
c cs.	mms.	c cs.	mms.	c.cs.	mms.
22 635	14,886	9 189	27,584	4 028	37,740
19 925	16,451	7·887	29,708	3 389	40,278
17 219	18,357	6 598	32,123	3 069	42,630
14 513	20,772	5 954	33,325	2 749	46,921
11 836	23,763	5·311	34,715	2 622	50,342
10 505	25,542	4 669	36,201	2 558	52,753

TEMPERATURE, 250°. Pressure of bromonaphthalene vapour, 386·35 mms.

Volume of 1 gramme.	Pressure	Volume of 1 gramme	Pressure	Volume of 1 gramme.	Pressure.
c.cs.	mms.	c.cs.	mms.	c cs.	mms.
22 651	16,223	10 513	29,074	5·315	43,259
19 938	18,058	9 195	31,741	4 672	46,155
17 231	20,269	7 892	34,815	4 351	47,884
14 523	23,110	6·602	38,513	4 031	50,129
11 844	26,789	5 958	40,664	3 711	52,599

TEMPERATURE, 280·35°. Pressure of bromonaphthalene vapour, 758·2 mms.

Volume of 1 gramme.	Pressure.	Volume of 1 gramme.	Pressure.	Volume of 1 gramme	Pressure.
c.cs.	mms.	c.cs.	mms.	c.cs.	mms.
22 669	17,806	10 521	33,349	5 963	49,552
19 954	19,881	9·203	36,777	5·642	51,598
17 245	22,499	7 899	41,084	5·319	53,841
14 535	25,834	7 253	43,537		
11 853	30,430	6 607	46,552		

At 175° and at 280° the readings of pressure were double, one set being made with rising and the other with falling pressures; the means are given.

IV. *On the Discrimination of Maxima and Minima Solutions in the Calculus of Variations*

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THE criteria for distinguishing between the maximum and minimum values of integrals have been investigated by many eminent mathematicians.* In 1786 LEGENDRE gave an imperfect discussion for the case where the function to be made a maximum is $\int f(x, y, dy/dx)\,dx$. Nothing further seems to have been done till 1797, when LAGRANGE pointed out, in his 'Théorie des Fonctions Analytiques,' published in 1797, that LEGENDRE had supplied no means of showing that the operations required for his process were not invalid through some of the multipliers becoming zero or infinite, and he gives an example to show that LEGENDRE's criterion, though necessary, was not sufficient. In 1806 BRUNACCI,† an Italian mathematician, gave an investigation which has the important advantage of being short, easily comprehensible, and perfectly general in character, but which is open to the same objection as that brought against LEGENDRE's method † The next advance was made in 1836

* This sketch is founded on TODHUNTER's valuable 'History of the Calculus of Variations'

† BRUNACCI's method may be explained as follows. Let

$$U = \iint\left(x,\ y,\ z,\ \frac{dz}{dx},\ \frac{dz}{dy}\right)dx\,dy$$

be the function to be made a maximum, and let us denote dz/dx and dz/dy by p and q respectively. Thus, the limits of z being supposed fixed,

$$\delta U = \iint\left(\frac{df}{dz}\delta z + \frac{df}{dp}\delta p + \frac{df}{dq}\delta q\right)dx\,dy$$
$$+ \tfrac{1}{2}\iint\left(\frac{d^2f}{dz^2}\delta z^2 + 2\frac{d^2f}{dz\,dp}\delta z\,\delta p + 2\frac{d^2f}{dz\,\delta q} + \frac{d^2f}{dp^2}\delta p^2 + 2\frac{d^2f}{dp\,dq}\delta p\,\delta q + \frac{d^2f}{dq^2}\delta q^2\right)dx\,dy. \quad . \quad . \quad (1)$$

The first integral must be made to vanish by a relation between z, x, and y. Eliminate z for the coefficients in the second integral of (1) by means of this relation, and let it be written—

$$\iint(A\,\delta z^2 + 2B\,\delta z\,\delta p + 2C\,\delta z\,\delta q + P\,\delta p^2 + 2R\,\delta p\,\delta q + Q\,\delta q^2)\,dx\,dy. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, remark that, the limits being fixed, $\int \delta z^2 (\alpha\,dx + \beta\,dy)$ vanishes, whatever α and β may be, for

by the illustrious JACOBI, who treats only of functions containing one dependent and one independent variable. JACOBI says (TODHUNTER, Art. 219, p. 243): "I have succeeded in supplying a great deficiency in the Calculus of Variations. In problems on maxima and minima which depend on this calculus no general rule is known for deciding whether a solution really gives a maximum or a minimum, or neither. It has, indeed, been shown that the question amounts to determining whether the integrals of a certain system of differential equations remain finite throughout the limits of the integral which is to have a maximum or a minimum value. But the integrals of these differential equations were not known, nor had any other method been discovered for ascertaining whether they remain finite throughout the required interval. I have, however, discovered that these integrals can be immediately obtained when we have integrated the differential equations which must be satisfied in order that the first variation may vanish."

JACOBI then proceeds to state the result of his transformation for the cases where the function to be integrated contains x, y, dy/dx, and x, y, dy/dx, d^2y/dx^2, and in this solution the analysis appears free from all objection, though, where he proceeds to consider the general case, the investigation does not appear to be quite satisfactory in form, inasmuch as higher and higher differential coefficients of δy are successively introduced into the discussion (see Art. 5). JACOBI'S analysis is much more complicated than BRUNACCI'S, its advantage being that the coefficients used in the transformation could be easily determined; hence it supplied the means of ascertaining whether they became infinite or not.

$\delta z = 0$ all along this curve, and it may, therefore, be added to the integral (2) without altering its value Now,

$$\int \delta z^2 (\alpha\, dx + \beta\, dy) = 0 = \iint \left(\frac{d}{dx}(\beta\, \delta z^2) + \frac{d}{dy}(\alpha \delta z^2)\right) dx\, dy$$
$$= \iint \left(\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right)\delta z^2 + 2\beta\, \delta z\, \delta p + 2\alpha\, \delta z\, \delta q\right) dx\, dy.$$

Adding this quantity to the integral, we get for the quantity under the integral sign in (2) the expression

$$\left\{\left(\mathrm{A} + \frac{d\alpha}{dy} + \frac{d\beta}{dx}\right)\delta z^2 + 2\,(\mathrm{B} + \beta)\,\delta z\,\delta p + 2\,(\mathrm{C} + \alpha)\,\delta z\,\delta q + \mathrm{P}\,\delta p^2 + 2\mathrm{R}\,\delta p\,\delta q + \mathrm{Q}\,\delta q^2\right\} dx\, dy.$$

This expression cannot change sign if $\mathrm{PQ} - \mathrm{R}^2$ is positive, and

$$(\mathrm{PQ} - \mathrm{R}^2)\left(\mathrm{P}\left(\mathrm{A} + \frac{d\alpha}{dy} + \frac{d\beta}{dx}\right) - (\mathrm{B} + \beta)^2\right) - (\mathrm{P}\,(\mathrm{C} + \alpha) - (\mathrm{B} + \beta)\,\mathrm{R})^2 > 0. \quad . \quad . \quad . \quad (3)$$

We can determine α and β so that (3) shall be true, and hence, if $\mathrm{PQ} - \mathrm{R}^2$ be positive, the second variation will be invariable in sign.

The objection to this method is that there is no means of ascertaining whether α and β remain finite or not. If the region of integration be small, they can always be determined so as to satisfy (3). But in general they become infinite when the integration extends over a large area. Thus, to complete the solution, it is necessary to have some means of finding within what range of integration the criteria are sufficient. It was because BRUNACCI'S method did not easily lend itself to the discussion of this problem that JACOBI devised his far more intricate method.

JACOBI did not himself give a detailed account of his process, but said that "the analysis just indicated requires a good knowledge of the Integral Calculus." Various demonstrations were subsequently given by different mathematicians. That of DELAUNAY has been adopted by JELLETT and other English writers.

In 1852 MAINARDI devised a method somewhat similar to that of BRUNACCI, but he endeavoured to remedy the omission in the latter by showing how to determine the coefficients used in the transformation, the equations for determining these coefficients being supplied by JACOBI's reasoning But in this he was not successful, even in some of the simple cases which he discussed, and in the more complicated cases the equations appear to be quite unmanageable.

In 1853 EISENLOHR extended JACOBI's method to double integrals. In 1854 a memoir by SPITZER was published which seems to have been more complete than MAINARDI's; but the most important advances were made a few years later when the Theory of Determinants was applied by HESSE and by CLEBSCH to simplify and extend JACOBI's methods.

From the foregoing sketch it will be seen that as early as 1806 the criteria had been correctly given and simply proved, with the exception of one point, namely, that there was no means of ascertaining for what range of integration the criteria ceased to be sufficient. JACOBI endeavoured, by the help of a complicated analysis, to remedy this defect; and, although all the efforts of later mathematicians have been directed to the extending or simplifying of JACOBI's method, the analysis is still very complicated, and requires an intimate acquaintance with other branches of mathematics.

All these methods, however, are open to the objection stated in Art. 5, and, furthermore, it appears to me, for the reasons briefly indicated in Art. 12, that, although the results arrived at by these mathematicians are undoubtedly correct, it would be impossible to give a *strict* proof of them by any method based on transformations. However this may be, I cannot find that anyone has yet given a proof of them. JACOBI merely states the limits within which the criteria hold.

The chief object of the present paper is to show that a rigorous discussion of the discriminating conditions can be given without introducing any analytical transformations whatever, the results being obtained by reasoning from the fundamental conceptions of the Calculus of Variations. In the *first* Part of the paper will, however, be found an analytical method leading to JACOBI's transformation, but free from any serious difficulty. It is inserted chiefly on account of the historic interest of the problem. I had extended this method to obtain the criteria for the case of any integral whatever before I was aware that the results were not altogether new. It was after finding the limits up to which the criteria were sufficient that I was led to the general method given in Part II.

For convenience, a summary of contents is given below; the numbers refer to the

articles. Those who desire to read only the general method will find Part II. complete in itself

It was originally intended that a tolerably complete account of the treatment of the problem, when the limits were not all fixed, should be inserted, but the length to which the paper has extended seems to render this inexpedient.

PART I.

Algebraic Transformations of the Second Variation.

1. In the following pages the word "fluxion" will be used instead of the long expression "differential coefficient." When there is but one independent variable, x, the successive fluxions of the dependent variables will be expressed in the known notation

$$\frac{dy}{dx} = \dot{y}, \qquad \frac{d^2y}{dx^2} = \ddot{y}, \quad \&c., \quad \&c., \quad \frac{d^ny}{dx^n} = y^{(n)}, \qquad \ldots \quad (1)$$

and the partial fluxions of a function (f) of these quantities with respect to any of them, say $y^{(n)}$, will be denoted by $df/dy^{(n)}$. It will often be convenient to use a bracket $[\ ]_0^1$ to denote the result of subtracting the value of the quantity in the bracket, when taken at the lower limit of integration, from its value for the upper limit, thus by $[y\,\delta y]_0^1$ is meant $y_1\,\delta y_1 - y_0\,\delta y_0$ In other cases, where it is unnecessary to write out the limiting terms, the letter L will be used as an abbreviation for the *expression* "terms depending on the limits only," as in the following equations :—

$$\int y\,dx = \mathrm{L} - \int x\dot{y}\,dx = \mathrm{L} + \tfrac{1}{2}\int x^2\ddot{y}\,dx, \qquad \ldots \quad (2)$$

where, though the letter L is the same in both equations, it does not necessarily denote the same quantity.

2. To obtain a clear insight into the nature of the problem before us, let us examine it in the most simple and familiar case, that in which there is but one dependent and one independent variable. Writing

$$\mathrm{U} = \int f(x, y, \dot{y}, \ddot{y} \ldots y^{(n)})\,dx, \qquad \ldots \quad (3)$$

let us call V.U the *total* variation of U due to a change of form in y, by which it becomes $y + \delta y$ (y being always a function of x):

$$\left.\begin{aligned}\mathrm{V.U} = &\int\left(\frac{df}{dy}\,\delta y + \frac{df}{d\dot{y}}\,\delta\dot{y} + \&c. + \frac{df}{dy^{(n)}}\,\delta y^{(n)}\right)dx\\ &+ \frac{1}{1\cdot 2}\int\left(\frac{d^2f}{dy^2}\,\delta y^2 + 2\,\frac{d^2f}{dy\,d\dot{y}}\,\delta y\,\delta\dot{y} + \&c. + \frac{d^2f}{dy^{(n)2}}\,\delta y^{(n)2}\right)dx\\ &+ \frac{1}{1\cdot 2\cdot 3}\int\left(\frac{d^3f}{dy^3}\,\delta y^3 + \&c.\right)dx + \&c.\end{aligned}\right\}, \quad \ldots \quad (4)$$

which we may write

$$\mathrm{V\,U} = \delta\mathrm{U} + \tfrac{1}{2}\,\delta^2\mathrm{U} + \frac{1}{1\cdot 2\cdot 3}\,\delta^3\mathrm{U} + \&c.,$$

where δU is the part of V. U depending on the first powers of δy and its fluxions, δ^2U that depending on their second powers, δ^3U on their third, and so on. As usual, these will be termed the first, second, and third variations of U, and so on.

3. The form of U having been determined so that the value of δU vanishes independently of the value of δy, the terms which we have to examine are those of δ^2U. This quantity, as obtained in the first instance, is explicitly a function of x, y, and δy. But, as the value of y obtained by making δU vanish is known in terms of x, δ^2U can be expressed in terms of x and δy. When so expressed, δ^2U may either remain of the same sign, whatever function δy may be of x, or else it may have one sign

when δy is a suitably chosen function of x, and the opposite sign when another expression is taken for δy. In the former case the integral must be either a maximum or a minimum; a maximum if the sign of $\delta^2 U$ is constantly negative, and a minimum if it is positive. In the latter case the integral is neither a maximum nor a minimum; its only characteristic is that the first variation vanishes. It will save much useless verbiage if we use a single word to express this latter class of integral, and the word "anticlastic," borrowed from geometry, seems a suitable one. It has also the advantage of suggesting another term, "synclastic," for those functions which give either a maximum or a minimum. With this explanation the problem before us is to ascertain whether the integral U is synclastic or anticlastic. If it be the former, a glance will enable us to determine which the result is, a maximum or a minimum.

4. We might reduce, by a real linear algebraic transformation, the part of $\delta^2 U$ under the integral sign to the sum of n squares, and say that, if their coefficients be positive all through the integration, then, whatever be the limiting values of the arbitrary variations, the second variation must be essentially positive (at least unless dx changes sign during the integration); but the converse of this is not true, for it does not follow that, if the coefficients of the squares have different signs, it is possible to make the second variation change sign. If $\delta y, \delta \dot{y} \ldots \delta y^{(n)}$ were all independent, this converse would be true; hence one element in the problem before us is to introduce, in a suitable form, the interdependence of the quantities $\delta y, \delta \dot{y} \ldots \delta y^{(n)}$. Again, if the limits are not all arbitrary, there will be further limitations to the range of values taken by δy and its fluxions; and the second element in the problem is to find how these limitations to δy and its fluxions affect the sign of the second variation.

5. JACOBI's method appears to me open to the serious objection that it is necessary to its validity that the first $(2n-1)$ fluxion of δy should be continuous; so that the discussion only proves that a curve AB fulfilling the synclastic condition gives a better result than any infinitely near curve fulfilling the same limiting conditions, *and which is continuous to the* $2n^{th}$ *fluxion of* y. And it would not show that it was not possible to find other broken curves fulfilling the same limiting conditions, and giving, at our pleasure, a value to the integral either greater or less than that given by the curve AB. Thus, for instance, in the case of least action, it would not show that the action in the free trajectory was less than in *any* constrained path (which it is, in fact), but only that it was less than that in any path for which the tangent had but one position at every point. To show for the general case that JACOBI's proof assumes

this continuity of δy would require a large amount of work, but it will be sufficient for our purpose to take the simple case where the function to be made synclastic contains only x, y, and $\dot{y}$.

Let us apply his *general* method to the integral *

$$\mathrm{U} = \int_{x_0}^{x_1} f(x, y, \dot{y})\, dx;$$

therefore

$$\delta\mathrm{U} = \int_{x_0}^{x_1} \left(\frac{df}{dy}\delta y + \frac{df}{d\dot{y}}\delta\dot{y}\right) dx;$$

integrating by parts,

$$\delta\mathrm{U} = \left[\frac{df}{d\dot{y}}\delta y\right]_0^1 + \int \left(\frac{df}{dy} - \frac{d}{dx}\frac{df}{d\dot{y}}\right)\delta y\, dx.$$

Hence

$$\delta^2\mathrm{U} = \left[\frac{d^2f}{dy\, d\dot{y}}\delta y^2 + \frac{d^2f}{d\dot{y}^2}\delta y\, \delta\dot{y}\right]_0^1 + \int \left(\frac{d^2f}{dy^2}\delta y + \frac{d^2f}{dy\, d\dot{y}}\delta\dot{y} - \frac{d}{dx}\left(\frac{d^2f}{dy\, d\dot{y}}\delta y + \frac{dy}{d\dot{y}^2}\delta\dot{y}\right)\right)\delta y\, dx.$$

It will save trouble if, in what follows, we use the abbreviations

$$\frac{df}{dy} = \mathrm{Y}_0, \quad \frac{df}{d\dot{y}} = \mathrm{Y}_1, \quad \text{and, in general, } \frac{df}{dy^{(r)}} = \mathrm{Y}_r,$$

$$\frac{d^2f}{dy^2} = \mathrm{Y}_{00}; \quad \frac{dy}{dy\, d\dot{y}} = \mathrm{Y}_{01}, \quad \text{and, in general, } \frac{d^2f}{dy^{(r)}\, dy^{(s)}} = \mathrm{Y}_{rs}.$$

Then we may write

$$\delta^2\mathrm{U} = \left[\mathrm{Y}_{01}\delta y^2 + \mathrm{Y}_{11}\delta y\, \delta\dot{y}\right]_0^1 + \int\left(\mathrm{Y}_{00}\delta y + \mathrm{Y}_{01}\delta\dot{y} - \frac{d}{dx}(\mathrm{Y}_{01}\delta y + \mathrm{Y}_{11}\delta\dot{y})\,\delta y\, dx\right),$$

of which the part under the integral sign is

$$\int\left(\mathrm{Y}_{00}\delta y + \mathrm{Y}_{01}\delta\dot{y} - \mathrm{Y}_{01}\delta\dot{y} - \frac{d\mathrm{Y}_{01}}{dx}\delta y - \frac{d\mathrm{Y}_{11}}{dx}\delta\dot{y} - \mathrm{Y}_{11}\delta\ddot{y}\right)\delta y\, dx$$

$$= \int\left\{\left(\mathrm{Y}_{00} - \frac{d\mathrm{Y}_{01}}{dx}\right)\delta y - \left(\frac{d\mathrm{Y}_{11}}{dx}\delta\dot{y} - \mathrm{Y}_{11}\delta\ddot{y}\right)\right\}\delta y\, dx. \quad . \quad . \quad . \quad . \qquad (5)$$

Now let z be a solution of the equation

$$\left(\mathrm{Y}_{00} - \frac{d\mathrm{Y}_{01}}{dx}\right)z - \frac{d\mathrm{Y}_1}{dx}\dot{z} - \mathrm{Y}_{11}\ddot{z} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

* JACOBI treats this case by a method not identical in form with the general method he gives.

Multiply (6) by $\delta y^2/z \, . \, dx$, and subtract it from the part under the integral sign in equation (5): then we have for the part under the integral sign

$$\int\left(-\frac{d\mathrm{Y}_{11}}{dx}\delta\dot{y}+\frac{d\mathrm{Y}_{11}}{dx}\frac{\dot{z}\,\delta y}{z}-\mathrm{Y}_{11}\ddot{y}+\mathrm{Y}_{11}\frac{\ddot{z}\,\delta y}{z}\right)\delta y\,dx=\int\frac{d}{dx}\left(\mathrm{Y}_{11}\left(\dot{z}\,\delta y-z\,\delta\dot{y}\right)\right)\frac{\delta y}{z}\,dx,$$

$$=\left[\mathrm{Y}_{11}\left(\dot{z}\,\delta y-z\,\delta\dot{y}\right)\frac{\delta y}{z}\right]_0^1-\int\mathrm{Y}_{11}\left(\dot{z}\,\delta y-z\,\delta\dot{y}\right)\frac{d}{dx}\left(\frac{\delta y}{z}\right)dx,$$

$$=\left[\mathrm{Y}_{11}\left(\dot{z}\,\delta y-z\,\delta\dot{y}\right)\frac{\delta y}{z}\right]_0^1+\int\mathrm{Y}_{11}\left\{\frac{d}{dx}\left(\frac{\delta y}{z}\right)\right\}^2 z^2\,dx,$$

and, writing out the complete variation, we get

$$\delta^2\mathrm{U}=\left[\mathrm{Y}_{01}\,\delta y^2+\mathrm{Y}_{11}\,\delta y\,\delta\dot{y}+\mathrm{Y}_{11}\left(\dot{z}\,\delta y-z\,\delta\dot{y}\right)\frac{\delta y}{z}\right]_0^1+\int\mathrm{Y}_{11}\left(\frac{d}{dx}\left(\frac{\delta y}{z}\right)\right)^2 z^2\,dx,$$

the final expression; from this it is evident that, if $\mathrm{Y}_{11}\,dx$ changed sign in the course of integration, the function could not be synclastic.

6. If the variation δy were such that $\delta\dot{y}$ suddenly changed from one finite value to another, $\delta\ddot{y}$ must be infinite at that point, and the integration would not be permissible, or, at least, it would require a justification which, so far as I am aware, has never been considered necessary.

Again, it will be observed that the limiting values of $\delta\dot{y}$ are introduced in the process (though for this case it is easily seen that in the final result the terms in which they appear are identically zero).

What has been shown in this case is true in general: the first $2n$ fluxions of δy are brought in under the integral sign, and of these the last n fluxions $y^{(n+1)} \ldots y^{(2n)}$ are got rid of by integration. Again, the limiting values of the first $2n - 1$ fluxions are introduced outside the integral, and, of these, the n fluxions $y^{(n)} \ldots y^{(2n-1)}$ disappear through their coefficients being identically zero; but the *direct* proof of this in the general case would, so far as I can see, require transformations of even greater length than those usually applied to the part under the integral sign.

7. It was with the object of ascertaining whether it was necessary, in the general case, to assume the continuity of the value of δy and its $2n - 1$ fluxions that I first considered the problem; and I communicated to the British Association at Montreal an account of a method which was free from this objection, and had the additional advantage of a simplicity which enabled it to be easily extended to any number of dependent variables; and it is, in its main principles, applicable to the most general case.

8. The application of this method to the preceding case is as follows. Taking the second variation in the form on p. 99, equation (4), in expanding V.U, we have

$$\delta^2 U = \int (Y_{00}\,\delta y^2 + 2Y_{10}\,\delta y\,\delta\dot{y} + Y_{11}\,\delta\dot{y}^2)\,dx.$$

Writing in this $\theta\,\delta_1 y$ for δy, where θ is a function of x at present unknown in form, and expanding the result, we get

$$\delta^2 U = \int (Y_{00}\theta^2\,\delta_1 y^2 + 2Y_{10}\theta\,\delta_1 y\,(\theta\,\delta_1\dot{y} + \dot{\theta}\,\delta_1 y) + Y_{11}\,(\theta\,\delta_1 y + \theta\,\delta_1\dot{y})^2)\,dx$$

$$= \int (\delta_1 y^2\,(Y_{00}\theta^2 + 2Y_{10}\theta\dot{\theta} + Y_{11}\dot{\theta}^2) + 2\delta_1 y\,\delta_1\dot{y}\,(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta}) + Y_{11}\theta^2\,\delta_1\dot{y}^2)\,dx.$$

Integrating by parts the term involving $\delta_1 y\,\delta_1\dot{y}$,

$$\delta^2 U = \left[(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta})\,\delta_1 y^2\right]_0^1 + \int \left(Y_{00}\theta^2 + 2Y_{10}\theta\dot{\theta} + Y_{11}\dot{\theta}^2 - \frac{d}{dx}(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta})\right)\delta_1 y^2\,\delta x$$
$$+ \int (Y_{11}\theta^2\,\delta_1\dot{y}^2)\,dx.$$

The multiplier θ is here quite arbitrary, and can therefore be determined so that the quantity multiplying δy^2 under the sign of integration shall vanish, and hence, when this value of θ is chosen, we have

$$\delta^2 U = \left[(Y_{10}\theta^2 + Y_{11}\theta\dot{\theta})\,\delta_1 y^2\right]_0^1 + \int Y_{11}\theta^2\,\delta_1\dot{y}^2\,dx$$

This is the same form as that in the other reduction, and it is easy to prove that θ is a solution of the equation (6) used in finding z in JACOBI's method.

9. We have now given examples of the simplest cases. The method of treating $\delta^2 U$, where

$$U = \int f(x, y, \dot{y}, \ldots y^{(n)})\,dx,$$

will now be given in its shortest form.

Using the notation already adopted, we may write

$$\delta^2 \int f(x, y, \dot{y}, \ldots y^{(n)})\,dx = \int \sum_0^n \sum_0^n Y_{rs}\,\delta y^{(r)}\,\delta y^{(s)}\,dx$$

If we transform this by writing $z_1\,\delta_1 y$ for δy, and expanding the fluxions, we shall get an expression which may be written

$$\delta^2 U = \int \sum_{(0)}^{(n)} \sum_{(0)}^{(n)} A_{rs}\,\delta_1 y^{(r)}\,\delta_1 y^{(s)}\,dx, \quad \ldots \ldots \ldots \quad (7)$$

where $\delta_1\dot{y}$, for instance, means d/dx, $\delta_1 y$ and A_{rs} contains z_1 as well as x. This expression consists of terms in which $\delta_1 y$ itself enters, and other terms in which only its fluxions appear. Integrate by parts the terms containing $\delta_1 y$; thus, in the previous expression

$$\int Y_{0s}\,\delta y\,\delta y^{(s)}\,dy = [Y_{0s}\,\delta y\,\delta y^{(s-1)}]_0^1 - \int\left(\frac{dY_{0s}}{dx}\,\delta y\,\delta y^{(s-1)} + Y_{0s}\,\delta\dot{y}\,\delta y^{(s-1)}\right)dx$$

$$= [Y_{0s}\,\delta y\,\delta y^{(s-1)}]_0^1 - \left[\frac{dY_{0s}}{dx}\,\delta y\,\delta y^{(s-1)}\right]^1$$

$$+ \int\left(\frac{d^2Y_{0s}}{dx^2}\,\delta y\,\delta y^{(s-2)} - \frac{dY_{0s}}{dx}\,\delta\dot{y}\,\delta y^{(s-2)} - Y_{0s}\,\delta\dot{y}\,\delta y^{(s-1)}\right)dx\,;$$

and, finally,

$$\int Y_{0s}\,\delta y\,\delta y^{s} = L \pm \int\left(\tfrac{1}{2}\,\frac{d^sY_{0s}}{dx^s}\,\delta y^2 - \delta\dot{y}\left(\frac{d^{s-1}Y_{0s}}{dx^{s-1}}\,\delta\dot{y} + \text{\&c.}\right)\right)dx,$$

where, with the exception of the term involving δy^2, the integral involves only fluxions of δy, and a similar reduction can, of course, be applied to any terms of the same form.

Reducing all the terms involving $\delta_1 y$ in (7), we get for $\delta^2 U$ an expression which we may write

$$\delta^2 U = L + \int\left(A_0\,\delta_1 y^2 + \sum_{(1)}^{(n)}\sum_{(1)}^{(n)} A'_{rs}\,\delta_1 y^{(r)}\,\delta_1 y^{(s)}\right)dx,$$

where, in the terms under the double summation sign, $\delta_1 y$ only appears through its fluxions. If we determine z_1 by the equation $A_0 = 0$, a differential equation of the $2n^{\text{th}}$ order, we have $\delta^2 U$ depending on the limits and on terms involving $\delta_1 y$ only through its fluxions. If this expression be transformed by writing $z_2\,\delta_2 y$ for $\delta_1\dot{y}$, then, after expanding, we get an expression which we may write

$$\delta^2 U = L + \int \sum_{(0)}^{(n-1)}\sum_{(0)}^{(n-1)} B_{rs}\,\delta_2 y^{(r)}\,\delta_2 y^{(s)}\,dx,$$

in which the highest fluxion of $\delta_2 y$ is of the $(n-1)^{\text{th}}$ order.

This expression can be reduced by the same method, and we get

$$\delta^2 U = L + \int\left(B_0\,\delta_2 y^2 + \sum_{(1)}^{(n-1)}\sum_{(1)}^{(n-1)} B'_{rs}\,\delta_2 y^{(r)}\,\delta_2 y^{(s)}\right)dx.$$

Determine z_2 so that $B_0 = 0$ for all values of x, then $\delta^2 U$ will depend on L and the fluxions of $\delta_2 y$: transform this by writing, in the part under the integral sign, $\delta_2\dot{y} = z_3\,\delta_3 y$, and reduce in the same way, and so on till we come to the last transformation but one, in which

$$\delta^2 U = L + \int\left(M_0\,(\delta_{n-1} y)^2 + \sum_{(1)}^{(2)}\sum_{(1)}^{(2)} M_{rs}\,\delta_{n-1} y^{(r)}\,\delta_{n-1} y^{(s)}\right)dx,$$

which, treated similarly, gives

$$\delta^2 U = L + \int(N_0\,\delta_n y^2 + N_{11}\,\delta_n\dot{y}^2)\,dx\,;$$

and, determining z_n so that $N_0 = 0$, we get finally

$$\delta^2 U = L + \int N_{11}\,(\delta_n\dot{y})^2\,dx.$$

When we substitute for $\delta_n \dot{y}$ its value in terms of δy and its fluxions, we must clearly have an expression of the form

$$\delta^2 U = L + \int N_{11} \{ f_0(x)\, \delta y^{(n)} + f_1(x)\, \delta y^{(n-1)} + \&c. + f_n(x)\, \delta y \}^2 \, dx ,$$

and, as it is evident that the coefficient of $\delta y^{(n)2}$ is unaltered by the transformations, the expression for $\delta^2 U$ may be written in the form

$$L + \int Y_{nn} (\delta y^{(n)} + \alpha_1 \delta y^{(n-1)} + \&c. + \alpha_n \delta y)^2 \, dx,$$

where L contains δy and its fluxions up to, but not including, δy^n.

10. This is the same form as that in JACOBI's method, and it has been obtained on the supposition that integrations may be performed on $\delta y, \delta \dot{y}, \ldots \delta y^{(n)}$; this requires that $\delta y \ldots \delta y^{(n-1)}$ should be continuous in every sense, and that $\delta y^{(n)}$ should not become infinite, but it may suddenly change from one finite value to another. It is very important to note this in connection with the condition determining the point at which a curve ceases to fulfil the synclastic condition

As yet, the equations determining the quantities $z_1, z_2, \ldots z_n$ used in the transformation have not been given, and it might be supposed that they would have to be found in order to determine the part of $\delta^2 U$ depending on the limits. It will, however, be seen that in this as well as in every other case under the Calculus of Variations it is only necessary to use these quantities to show that the reduction *can be made*. As a matter of interest, however, a short discussion is given below.*

* It may be interesting, though not required for our immediate purpose, to examine the relation which the coefficients z_1, &c, in the transformation bear to the value for y which gives the synclastic value to the integral For this purpose it will be convenient to use $\Delta\, \delta U$ instead of $\delta^2 U$

In the first place, it is evident that, if we have any expression for $\delta^2 U$ as a quadratic function of δy and its fluxions, we shall get $\delta \Delta U$ by taking the polar, with respect to the quadratic function, of the point whose coordinates are $\Delta y, \Delta \dot{y}, \Delta \ddot{y}$, and so on, using the word "point" in an extended sense. If a proof of this is desired, write $(\delta + K\Delta)$ for δ in

$$\delta^2 U = f(\delta y, \delta \dot{y}, \quad \delta y^{(n)}),$$

and get

$$(\delta + K\Delta)^2 U = f\{(\delta + K\Delta)\, y,\ (\delta + K\Delta)\, \dot{y} \,..\ \ (\delta + K\Delta)\, y^n\},$$

and, after expanding, equate coefficients of K, which is quite arbitrary. Hence, from equation (7) we obtain

$$\Delta\, \delta U = L + \int \sum_1^n \sum_1^n A_{rs}\, \Delta_1 y^r\, \delta_1 y^s \, dx ,$$

and, since the integral contains only fluxions of $\Delta_1 y_1$, $\Delta\, \delta U$ will depend only on the limits, whatever be the form of δy, provided $\Delta_1 y$, that is, $\Delta y / z_1$, is constant; and therefore z_1 is one of the values of Δy for which $\Delta\, \delta U$ is independent of the form of δU. Now,

$$\delta U = L + \int \left(Y_0 - \frac{dY_1}{dx} + \frac{d^2 Y_2}{dx^2} - \&c. \right) \delta y \, dx,$$

11. It is important to observe that this transformation is only valid, provided none of the quantities $z_1, z_2, \ldots z_n$ used in the transformation vanish for any value of x passed through in going from x_0 to x_1. For, suppose z_r vanishes, then, as $\delta_{r-1} y = z_r \delta_r y$, the corresponding value of $\delta_r y$ must be infinite. Now, since $z_1 \ldots z_n$

and therefore

$$\Delta\, \delta U = L + \int \Delta \left(Y_0 - \frac{dY_1}{dx} + \ \ldots \right) \delta y \, dx ,$$

and this can only be independent of the form of δy when

$$\Delta \left(Y_0 - \frac{dY_1}{dx} + \ldots \right) = 0, \qquad (a)$$

that is to say, when $y + \Delta y$ satisfies the equation satisfied by y, *i.e.*,

$$Y_0 - \frac{dY_1}{dx} + \&c = 0 \ . \qquad (b)$$

If, then, the solution of (b) be $y = f(x, c_1, c_2, \ \ldots c_{2n})$, that of (a) is obtained in the well-known form (see TODHUNTER, 'History of the Calculus of Variations,' p. 271, or JELLETT, 'Calculus of Variations,' p. 84)—

$$\Delta y = \frac{df}{dc_1} \Delta c_1 + \frac{df}{dc_2} \Delta c_2 + \ldots + \frac{df}{dc_{2n}} \Delta c_{2n}. \qquad (c)$$

It follows that z_1 must be $C_1\, df/dc_1 + C_2\, df/dc_2 + \&c.$, or, shortly, $z_1 = y_1$; $y_1, y_2, \ \ldots y_{2n}$ being independent solutions of (c).

In finding z_2, we employ a similar process. Since the part of $\delta^2 U$ under the integral sign has been expressed in terms of the *fluxions* of $\delta_2 y$, that of $\Delta\, \delta U$ can be expressed in terms of the fluxions of $\delta_2 y$ and $\Delta_2 y$. Hence, if we choose Δy, so that $\Delta_2 y =$ constant, the integral vanishes identically, and the whole variation $\Delta\, \delta U$ depends only on limiting values of δy, and not on the general value. Hence, as before, Δy must be a solution of (a), and evidently it must not be the y_1 solution. Let us denote it by y_2 (observing that y_2, however, cannot be quite arbitrarily chosen from the remaining $2n - 1$ solutions because the equation for Δy is only of the $(2n-2)^{\text{th}}$ order, and can only have $2n - 2$ solutions). Then, since $z_2 = \Delta_1 \dot{y}/\Delta_2 y = 1/\Delta_2 y \,.\, d/dx\, (\Delta y/z_1)$, it easily follows that $z_2 = c\, d/dx\, (y_2/y_1)$. Similarly, when we come to the third transformation, we have $z_3 \Delta_3 y = \Delta_2 \dot{y}$, whence

$$z_3 = \frac{\Delta_2 \dot{y}}{\Delta_3 y} = \frac{1}{\Delta_3 y} \frac{d}{dx} \left[\frac{\frac{d}{dx}\left(\frac{\Delta y}{y_1}\right)}{\frac{d}{dx}\left(\frac{y_2}{y_1}\right)} \right]$$

and, when $\Delta_3 y =$ constant, Δy is a solution y_3 of (a). Hence,

$$z_3 = \frac{d}{dx} \left[\frac{\frac{d}{dx}\left(\frac{y_3}{y_1}\right)}{\frac{d}{dx}\left(\frac{y_2}{y_1}\right)} \right],$$

and similarly for the remainder. As nothing hangs on the discussion of these quantities, it is not worth writing out any more. In fact, the only application of the investigation to the present case would be to show that the transformations in Art. 9 are always possible, provided a sufficiently short length of the curve be taken; that is, to show that it is always possible to choose z_1, z_2, z_3, &c., so that none of them vanish for any value of x between the limits of integration. But it is easier to see that this is, in general, the case from the equations obtained for z_1, z_2, z_3, &c., in the course of the transformations themselves. For, z_r being a function of x and arbitrary constants, it follows that, if we put $z_r = 0$, we can solve for the value of x in terms of those arbitrary constants, and hence, by taking suitable values for the constants, we can, in general, ensure that z_r does not vanish for a value $x = x_0$.

are definite functions of x and the arbitrary constants introduced in the solution, it is evident that, x_0 being the initial value of (x), there will in general be some value x_1 at which it becomes impossible to determine the arbitrary constants, so that some one at least of the solutions $z_1 \ldots z_n$ shall not have changed sign. Up to this point the transformation must hold, and the conditions for synclasticism derived from it must be sufficient and necessary.

12. Confining our attention to integrals for which the transformation does hold (that is, integrals whose limits are not too widely separated), it is easy to see by the usual method that, unless $Y_{nn}dx$ retains the same sign throughout the integration, the integral cannot be synclastic.* For the integral

$$\delta^2 U = \int Y_{nn} (\delta y^{(n)} + a\, \delta y^{(n-1)} + \&c\,)^2\, dx$$

may then be divided into two parts, one negative and the other positive, and, as the form of δy is arbitrary, we could make the numerical value of either of these parts exceed that of the other, and therefore $\delta^2 U$ would be capable of either sign.

But the condition that $Y_{nn}dx$ remains of the same sign throughout the integration is not sufficient to ensure that this integral shall be synclastic. This would be the proper place to examine the further condition if it could be derived from the preceding transformation, but it does not appear to me that we can avail ourselves of the analysis, for the following reasons —Some of the quantities z_r used in the tranformation may vanish for some value of x included in the integration, and the investigation would not apply. Hence we should have to give an independent discussion to discover the limits of the integration within which the transformation does apply. But even then we should only have proved that the function was synclastic up to those limits at least, and we should have still to discover whether it might not be

* It is usually stated that, unless Y_{nn} preserves its sign, the integral could not be synclastic, but this is a mistake arising from the supposition that, because dx increases from the lower to the higher limit, it must have the same sign throughout the integration. But it is evident from the figure

that dx may change sign for a value of x *between* the limits, in which case there must be an even number of changes, or it may change sign an uneven number of times for values not numerically between the lower and higher limit, but yet passed through in going from the one to the other *vid* the curve It would be easy, by transforming the axes, to multiply examples of the latter, and in these Y_{nn} and dx will change sign together.

synclastic for wider limits. For it does not seem by any means evident that the function cannot be synclastic unless the transformation is valid. It might still be possible to reduce it to a sum of *several* squares, for instance, after it had ceased to be possible to reduce it to a single square term. At first sight this would appear not improbable, for it would mean that it was still possible to determine the first few coefficients, $z_1, z_2, \ldots z_r$, &c., so that none of them vanished, although it was impossible to determine z_{r+1} so that it did not vanish; then the integral would be reduced to the sum of $(n-r)$ squares, and the conditions would be that all their coefficients were positive.

These difficulties seemed so great that it appeared better to attempt the problem by an altogether different method, namely, that of supposing that the synclastic property does hold for a given length of the curve and then ascertaining where the property ceases to hold.

It is evident that if this could be done it would be sufficient to find the synclastic condition for an infinitely small range of integration, and this suggested the method now to be given. As already stated, the discussion of the further condition for synclasticism will be postponed to Art. 21.

PART II.

The General Method.

13. A full account will now be given of the general investigation as applied to the case of two variables, and a somewhat shorter discussion of the general case will be found in Arts. 20, 21.

Consider the conditions under which the equation

$$\begin{aligned}\int_{x_0}^{x_1} f(x, y + \delta y, \dot{y} + \delta\dot{y}, \ldots y^{(n)} + \delta y^{(n)})\, dx &= \int_{x_0}^{x_1} f(x, y, \dot{y}, \ldots y^{(n)})\, dx \\ &+ \int_{x_0}^{x_1} \left(\frac{df}{dy}\,\delta y + \frac{df}{d\dot{y}}\,\delta\dot{y} + \ldots + \frac{df}{dy^{(n)}}\,\delta y^{(n)}\right) dx \\ &+ \tfrac{1}{2}\int_{x_0}^{x_1} \left(\frac{d^2f}{dy^2}\,\delta y^2 + 2\frac{d^2f}{dy\,d\dot{y}}\,\delta y\,\delta\dot{y} + \frac{d^2f}{d\dot{y}^2}\,\delta\dot{y}^2 + \ldots + \frac{df}{dy^{(n)2}}\,\delta y^{(n)2}\right) dx \\ &+ \tfrac{1}{2\cdot 3}\int_{x_0}^{x_1} \left(\frac{d^3f}{dy^3}\,\delta y^3 + 3\frac{d^3f}{dy^2\,d\dot{y}}\,\delta y^2\,\delta\dot{y} + \ldots + \frac{d^3f}{dy^{(n)3}}\,\delta y^{(n)3}\right) dx + \&c.\end{aligned}$$

is valid. It is obtained by writing $y + \delta y$, $\dot{y} + \delta\dot{y} \ldots$ for $y, \dot{y} \ldots$ in $f(x, y, \dot{y} \ldots y^{(n)})$, expanding by TAYLOR'S Theorem and then integrating. TAYLOR'S Theorem requires that numerical values of the quantities δy, $\delta\dot{y}$, &c., shall not exceed certain limits, and that the values of $x, y, \dot{y}$, &c., shall not be such as to make the coefficients in the expansion infinite. Hence, if $f(x, y, \dot{y} \ldots y^{(n)})$ satisfy the latter condition for every value of x included in the range of integration, and if we take δy, $\delta\dot{y}$, &c., small

enough, we can always ensure that the above expansion holds. So far, nothing has been said as to the continuity of δy, δy, &c., but, δy, &c., being the successive fluxions of a single quantity δy, they must all be continuous functions of x except $\delta y^{(n)}$, the highest fluxion, whose magnitude is not so restricted. For, if $\delta y^{(r)}$ changes suddenly from one finite value to another, for change of x from x' to $x' + dx$, *its* differential coefficient, $\delta y^{(r+1)}$, would become infinite for that value of x. But, as $\delta y^{(n+1)}$ does not occur in $f(x, y, \dot{y} \ldots y^{(n)})$, the validity of the expansion will not be affected by its becoming infinite, and therefore $\delta y^{(n)}$ may change from one finite value to another.

14. Hence, if we discuss the problem of maxima and minima by the usual method, the variation we give is of necessity restricted as follows: $\delta y, \delta\dot{y}, \ldots \delta y^{(n-1)}$ must all be continuous functions of x. $\delta y^{(n)}$ need not be continuous, but the magnitude of each fluxion must be restricted with certain limits, which will vary with the nature of the problem under discussion, but it will in all cases be sufficient to make them infinitely small. It will be convenient to consider δy as $\alpha\phi x$, where α is a small numerical coefficient, and ϕx is a function of x, such that it and any number of its fluxions may become zero, though in general they will be finite, while neither the function itself nor any of its fluxions up to and including the n^{th} can become infinite for any value of x occurring in the integration.

The coefficient α must be taken sufficiently small to ensure that, when considering only the sign and not the value of an expression involving it, we may neglect terms depending on α^2 or higher powers in comparison with those depending on α. Denoting, as usual, by δU, $\delta^2\text{U}$, &c., the part of the expansion depending on the first, second, &c., powers of δy and its fluxions, we may say that δU is of the order α, $\delta^2\text{U}$ of the order α^2, and so on. Hence, in the absence of special determinations of the form of δy, $\delta^2\text{U}$, the part depending on α^2 will exceed all terms depending on α^3 and higher powers of α, and then the sign of the whole variation will be the same as that of $\delta^2\text{U}$ (δU being zero when y has its synclastic value).

It is convenient to have a geometric representation, and the function y will be taken as the ordinate of a curve of which x is the abscissa, and the curve corresponding to the synclastic form for y will be called the synclastic curve.

15 We may now easily prove the following proposition.—

Let
$$\text{U} = \int_{x_0}^{x_1} f(x, y, \dot{y} \ldots y^{(n)})\, dx,$$

y being *any* function of x. Let the second variation $\delta^2\text{U}$ be taken, subject to the condition that $\delta y, \delta\dot{y}, \ldots \delta y^{(n-1)}$ are zero at each limit. Then the sign of $\delta^2\text{U}$ is the same as that of the term involving $\delta y^{(n)2}$ in the integral, provided the range of the integration be sufficiently small.

The second variation being written

$$\delta^2\text{U} = \int_{x_0}^{x_1} (\text{Y}_{00}\, \delta y^2 + 2\text{Y}_{01}\, \delta y\, \delta\dot{y} + \text{Y}_{11}\, \delta\dot{y}^2 + \&\text{c} + 2\text{Y}_{n-1,n}\, \delta y^{(n-1)}\, \delta y^{(n)} + \text{Y}_{nn}\, \delta y^{(n)2})\, dx,$$

the proposition will be proved when it is shown that throughout the integration $\delta y^{(n-1)}/\delta y^{(n)}$, $\delta y^{(n-2)}/\delta y^{(n-1)}$, &c., are all negligible, for then the term $Y_{nn}\,\delta y^{(n)2}$ is obviously the most important.

Now

$$\delta y_x^{(n-1)} = \int_{x_0}^{x} \delta y^{(n)}\, dx,$$

no constant being added, as $\delta y^{(n-1)}$ vanishes when $x = x_0$. Let the numerically greatest value of $x - x_0$ in the integration be β, and that of $\delta y^{(n)}$ be γ; then, numerically, $\delta y^{(n-1)} < \beta\gamma$.

A fortiori, $\delta y^{(n-1)} < \beta^2\gamma$, for, if β^2 be the greatest value of $\delta y^{(n-1)}$, $\delta y^{(n-2)} < \beta\,\beta'$. But $\beta' < \beta\gamma$, similarly $\delta y^{(n-3)} < \beta^3\gamma$, and so on. Hence $\delta y^{(n-1)}/\delta y^{(n)}$ is of the order β, and similarly for each of the fractions.*

It follows from this that the only term of the order α^2 in the expression for $\delta^2 U$ is

$$\int_{x_0}^{x_1} Y_{nn}\,\delta y^{(n)2}\, dx.$$

It follows that, if $Y_{nn}\, dx$ does not change sign in passing from x_0 to x_1, neither can $\delta^2 U$ change sign, whatever be the form of δy, and it is clear that $\delta^2 U/U$ is of the order α^2, exactly as if the integral were taken over a finite portion of the curve; and it should be remarked that the value of $\delta^2 U$ is of exactly the same order as if the limiting variations $\delta y \ldots \delta y^{(n-1)}$ had not been zero—that is, the order of $\delta^2 U$ is the same as if the most general variation possible had been given to y, although the actual variation is zero, as also are all its fluxions up to, but not including, the n^{th}.

It might possibly be objected that, as $\delta y^{(n-1)}$ is zero at each limit, therefore $\int_{x_0}^{x_1} \delta y^{(n)}\, dx$ is also zero. Hence, as dx may be taken to *increase* uniformly, $\delta y^{(n)}$ must change sign between $x = x_0$ and $x = x_1$; and, as these values are very close, $\delta y^{(n)}$ is everywhere very near its vanishing point, and is therefore everywhere very small. But this proceeds on the idea that $\delta y^{(n)}$ must be continuous. There is no difficulty when it is remembered that $\delta y^{(n)}$ may change suddenly from a positive value to a negative value.

16. Thus, without any analytical transformations, it has been shown that if Y_{nn} be positive the integral is a true minimum, and if Y_{nn} be negative it is a true maximum when the integration is extended over a very small range. We have still to consider how far the integration may be extended without annulling this property. Withdrawing the restriction that $x_1 - x_0$ is small, let us consider the continuity of the second variation

$$\delta^2 U = \int_{x_0}^{x_1} (Y_{00}\,\delta y^2 + 2Y_{01}\,\delta y\,\delta\dot{y} + \&c. + Y_{nn}\,\delta y^{(n)2})\, dx.$$

* It follows similarly that, provided β, the range of the integration, be not too great, the sign of $\delta^3 U$ is the same as that of $\int_{x_0}^{x_1} (d^3 f/\delta y^{(n)3}\,\delta y^{(n)3})\, dx$, that of $\delta^4 U$ the same as that of $\int_{x_0}^{x_1} d^4 f/dy^{(n)4}\,(\delta y^{(n)})^4\, dx$, and so on; and this does not in any way depend on y having its synclastic value.

It is, in general, a quantity of the order α^2 (Art. 14), and it is continuous if we can alter it by amounts infinitely small compared to α^2. This is always possible. First let the limiting values of x be unchanged; then, writing, for δy, $\delta y + \delta^2 y$, where $\delta^2 y$ is infinitely small compared to δy or α, it is evident that the change in the integral is infinitely small compared to α. If, still keeping the same variation $\delta y + \delta^2 y$, we change the limits by writing $x_1 + dx_1$, $x_0 + dx_0$, for x_1 and x_0, the most important additional terms due to this change,

$$\left[(\mathrm{Y}_{00}\,\delta y^2 + 2\mathrm{Y}_{01}\,\delta y\,\delta\dot{y} + \ldots + \mathrm{Y}_{nn}\,\delta y^{(n)2})\,dx\right]_0^1,$$

are also infinitely small compared to the original integral.

Now, as in the original expression for $\delta^2\mathrm{U}$ we may suppose δy, $\delta\dot{y}$, .. $\delta y^{(n-1)}$ zero at each limit x_1 and x_0, so we may suppose $\delta^2 y$ so determined that $\delta y + \delta^2 y$, $\delta\dot{y} + \delta^2\dot{y}$, ... $\delta y^{(n-1)} + \delta^2 y^{(n-1)}$, shall be zero for the values $x_0 + dx_0$, $x_1 + dx_1$.

Hence, representing by $\delta^2\mathrm{U}_{x_0}^{x_1}$ the value of the second variation when the value of δy is such that it and all its fluxions, up to the n^{th}, vanish for both limits of integration x_0 and x_1, we see that we can always alter the value of δy so that

$$\frac{\delta^2\mathrm{U}_{x_0+dx_0}^{x_1+dx_1} - \delta^2\mathrm{U}_{x_0}^{x_1}}{\delta^2\mathrm{U}_{x_0}^{x_1}}$$

is infinitely small. (Of course we might also alter it so that it should be finite.)

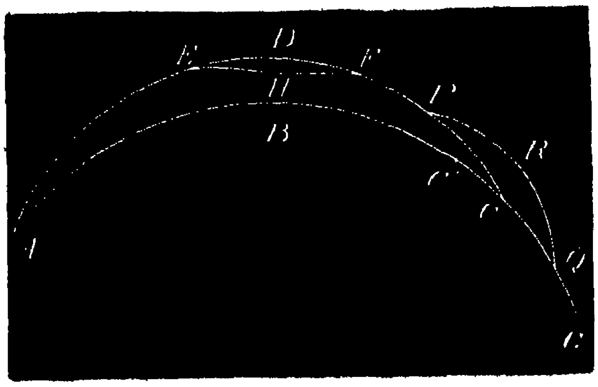
It will simplify the further explanation if we represent the values of y by ordinates of a curve of which x is the abscissa,* and we will suppose that the value of Y_{nn} at the lower limit is negative, so that the curve obtained by making $\delta\mathrm{U}$ vanish is a maximum when the upper limit is very near the lower one.

Considering the lower limit of integration x_0 as a fixed point A, and the higher one x_1 as an arbitrary point M, on the curve ABC, we know from § 15 that when M is sufficiently close to x_1 the integral is a maximum, *i.e.*, $\delta^2\mathrm{U}_{x_0}^{x_1}/(x_1 - x_0)$ is a negative quantity of the order α^2. Suppose the curve first ceases to give a maximum when M coincides with C. Then we may easily see that C is the point to which it first

* It is evident that what has been said about the admissibility of a variation may be expressed thus: δy is the difference between the ordinates of the new curve and old curve, and any curve is admissible, provided δy^n is nowhere infinite Thus the broken curve ABC is admissible provided that at D and E,

the points of junction with AFC, it has contact of the $(n-2)^{\text{th}}$ order Furthermore, the difference of the integrals taken along ABC and AFC is the same as the difference of the integrals along DBE and DFE, as is seen at once by regarding the sign of integration as one of summation

becomes possible to draw a second curve such that $\delta^2 U_A^C = 0$. For, if $\delta^2 U_A^C$ could be a *positive quantity of the order* α^2, we could alter δy so that $\delta^2 U_A^{C'}$ should be positive and of the same order, C′ being between A and C, infinitely near C. Hence the



maximum property would have ceased at C′, but by supposition it does not cease till C, and therefore we have to find C as being the first point for which $\delta^2 U_A^C$ or $\delta^2 U_{x_0}^{x_1} = 0$.

Now $\delta^2 U_{x_0}^{x_1} = 0$, 1st, because it vanishes independently of the form of δy, or 2nd, because a particular form is assigned to δy, causing it to vanish, or 3rd, because $x_1 - x_0$ can be divided into separate parts, some of which vanish from the first cause and the rest from the others.

The first case can only occur if $\delta^2 U$ vanishes identically, *i.e.*, if its coefficients vanish, and will not be further discussed. The second case will occur when Y_{nn} does not vanish between A and the point for which $\delta^2 U_A^C = 0$ (this will be evident presently). The third case must be investigated on the supposition that the parts which vanish independently of the form of δy are infinitely small (for otherwise $\delta^2 U$ would vanish identically), and for these parts Y_{nn} must vanish, for when the limits of integration are infinitely close, and the limiting values of $\delta y, \delta \dot{y}, \ldots \delta y^{(n-1)}$ zero, $\delta^2 U = \int Y_{nn} \, \delta y^{(n)2} \, dx$, and this vanishes independently of the form of δy when $Y_{nn} = 0$.

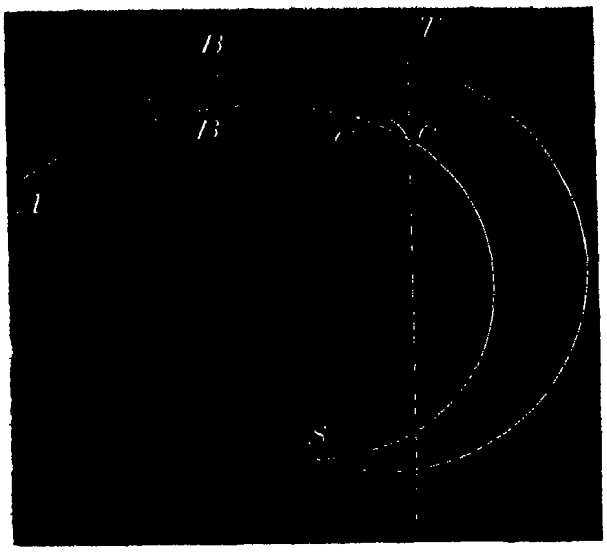
We have then only two cases to discuss—(a) Y_{nn} does not vanish throughout the integration, so that $\delta^2 U_A^C$ vanishes in consequence of a particular form being assigned to δy throughout the integration, and (b) that in which Y_{nn} does vanish.

First suppose that Y_{nn} does not vanish. Then C is evidently determined at the first point to which a second synclastic curve can be drawn, having at each limit contact of the $(n-2)^{\text{th}}$ order with ABC.*

* For it is evident that if $\delta^2 U$ vanishes in passing from ABC to ADC, and if any portion as EF of ADC were not itself synclastic, we could obtain an integral greater than that along ADC, and therefore greater than that along ABC, by joining EF by a synclastic curve EHF, having at E and F contact of the $(n-2)^{\text{th}}$ order with ADC; and therefore the preceding reasoning shows that C would not be the *first* point at which a curve can be drawn so that $\delta^2 U_A^C = 0$. Hence we must use the synclastic curve to get this limit.

Next suppose that, in fig. 4, the point S represents the first intersection of a consecutive synclastic curve AB′S with ABS, and that the dotted line, through T, is a line whose equation is $x = x'$, x' being the value of x for which $Y_{nn} = 0$ (if there is more than one value, take the first one you pass through in going from A *viâ* the curve). Then, in that figure, $\delta^2 U_A^C$ first becomes capable of the value zero at C, where the curve meets the ordinate through T. For the second variation vanishes in passing from ABC to a curve ABC′C coincident with ABC up to C′, a point infinitely close to C, and having at C and C′ contact of the $(n-2)^{\text{th}}$ order with ABC, since from A to C′ δy is zero, and from C to C′ Y_{nn} is zero.

Hence $\delta^2 U_A^C$ first becomes capable of a zero value when C coincides with the intersection (S) of a consecutive synclastic curve having contact of the $(n-1)^{\text{th}}$ order with ABC, or with the point (T), given by $Y_{nn} = 0$, *whichever is first reached in passing from A along the curve.*



It follows, rigorously, that the synclastic property cannot cease *until* the first of these points is reached. It does not follow that, if the integration is extended beyond this limit, the integral is anticlastic.

Three cases arise :—1st, S is nearer A than T is; or, 2nd, T is nearer to A than S; or, 3rd, T and S coincide.

Case 1.—In this case we can easily show (see fig 3) that, C being the intersection of a consecutive synclastic curve ADC with ABCG, the synclastic property does cease at C, that is, we can join A and G by curves the integral along which is either greater or less than that along ABCG. Let ADC be a consecutive synclastic curve for which $\delta^2 U_A^C = 0$, then evidently, representing by I (ABC . . .) the integral along the curve ABC . . . , we have

$$I(ABC) = I(ADC);$$

add

$$I(CG) \equiv I(CG),$$

and get

$$I(ABCG) = I(ADCG)$$

(which is legitimate, as δy^n is nowhere infinite). Now take two points P and Q on ADC and CG respectively, and join them by a synclastic curve PRQ having contact of the $(n-2)^{\text{th}}$ order with ADC and CG, P and Q being sufficiently close for the synclastic property to hold between them. Then, since I (PRQ) is a maximum,

$$\mathrm{I}\,(\mathrm{PRQ}) > \mathrm{I}\,(\mathrm{PCQ})\,;$$

add

$$\mathrm{I}\,(\mathrm{ADP}) + \mathrm{I}\,(\mathrm{QG}) \equiv \mathrm{I}\,(\mathrm{ADP}) + \mathrm{I}\,(\mathrm{QG}),$$

and get

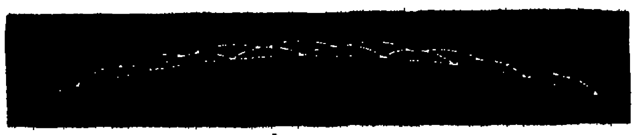
$$\mathrm{I}\,(\mathrm{APRQG}) > \mathrm{I}\,(\mathrm{ADCG}), \text{ and therefore } > \mathrm{I}\,(\mathrm{ABCG}),$$

showing that the integral taken along ABCG is not a maximum, and evidently it is not a minimum, for, if it were, every part of it would have to be a minimum, and the part from A to B, for instance, is not a minimum, but a maximum.*

Case 2.—In this case, if Y_{nn} changes sign as well as vanishes, the integral becomes anticlastic when the limit is beyond T (the point where $\mathrm{Y}_{nn} = 0$). This is obvious, as the part immediately beyond T then gives a minimum value to the integral whose lower limit is T, while from A to T it gives a maximum value. If Y_{nn} does not change sign, it is easy to see that the maximum property does hold beyond C, with this nominal exception. A curve ABC′C″C‴G can be found the integral along which is *equal* to that along ABCG, C′ and C‴ being infinitely near C, and C′C″C‴ being any curve having contact of the $(n-1)^{\text{th}}$ order with ABCG at C′ and C‴. It is not difficult to see this by reasoning similar to the above, but it is shorter to observe that if we alter very slightly the value of Y_{nn}, so as just to make it preserve its sign without vanishing, we alter the value of the integral *very* slightly, and, therefore, &c.

Case 3.—Very slight consideration shows that the synclastic property ceases at T.

17. Before passing to the general case, it will not be amiss to add a few explanations.



The argument is not that it is possible to pass from the original curve to any other infinitely near curve by repeating again and again for each part of the curve a

* It is sometimes considered sufficient to say that, as $\delta \mathrm{U}_{\mathrm{A}}^{0} = 0$, and $\delta^2 \mathrm{U}_{\mathrm{A}}^{0} = 0$, while $\delta^3 \mathrm{U}_{\mathrm{A}}^{0}$ changes sign with δy, the maximum property must cease at A. In TODHUNTER'S "ADAMS' Prize Essay on the Calculus of Variations" this reasoning is employed (Art. 24, p. 25). But it is invalid for two reasons. 1st, $\delta^3\mathrm{U}$ and all higher variations may vanish, as in the case of great circles on a sphere; and 2nd, it shows only that a curve can be got giving an integral greater than that corresponding to ABCG by terms of the *third*, not of the *second*, order. Now, as the second and third variations are quite independent of each other, there is nothing whatever to show that when a value is given to δy other than that which makes $\delta^2\mathrm{U} = 0$ this variation $\delta^2\mathrm{U}$ can be made to change sign. It is absolutely necessary to show this, for otherwise the integral from A to G would really be a true maximum, though of a very curious nature. —See the remarks at the end of Art. 17.

variation such as we have just given. No doubt, if we were to do this, we should at last get a curve differing from the original one, and, since Y_{nn} is the same if x be the same, it seems that we should get the variations all of the same sign, and that the result could thus at once be extended to a finite length of the curve; but, although it would be true that the sign of the *second* variation in passing from any one of the curves thus found to the consecutive one would have the same sign as $Y_{nn}\,dx$, yet the *first* variation would not vanish.

It is not difficult to see that these two alternative conditions, treated of in Cases 1 and 2 respectively, are independent. For, if the points obtained by the two criteria coincided, the least root of the determinant equation (a) in the note* must coincide with the least value of x for which $Y_{nn} = 0$, and it seems evident that there can be no such connection. It may, however, be as well to give an example, to show that the condition in Case 2 may cease to be fulfilled, while that in Case 1 still holds. Taking for U the expression $\int \dot{y}^4 x\,dx$, we have

$$\delta U = 4 \int \delta\dot{y}\,\dot{y}^3 x\,dx = \left[4\dot{y}^3 x\,\delta y\right]_0^1 - 4 \int \frac{d}{dx}(\dot{y}^3 x)\,\delta y\,dx,$$

and the equation given by the calculus is

$$\frac{d}{dx}\dot{y}^3 x = 0\,;$$

integrating, we get

$$y = cx^{2/3} + c',$$

whence it is easy to see that $Y_{nn}\,dx$ or $4.3.\dot{y}^2 x\,dx$ changes sign as x passes through the value zero. (It might be thought sufficient to say that, as $\dot{y}^2$ cannot change sign,

* If $y = f(x, c_1, c_2, \ldots c_{2n})$ be the general solution obtained by making

$$Y_0 - \frac{dY_1}{dx} + \frac{d^2Y_2}{dx^2} + \&c \pm \frac{d^nY_n}{dx^n} = 0,$$

then, denoting df/dc_1, df/dc_2, &c, by y_1, y_2, &c., it is well known that we have for the value of x_1, to which a second curve of the species can be drawn from x_0, so as to have, at x_0 and the point we seek, contact of the $(n-2)^{\text{th}}$ order with the curve $y = f(x, c_1, c_2, \ldots c_{2n})$, the equation

$$\begin{vmatrix} y'_1, & y'_2, & \cdot & \cdot & y'_{2n} \\ \dot{y}_1, & \dot{y}_2, & & \cdot & \dot{y}_{2n} \\ \cdot\,, & \cdot\,, & & \cdot\ \cdot & \\ y'^{(n-1)}_1, & y'^{(n-1)}_2, & & \cdot & y'^{(n-1)}_{2n} \\ y_1, & y_2, & & & y_{2n} \\ \dot{y}_1, & \dot{y}_2, & \cdot\ \cdot\ \cdot & \cdot & \dot{y}_{2n} \\ \cdot\,, & \cdot\,, & \cdot\ \cdot\ \cdot & \cdot\ \cdot & \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \cdot\ \cdot & \cdot & y_{2n}^{(n-1)} \end{vmatrix} = 0, \quad \ldots\ldots\ldots \quad (a)$$

where y' means the value of y when x_0 is substituted for x. The least value of x satisfying this, or more properly the value first reached in going from x_0 *viâ* the curve, gives the point in question.

this quantity must change sign with x, as dx is a constant quantity; this, however, is a mistake. it is necessary to know the nature of the curve in order to be sure that dx does not change sign when x does) Hence, according to the first rule, the integral will be anticlastic, should the value $x = 0$ be included in the limits of integration. But, if we seek the limits within which the condition in the second proposition is fulfilled, we find that, if we start from a point $x_0 = -\alpha$, a second curve of the same species does not intersect the original curve until $x = +\alpha$. It follows that, so far as regards the second condition, the integration might extend from $x_0 = -\alpha$ to $x_1 = +\alpha$.

It may be well to observe that the proof given in Case 1, that when the integration is extended beyond the limits stated, *i.e.*, those for which δ^2U can vanish, the synclastic property ceases to hold, does not in any way depend on the supposition that δ^3U does not vanish. It is shown absolutely, and without exception, that when the limit stated is passed, δ^2U can change its sign (§ 16), and the values of δ^3U, δ^4U, for those limits will only enable us to find whether the synclastic property holds *at* the limit *up to which* it is known to hold, namely, whether it holds up to *and including*

the limits found in Prop. 2. Consider the case of a curve, and let ACB and AC′B be two consecutive curves satisfying the limits and the differential equation

$$Y_0 - \frac{dY_1}{dx} + \frac{d^2Y_2}{dx^2} - \&c.;$$

then both the first and second variations, δU and δ^2U, vanish in passing from the curve ACB to AC′B. But the third variation, in passing from A to B, will not, in general, vanish, and may be expressed as a function of the coordinate of A, $= f(x)$, suppose; now, as A moves along the curve, $f(x)$ will, in general, vanish at one or more points. Hence it follows that, in general, the synclastic property only holds *between* A and B, and does not hold *for the limits* A *and* B *actually*, though there the difference is only of the third order; but there may be certain points for which δ^3U vanishes and δ^4U is of the same sign as $Y_{nn}dx$, and for these the curve joining A and B gives a truly synclastic value to the integral.

If the synclastic property ceases because Y_{nn} changes sign, it will hold up to (and including) the limit, and only cease as you pass beyond the limit.

18. It is often convenient to borrow a few terms from geometry when treating of functions depending on a number of independent variables $x_1, x_2, \ldots x_n$. In the following paragraphs the word "point" will be used to denote any single set of values of the independent variables, while "region" will mean a continuous collection of

points whose boundary is defined by those values of $x_1 \ldots . x_n$ which make a certain function or functions vanish; as, for instance, all those sets of values which satisfy the two sets of inequalities

$$x^2_1 + x^2_2 + \quad . \quad . \quad . \quad + x^2_m - r^2 < 0$$

and

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \quad . \quad . \quad . \quad + (x_m - a_m)^2 - r'^2 < 0$$

The dependent variables being $y_1 \ldots y_n$, the word "surface" will be used as an abbreviation for the term "set of equations expressing the dependent variables in terms of the independent ones." Not only is there much saving in labour, both to the writer and to the reader, in the adoption of these terms, but there is the additional advantage that the same explanation is applicable alike to the most general case and to that in which geometrical conceptions enable the argument to be grasped with a clearness unattainable in reasoning of a purely analytic character. The reason for adopting the word "surface" instead of "curve" is that, as the explanation for the curve has been already given, it would be superfluous to repeat it, while it seems a real advantage to give the investigation for the case of one dependent and two independent variables

It will be necessary, in the first place, to examine the conditions under which the solution supplied by the rules of the Calculus of Variations is applicable.

Let the function to be made a minimum be

$$\mathrm{U} = \int dx_1 \ldots \int dx_m f(x_1, \ldots x_m, y_1, \ldots y_n,),$$

where $f(x_1, \ldots x_m, y_1, \ldots y_n)$ includes the fluxions of $y_1 \ldots y_n$ with regard to the independent variables.

To find the variation in this expression, let us increase $y_1, y_2, \ldots y_n$ to $y_1 + h_1$, $y_2 + h_2, \ldots y_n + h_n$ (h_1, &c., being functions of $x_1, x_2, \ldots x_m$); then the fluxions of y_1, &c., will be increased by the corresponding fluxions of h_1, &c.

For the purpose of ascertaining the limits within which the quantities h_1, h_2, &c., must be confined in order that our reasoning may be valid, let us write, in place of h_1, h_2, &c., $\alpha\Delta y_1, \alpha\Delta y_2$, &c., where α is a constant, the same for all the variables, and $\Delta y_1, \Delta y_2$, &c., are finite functions of $x_1, x_2, \ldots x_m$, or, more accurately, are functions which, though they (and their fluxions contained in the integrals) may vanish, neither become infinite, nor give infinite values to these fluxions, for values of the independent variables included in the region of integration; they are *not-infinite* functions. If we represent by z, z', &c., any of the dependent variables *or* their fluxions, we may represent the corresponding change by $\alpha\Delta z, \alpha\Delta z'$. By TAYLOR's theorem we may write the new value of U corresponding to the new values for the dependent variables in the form

$$\mathrm{U}_{y+h} = \mathrm{U}_y + \int \ldots \int \left\{ \alpha\Sigma\left(\frac{df}{dz}\Delta z\right) + \tfrac{1}{2}\alpha^2\Sigma\left(\frac{d^2f}{dzdz'}\Delta z\Delta z'\right) + \frac{\alpha^3}{\underline{|3}}(\&c.)\right\} dx_1 \ldots dx_m. \quad (8)$$

For our purposes it will be necessary to take α of such a magnitude that the part depending on α^2 is greater than all the subsequent terms (It seems well to observe that this does not, in general, imply that α is *infinitely* small.) But there are restrictions to the values of the quantities Δz, $\Delta z'$, &c., and in order to find them it will be necessary to give an outline of the method which is usually employed in the calculus. The part of (8) depending on the first power of α is reduced by successive integration by parts, so that the part of the integral not solely depending on the limiting values contains only the variations Δy_1, Δy_2, &c., and does not contain their fluxions. Thus, if D represent the operation by which z is got from y,* we shall get from the term

$$\alpha\int\ldots\int\frac{df}{dz}\Delta z\,dx_1\ldots dx_m$$

such terms as

$$\alpha\int\ldots\int(\mathrm{B}_1\,dx_2\,dx_3\ldots dx_m+\mathrm{B}_2\,dx_1\,dx_3\ldots dx_m+\&\text{c.})$$
$$\pm\alpha\int\ldots\int\left[\mathrm{D}.\left(\frac{df}{dz}\right)\right]dx_1\ldots dx_m,$$

where the first part of the right-hand side depends only on limiting values. Applying similar reductions to all terms containing *fluxions* of variations, we get an expression of the form

$$\left.\begin{aligned}\delta\mathrm{U}=\alpha\int\ldots\int\{\mathrm{L}_1\,dx_1\,dx_2\ldots dx_m+\mathrm{L}_2\,dx_1\,dx_2\ldots dx_m+\&\text{c.}\}\\+\alpha\int\ldots\int\{\mathrm{A}_1\,\Delta y_1+\mathrm{A}_2\,\Delta y_2+\&\text{c.}\}\,dx_1\,dx_2\ldots dx_m\end{aligned}\right\}\quad . \quad . \quad (9)$$

This integration by parts depends for its validity on the supposition that no one of the quantities Δz becomes infinite for any values of $x_1\ldots x_m$ within the limits of integration. But it is very important to remark that the integration is legitimate, whether the variations of the *highest* fluxions are or are not discontinuous in the sense of suddenly changing from one *finite* (properly, *not-infinite*) value to another. It is to be observed that, in discussing the sign of $\delta^2\mathrm{U}$, we introduce no limitations except those already implied in the usual treatment of $\delta\mathrm{U}$.

Again, it is to be observed that the terms in the limiting integrals in (9) will contain the limiting values of all but the highest fluxions of the variations. Hence, when we say that the limits are given, we mean that the values of the dependent variables and of all but their highest fluxions are given for all points on the boundary of the region of integration. Hence, if we determine the forms of the functions $y_1\ldots y_n$ so as to satisfy the equations—

* Thus, when z represents $\dfrac{d^8y_1}{dx_1^{\,2}\,dx_2^{\,3}\,dx_4^{\,3}}$, D will represent $\dfrac{d^8}{dx_1^{\,2}\,dx_2^{\,3}\,dx_4^{\,3}}$.

$$\left.\begin{aligned}
A_1 &\equiv \frac{df}{dy_1} - \frac{d}{dx_1}\frac{df}{d\left(\dfrac{dy_1}{dx_1}\right)} - \frac{d}{dx_2}\frac{df}{d\left(\dfrac{dy_1}{dx_2}\right)} - \&c. + \&c = 0 \\
A_2 &\equiv \frac{df}{dy_2} - \frac{d}{dx_1}\frac{df}{d\left(\dfrac{dy_2}{dx_1}\right)} - \dots \dots = 0 \\
\cdot &\equiv \cdot \quad \dots \dots \dots = 0 \\
A_n &\equiv \frac{df}{dy_n} - \frac{d}{dx_1}\frac{df}{d\left(\dfrac{dy_n}{dx_1}\right)} - \&c. \quad \dots \dots = 0
\end{aligned}\right\}, \quad . \quad . \quad . \quad (10)$$

and at the same time fulfil the conditions supplied by the given limiting values of the dependent variables and of all but their highest fluxions, the difference between the integrals for the surface so determined and that for any other surface will vanish so far as regards terms involving only the first power of α, provided only that the second surface can be obtained from the first by a change such that none of the Δ variations for A become infinite when $x_1, \dots x_m$ have the values corresponding to any point within the region of integration. This, indeed, is true whatever be the order of α, but, in order that the sign of the difference between the integrals shall be the same as that of the second variation (the part depending on α^2), it is, in general, necessary that the quantities $\alpha \Delta z$, $\alpha \Delta z'$, &c., be small.

19. To determine whether the integral is a true maximum or minimum, we have now only to find whether the terms of this order α^2 in (8) will be always of the same sign when the variations are given in any values consistent with the conditions given in Art. 18. For it is evident that, if we restrict ourselves to a less general variation in examining the sign of the second variation, we could neither be sure that the conditions obtained were sufficient to ensure that the integral was synclastic, though they would be necessary; nor that the conditions that it should be anticlastic were necessary, though they would be sufficient. If, on the other hand, we were to admit a more general variation, the conditions for synclasticism would be sufficient, but not necessary, and those for anticlasticism would be neither sufficient nor necessary. In fact, it will be found that the conditions under which we are discussing the problem are really those necessary in order that it shall have a

meaning. For instance, in the case of least action, when we say that the action in the free path is less than in any other, we imply that there is to be no sudden change

in the value of y, such as would occur in a path AC . . . DB, but there may be sudden changes in $\dot{y}$, the inclination of the tangent (as in AEFB).

20. To facilitate the discussion, a fluxion will be said to be one of the "highest fluxions" when no fluxions of that fluxion appear in the function whose integral is to be made synclastic. Thus, $d^2y_1/dx_1\,dx_2$ may be one of the "highest" fluxions, although d^7y/dx^7 is not; for there may be no terms which can be written $D\,.\,d^2y/dx_1\,dx_2$, where D represents any combination of d/dx, d/dx_2, &c., while there might be a term d/dx_2, $d^7y/dx_1{}^7$.

The limits being supposed fixed, the following proposition can be easily proved.—If the highest fluxions of the variable y_1 which occur in U be all of the same order n_1, those of y_2 of the same order n_2, and so on, then the conditions that U shall be synclastic when the integral is only extended over a *small* region R are the same as those that the quantity

$$\left(\frac{d^2f}{da^2}\mathrm{X}^2+\frac{d^2f}{db^2}\mathrm{Y}^2+\text{\&c.}+2\frac{d^2f}{da\,db}\mathrm{XY}+\text{\&c.}\right)dx_1\,dx_2\ldots dx_m$$

shall be incapable of a change of sign, a, b, &c., representing the highest fluxions, and X, Y, &c., being any arbitrary quantities.

The conditions for the case where the highest fluxions of any dependent variable are not all of the same order will be discussed afterwards. When it is said that the region R of integration is small it is meant that the greatest *ranges of value* of the coordinates $x_1 \ldots x_m$ are small. Thus, if the region is given by

$$x_1{}^2+x_2{}^2+\ldots+x_m{}^2-r^2<0,$$

then r must be small; of the order β, suppose. This being so, it is easy to show that we may neglect the variations of all but the highest fluxions of the dependent variables when finding the sign of δ^2U. For the change in Δz in passing from a point P_0 on the boundary of R to any point P within it is

$$(\Delta z)_{\mathrm{P}}-(\Delta z)_{\mathrm{P}_0}=\int_{\mathrm{P}_0}^{\mathrm{P}}d\,\Delta z=\int_{\mathrm{P}_0}^{\mathrm{P}}\left(\frac{d\Delta z}{dx_1}dx_1+\frac{d\Delta z}{dx_2}dx_2+\text{\&c.}\right);$$

and, since the total range of $dx_1, \ldots dx_m$, in the integration is of order β, the order of the integral will be that of the quantities $\beta\,d\Delta z/dx_1$, $\beta\,d\Delta z/dx_2$, &c., or at least it cannot be greater. Again, the limits being fixed, $(\Delta z)_{\mathrm{P}_0}$, the limiting value of Δz, must be zero unless z is one of the "highest fluxions." Hence it follows that all the other fluxions are small (of the order β at least) compared to the highest; and that they can be omitted from the integral when all we wish to determine is its sign. It is, however, necessary to show that the entire integral does not vanish, for, as the values of all but the highest fluxions, and therefore of the highest but one, are given for all points of the boundary, it is evident that

$$\int_{P_0}^{P'_0} \frac{d\Delta z}{dx_1} dx = 0,$$

where $d\Delta z/dx_1$ is one of the highest fluxions, and P_0 and P'_0 are two points on the boundary for which the value of x_1 alone is different, those of $x_2, \ldots x_m$ being the same. It follows from this that $d\Delta z/dx_1$ must change sign once at least in passing from P_0 to P_0', and therefore, if it varied continuously, must everywhere be infinitely small. Since, however, there is nothing to prevent d/dx, $d\Delta z/dx_1$, being infinite (§ 18), $d\Delta z/dx_1$ may have a finite value within the region.

Remembering that the region is small, and that the order of magnitude of Δa, Δb, &c., a, b, &c., being "highest fluxions," does not depend in any way on that of the region of integration, we see that the sign of $\delta^2 U$ will be the same as that of

$$\int \ldots \int \left(\frac{d^2 f}{da^2} (\Delta a)^2 + 2 \frac{d^2 f}{da\, db} (\Delta a\, \Delta b) + \&c.\right) dx_1, dx_2, \ldots dx_m, \quad . \quad . \quad . \quad (11)$$

Δa and Δb representing highest variations. As the region is small, d^2f/da^2, $d^2f/da\, db$, &c., may be considered as constants throughout the integration, a supposition which again involves neglect of small terms. (Δa, Δb, &c., cannot be regarded as constants, however small the region may be, for their differential coefficients may be infinite.) Now, the part inside the bracket can be resolved into a sum of squares, and, if the coefficient of any one (or more) of these squares is negative, the expression can be given either sign. For, although the quantities Δa, Δb, &c., are not independent of each other to the extent that all the other square terms could be made to vanish, yet they are so to the extent that any one term may be made to exceed all the others; for instance, if the region of integration be that for which

$$x_1^2 + x_2^2 + \&c. \ldots - r'^2 = < 0 \quad \text{or} \quad \phi' = < 0,$$

and

$$x_1^2 + x_2^2 + \&c. \ldots - r''^2 = < 0 \quad \text{or} \quad \phi'' = < 0,$$

r' and r'' being small quantities of the order β, we may assume

$$\Delta y_1 = \frac{1}{\beta^{2n_1}} (\phi')^{n_1} (\phi'')^{n_1} f_1 \left(\frac{x_1}{\beta}, \frac{x_2}{\beta}, \ldots \frac{x_m}{\beta}\right),$$

$$\Delta y_2 = \frac{1}{\beta^{2n_2}} (\phi')^{n_2} (\phi'')^{n_2} f_2 \left(\frac{x_1}{\beta}, \frac{x_2}{\beta}, \ldots \frac{x_m}{\beta}\right),$$

and so on for the others. For these assumptions will satisfy the limiting conditions, whatever be the forms of f_1, f_2, &c., which may be regarded as quite arbitrary, and they will give finite values to the "highest fluxions" of the dependent variables. If we resolve the quadratic expression in (11) into a sum of squares, and substitute for Δa, Δb, &c., their values in terms of f_1, f_2, &c., we shall, by solving a differential

equation, be able to make any one of these terms larger than the sum of the others. It is necessary, however, to show that such an equation has at least one real solution, and this can be done by the method of expansion in series just as it is usually done for the case of two variables only.

Thus the criterion for a maximum or minimum value of the function has been found when the region of integration is small. It is the same as that obtained by the methods of transformation; but the proof now given is free from the uncertainty which is connected with analytical proofs.

21. The criteria for the case where the region of integration has any finite magnitude can be derived from the preceding by considerations depending on the *continuity of the integrals.* Remembering that we are still treating of the case where the limits are fixed, we may prove the following proposition :—

If it is possible to take, around every point P in a region R of finite magnitude, a minor region (p), no matter how small, such that the integral U for that region is synclastic, then the integral for the entire region R will be synclastic, provided the further condition be fulfilled that it is impossible to take within the region R a second synclastic surface V′, having at all points of its limiting intersection with the first the same values for the dependent variables and for all their fluxions, *with the exception of the highest.* If it be possible to find such a surface, the integral U will be anticlastic for the region R.

Let us consider how it could happen that δ^2U became capable of either sign at pleasure when the region of integration is extended. Let S be the region for which this change of sign *first* becomes possible. Hence (restricting ourselves to the case where the function U has a *minimum* value), when the integration extends over *any* region wholly contained in S, δ^2U is positive, while, if it be extended over a region including S, δ^2U can change sign. It is clear that this can only happen if the least value of δ^2U is zero when the integration is extended over S. This may be shown thus. The second variation being written

$$\delta^2 U = \alpha^2 \int \ldots \int \Sigma \frac{d^2 f}{dz\, dz'} \Delta z\, \Delta z'\, dx_1 \ldots dx_n$$

where Δz, $\Delta z'$, &c., are finite quantities, it is evidently capable of being changed by an amount infinitely small compared to α^2, by an infinitely small change in Δz, $\Delta z'$, &c., and the *new* values of Δz can be made to satisfy limiting conditions obtained from the previous ones by infinitely small changes. It follows hence that, if it were possible to take such values for Δz, $\Delta z'$, &c., that δ^2U should be a negative quantity of the order α^2 when the integration extends over a region S + dS, greater than, but differing infinitely little from, S, we could, by an infinitely small change in the variations, obtain values making δ^2U negative and of the order α^2 within the region S − dS, which by the supposition is impossible, as within S the integration of δ^2U gives a result which is always positive. It follows that the greatest negative value of δ^2U

for the region $S + dS$ can only differ from zero by a quantity infinitely small compared to α^2, and therefore the least positive value of $\delta^2 U$ must vanish for the region S

It has now to be proved that $\delta^2 U$ must be capable of a negative value of order α^2 when the region of integration is extended beyond the region S. It sometimes seems to be considered that this may be inferred from the fact of $\delta^3 U$ changing sign when $\delta^2 U = 0$. But, as $\delta^3 U$ does not represent the increment of $\delta^2 U$ *due to extending the region of integration beyond S*, this is not admissible. If we draw an imperfect analogy from algebra, we may say that what we have to prove is that in no case does $\delta^2 U$ behave as if it had a square factor the value of which, after vanishing, remains of the same sign, but that, if S be a region of integration of the character supposed in Proposition 2, for which it is possible to make $\delta^2 U$ zero, then for any region including S it is possible to make $\delta^2 U$ take either sign, the limits being in each case supposed fixed

To prove this, let us suppose that $y_1, y_2, \ldots y_n$ represent values of the dependent variables which make the first variation vanish. Let $\alpha \Delta_1 y_1, \alpha \Delta_1 y_2$, &c., represent the variations for which $\delta^2 U = 0$ (that is, $\Delta_1{}^2 U = 0$) when the integration is extended over S. The limiting values of all variations, except those of the highest fluxions, are zero. Let S′ be a region including S, and let $\alpha \Delta_2 y_1, \alpha \Delta_2 y_2$, &c., be variations having at all points of S the values $\alpha \Delta_1 y_1, \alpha \Delta_1 y_2$, &c., for all the variations, and having at all points of S′ not common to it and S the values zero for all the variations.

Now take a third region S″, wholly included in S′, and of which a portion Σ_1 is included in S, and the remainder Σ_2 excluded from S; and let $\alpha \Delta_3 y_1, \alpha \Delta_3 y_2$, &c., be a variation having at all points of S′ — S″ (representing in that way the points contained in S′ and excluded from S″) the values $\alpha \Delta_2 y_1, \alpha \Delta_2 y_2$, &c., while, over the region S″, $\alpha \Delta_3 y_1, \alpha \Delta_3 y_2$, &c., are determined by the conditions that $y_1 + \alpha \Delta_3 y_1, y_2 + \alpha \Delta_3 y_2$, &c., are the values which make U a true minimum when the integration extends over the region S″, and the limiting conditions are that, for all except the highest fluxions of the variations, $\Delta_3 z = \Delta_2 z$ all over the boundary of S″, z representing, as before, any dependent variable or fluxion (By taking the region S″ small enough, these conditions can always be satisfied)

Then the variations represented by Δ_1, Δ_2, and Δ_3 are admissible ones (Art 18). Then $\Delta_1{}^2 U = 0$ when the integration extends over S, and $\Delta_2{}^2 U$, when the integration extends over S′, is identically equal to it, and therefore vanishes (for the Δ_2 variations are zero except over S, where they equal the Δ_1 variations). Again the Δ_3 variations are the same as the Δ_2 ones, except over S″; and over S″ the functions $y_1 + \alpha \Delta_3 y_1$, &c., give a smaller value to the integral than do $y_1 + \alpha \Delta_2 y_1$ (because they make it an absolute minimum compared to all near values), and therefore $\Delta_3{}^2 U$ is smaller than $\Delta_2{}^2 U$, the integration being extended over S′. But $\Delta_2{}^2 U = 0$, and therefore $\Delta_3{}^2 U$ is negative. But clearly $\delta^2 U$ may be positive, and we have now shown that it may be negative, for one value is $\alpha^2 \Delta_3{}^2 U$; hence it is capable of either sign.

It may be objected that exceptional cases might occur, in which $\Delta_3 y$ and $\Delta_2 y$ coincided over S″ as well as over S′—S″. But it is directed that, of S″, Σ_1 is in the part common to S and S′, and Σ_2 is outside S. The surface $y_1 + \alpha\Delta_2 y_1$, &c., is therefore one in which discontinuous values for the fluxions of $y_1 + \alpha\Delta_2 y_1$, $y_2 + \alpha\Delta_2 y_2$, &c., would appear in the equations (10), Art. 18 ; and therefore that equation cannot be satisfied by these values. Hence Δ_2 is not the same operation as Δ_3.

Observe that the whole point of the proof consists in the fact that the Δ_3 variations are sufficiently continuous, notwithstanding the discontinuity in the highest fluxions.

To complete the proof, it only remains to show that the surface V′, for which $\delta^2 U = 0$, satisfies the equations (10), Art. 18, for every point of the region S. Suppose that it did not do so for a portion S_1 of S. Take a compound surface made up of V_1 over S_1 and V′ over $S - S_1$, where V_1 is the synclastic surface, having at all points of the boundary of S_1 the same values of Δy_1, Δy_2, &c., and all but their highest fluxions, as those of the function V′. This compound surface gives us an admissible variation, and the integral over it is less than that over V′. But, by hypothesis, it is impossible within the region S to find a surface giving a smaller value to the integral than that given by V. Hence V′ must be a synclastic surface, and the proposition is proved.

Hence a function will be synclastic provided, first, that the condition given in Art. 20 is fulfilled for every point in the region of integration; and second, that it is impossible, within that region, to draw another synclastic surface with the same limits. It will be easily seen that these conditions are independent of each other, and that, if either or both fail, the function becomes anticlastic.

22. When the highest fluxions of any dependent variable are not all of the same order of differentiation, the conditions found in Art. 20, although sufficient, are not all necessary. For, as will be proved presently, the values of the highest fluxions of the Δ variations are not all of the same order of magnitude. To ascertain the comparative orders of different fluxions, let us consider the equations

$$\Delta z_P - \Delta z_{P_{01}} = \int_{x_1'}^{x_1} \frac{d\Delta z}{dx_1} dx_1,$$

$$\Delta z_P - \Delta z_{P_{02}} = \int_{x_2'}^{x_2} \frac{d\Delta z}{dx_2} dx_2, \text{ and so on,}$$

where Δz_P means the value at the point P, or $x_1, x_2, \ldots x_m$; $\Delta z_{P_{01}}$ that at the point on the boundary which has the same values for all the coordinates except x_1, which has the value x_1' given by $\phi(x_1' x_2 \ldots x_m) = 0$, where $\phi = 0$ is the equation of the boundary; and similarly for the other coordinates. As $x_1 - x_1'$, $x_2 - x_2'$, &c., are everywhere of the same order, β, it might appear that $d\Delta z/dx_1$, $d\Delta z/dx_2$, must each be of the order $\Delta z/\beta$. This, however, is only the case when $d\Delta z/dx_1$, $d\Delta z/dx_2$, &c., do not rapidly change sign. For, while it is clear that $d\Delta z/dx$ must exceed Δz in the ratio $1/\beta$ at least, it is easily seen that the ratio may be greater if the terms in the

integral change sign rapidly. In fact, if the order of $d\Delta z/dx_1$ is unity, and its fluctuations recur at intervals of order β^n, then the terms in the integral

$$\int_{x_1'}^{x_1} \frac{d\Delta z}{dx_1} dx_1$$

will cancel, provided $x_1 - x_1' = m\beta^n$, m being any integer; and so the integration up to any point can only contain terms of the order β^n. Moreover, since, if any function is periodic with respect to any variable, all its fluxions with respect to that variable are of the same period, it follows that, if *one* differentiation introduces a coefficient of the order β^{-n}, *two* will bring in β^{-2n}, and so on.

It thus appears that, if we assume $\Delta y_1 = \beta^p$, multiplied by a function whose fluctuation-periods with regard to x_1, x_2, &c., are p_1, p_2, &c., the order of

$$\frac{d\Delta y_1}{dx_1} \text{ will be that of } \beta^{(p-p_1)};$$

$$\frac{d\Delta y_2}{dx_2} \quad \ldots\ldots \quad \ldots \beta^{(p-p_2)},$$

$$\ldots\ldots\ldots\ldots\ldots,$$

and, in general, that of $d^{a_1+a_2+\&c.}\Delta y_1/dx_1^{a_1}\,dx_2^{a_2}\ldots$ will be $\beta^{p-a_1p_1-a_2p_2-\&c}$. Now, by the conditions in Art. 18, none of the quantities Δz, &c., can be infinite; hence we must reject any values of Δy_1, Δy_2, &c., which do not fulfil this condition. Thus the term just considered must be rejected if it makes any of the expressions

$$p - a_1p_1 - a_2p_2 - \&c. - a_mp_m < 0,$$

where $a_1, a_2, \ldots a_m$ have the values corresponding to any fluxion occurring in U. And it may be rejected as useless unless it makes some one, at least, of these expressions zero. An example will render this more intelligible. Suppose we consider a function in which d^8y_1/dx_1^8, $d^4y_1/dx_1^3\,dx_2$, $d^4y_1/dx_1^2\,dx_2^2$, d^5y_1/dx_2^5, are the highest fluxions. Here the equations to be satisfied are

$$p - 8p_1 = > 0 \quad \ldots (a); \qquad p - 3p_1 - p_2 = > 0 \quad \ldots (b);$$
$$p - 2p_1 - 2p_2 = > 0 \quad \ldots (c), \qquad p - 5p_2 = > 0 \quad \ldots (d),$$

together with those obtained from the lower fluxions. But, as the lower fluxions must be smaller than the higher in the ratio $1/\beta$ *at least*, it is unnecessary to consider the conditions obtained from them. If we put $p_1 = 1$, we get, from (a), $p = 8$, and it is obvious that $p_2 = 1$ will make (b), (c), and (d) each >0. If we put $(b) = 0$ and eliminate p from (1), (c), and (d), we get, as our conditions from (a), $p_2 - 5p_1 > 0$; therefore $p_2 > 5$, as $p_1 = 1$; from (3), $p_1 - p_2 > 0$, and therefore $p_2 < 1$. Hence no value for p_1, p_2, or p can satisfy $(b) = 0$ and the other inequalities. Hence we may leave $d^4\Delta y/dx_1^3\,dx_2$ out of the final condition, as it could only give terms

multiplied by positive powers of α. Similarly, if $(c) = 0$, $p = 2p_1 + 2p_2$, and from (a) $2p_2 - 6p_1 = > 0$, but from (d) $2p_1 - 3p_2 = > 0$, and, as these are incompatible for positive values of p_1 and p_2, $d^4\Delta y_1/dx_1^2\,dx_2^2$ cannot appear in the quadratic function, on whose sign the problem depends For, if we make any assumption which gives to it a finite value, we give, to other quantities in the integral, values involving negative powers of β If β be infinitely small, this would imply infinite values for the Δ variations. But the values $p = 8$, $p_1 = 1$, $p_2 = \frac{8}{5}$, will satisfy $(d) = 0$, and the rest either $=$ or > 0. Hence, in the case in question, the sign of the variation depends on that of

$$\left\{\frac{d^2f}{\left(d\left(\frac{d^8y_1}{dx_1^8}\right)\right)^2}\mathrm{X}^2 + 2\frac{d^2f}{\left(d\left(\frac{d^8y_1}{dx_1^8}\right)\right)\left(d\left(\frac{d^5y_1}{dx_2^5}\right)\right)}\mathrm{XY} + \frac{d^2f}{\left(d\left(\frac{d^5y_1}{dx_2^5}\right)\right)^2}\mathrm{Y}^2\right\} dx_1\,dx_2.$$

If the highest fluxions had been

$$\frac{d^8y_1}{dx_1^8}, \quad \frac{d^8y_1}{dx_1^7\,dx_2}, \quad \frac{d^7y_1}{dx_1^5\,dx_2^2}, \quad \frac{d^3y_1}{dx_2^3},$$

the conditions would be, if we write $p_1 = 1$ all through, $p - 8 = > 0\,(a)$; $p - 7 - p_2 = > 0\,(b)$; $p - 5 - 2p_2 = > 0\,(c)$; $p - 3p_2 = > 0\,(d)$. Putting $p = 8$, $p_2 = 1$, (a) and $(b) = 0$, and (c) and $(d) > 0$: hence $d^8\,\Delta y_1/dx_1^8$ and $d^8\,\Delta y_1/dx_1^7\,dx_2$ will remain in the condition. Putting $p = 9$, $p_2 = 2$, (b) (and $(c)) = 0$, while (a) and $(d) > 0$: hence $d^7\,\Delta y_1/dx_1^5\,dx_2^2$ remains. Again, putting $p_2 = 5$ and $p = 15$, we get $(d) = 0$, while $(c) = 0$ and (a) and $(b) > 0$, and hence $d^3\,\Delta y_1/dx_2^3$ appears. Hence in this case all the fluxions remain.

To complete the proof, we must show that it is possible to assign values for the variations Δy_1, Δy_2, &c., which shall satisfy the limiting conditions as well as those given above. In order to show what kind of assumption must be made, it is necessary to remark that, if z be any quantity such that $z = 0$, and $dz/dx_1 = 0$ all over any surface $\phi(x_1, x_2, \ldots x_m) = 0$, then will every other fluxion, as dz/dx_r, $= 0$ over that surface. For, as $z = 0$,

$$0 = dz = \frac{dz}{dx_1}dx_1 + \frac{dz}{dx_2}dx_2 + \ldots \frac{dz}{dx_m}dx_m$$

whenever $dx_1, dx_2, \ldots dx_m$ satisfy the equation $\phi = 0$. The only relation among $dx_1, dx_2, \ldots dx_m$ imposed by this limitation is

$$\frac{d\phi}{dx_1}dx_1 + \frac{d\phi}{dx_2}dx_2 + \&c. = 0.$$

Hence, comparing coefficients,

$$\frac{dz/dx_1}{d\phi/dx_1} = \frac{dz/dx_2}{d\phi/dx_2} = \&c.;$$

but, as dz/dx_1 vanishes, so do all the other fluxions It follows that, if the boundary conditions are $\Delta y_1 = 0$, $d/dx_r\ \Delta y_1 = 0$, $d/dx_s\ d/dx_t \,.\, \Delta y_1 = 0$, and so on up to $d^{a_1+a_2}\ /dx_1^{a_1}\, dx_2^{a_2}\ \ .\ \Delta y_1$, all the fluxions whose order does not exceed $a_1 + a_2 \ldots + a_m$ must vanish at the boundary. For, as Δy_1 and $d\Delta y_1/dx_r = 0$, so do all other first fluxions, and therefore $d\Delta y_1/dx_s = 0$: but this, with $d/dx_t \,.\, d\Delta y_1/dx_s$, shows that all second fluxions with dx_s in them vanish. Hence any fluxion $d^2\Delta y_1/dx_a\, dx_b = 0$ for $d\Delta y_1/dx_a = 0$ and $d^2\Delta y_1/dx_s\, dx_a = 0$, and hence all second fluxions of $d\Delta y_1/dx_a$ vanish, and therefore, &c.

It follows from the preceding that in the examples in question all fluxions up to and including those of the 7th order must vanish for points on the boundary, and if we assume for Δy_1 an algebraic form we must write

$$\Delta y_1 = (\phi\,(x_1,\ x_2,\ .\ .\ .\ ,\ x_m))^8 f(xy),$$

where $\phi = 0$ is the equation of the bounding surface. Suppose we adapt this to the last example, the origin being taken as the point P in Proposition 1, and the bounding surface as

$$x_1^2 + x_2^2 - r^2 = 0,$$

so that the integration extends over all values of x_1 and x_2 which make the left-hand negative. Hence r is to be a quantity of the order β, and, to adapt the expression to the preceding formula, we must write

$$\Delta y_1 = \left[\left(\frac{x_1}{r}\right)^2 + \left(\frac{x_2}{r}\right)^2 - 1\right]^8 \left[\beta^8 f\left(\frac{x_1}{r},\ \frac{x_2}{r}\right) + \beta^9 \cos\left(\frac{x_2}{\beta^2}\right) + \beta^{15} \cos\left(\frac{x_2}{\beta^5}\right)\right],$$

where, however, $\cos\,(x_2/\beta^2)$ and $\cos\,(x_2/\beta^5)$ are to be considered as abbreviations for any fluctuating functions of periods β^2 and β^5 respectively

Similar assumptions can easily be made when other fluxions appear. The convenience in choosing $p_1 = 1$ is now evident, though, as far as the equations were concerned, it made no difference, as only the ratios of p, p_1, and p_2 entered into them.

The limits within which the property holds are evidently given by the discussion of Art. 21.

23 If all the highest of all the fluxions of any of the dependent variables appear in U in the first degree only, the foregoing reasoning would not hold, as *all* the variations of that dependent variable appearing in δ^2U would then vanish at the limits, and therefore the variation δ^2U when taken over a small region would be zero compared to quantities of the order α^2. (When there is only one independent variable there *must* be some *one* highest fluxion, but in general there is a group of fluxions higher than any others, not usually identical with what have been called the "highest fluxions," Art. 20, but included in them; it is only when each member of this group vanishes that the exception occurs) It is known, however, that in this case it is, in general, impossible to fulfil the limiting conditions by means of the arbitrary

functions arising from the solution of the partial differential equations. This exception is well known in the simpler cases, but I am not aware that it has been generally discussed.

A partial differential equation between independent variables $x_1, \ldots x_m$ and dependent variables $y_1, \ldots y_n$ will, in general, require for the complete determination of $y_1, \ldots y_n$ several sets of limiting conditions, for instance, one set when x_m has its limiting values $x_m = f_0(x_1, \ldots x_{m-1})$ and $f_1(x_1, \ldots x_{m-1})$, and another set when $x_{m-1} = f_0\ (x_1, \ldots x_{m-2})$ and $f_1(x_1, \ldots x_{m-2})$, and so on until, finally, there is a set derived from $x_1 = c_0$ and $x_1 = c_1$. But in the particular case in which the conditions at the limits are the values of $y_1, \ldots y_n, dy_1/dx_m, \ldots dy_n/dx_m, d^2y_1/dx^2_m$, &c., for a single surface $f(x_1, x_2, \ldots x_m) = 0$ the functions $y_1, \ldots y_n$ will be completely determined without any further limiting conditions (provided the proper number of conditions be given), and when the limiting conditions are of this character there is no difficulty in finding the requisite number of conditions relative to each variable. The limiting equations furnished by the Calculus of Variations are not, however, so simple as this, being of the dual character above. But, as this does not affect the *number* of the conditions at the limiting values of x_m, it will serve our purpose to find the number of conditions necessary in the simpler case. Let there be n equations, represented by (1), (2), . . . (n), between the n dependent variables $y_1, y_2, \ldots y_n$, and let the highest order of differentiation with respect to any independent variable in which y_r appears in (s) be $[r, s]$. Then, provided each dependent variable appears in each equation, it can easily be shown that the number of conditions necessary to determine the dependent variables is the greatest of the sets of numbers $\Sigma\ [r, s]$, so chosen that in each set there is one term corresponding to each variable and one to each equation. For from § 22 it is evident that the number of functions required is *the same as if there was but one independent variable*, for the values of the single set $y, dy/dx, d^2y/dx^2$, &c., at the boundary determine those of all the other fluxions of y. We may, therefore, discuss the question on the supposition that there is but one independent variable. Now it is evident that, if we could determine all the successive differential coefficients of each function for each point of the bounding surface, we could, by TAYLOR'S theorem, expand the function in a series; and we know the limiting values of the differential coefficients of the y functions with regard to all variables x_1, x_2, &c , when we know those for any variable (Art. 22). Hence the problem will be solved if we show how many of the limiting values we must assume in order to determine all the rest. But we can show that it is possible to find the first Y_1 quantities of the series $y_1, dy_1/dx_1, \ldots d^{Y_1-1}y_1/dx_1^{Y_1-1}$; the first Y_2 quantities $y_2, dy_2/dx_1, \ldots d^{Y_2-1}y_2/dx_1^{Y_2-1}$; and so on for the rest, provided we assume $\Sigma\ [r, s]$ of these functions; Y_1 and Y_2, &c., being, if necessary, indefinitely large. For differentiate the equation (1) a_1 times, the equation (2) a_2 times, and so on. Since we are not to introduce any differential coefficients of orders higher than $Y_1 - 1$, $Y_2 - 1$, these being the orders of the highest fluxions in

Y_1, Y_2, &c, we must have, considering only fluxions of y_1, $a_1 + [1, 1] \overline{\gtrless} Y_1 - 1$, $a_2 + [1, 2] \overline{\gtrless} Y_1 - 1$, and so on, one inequality from each equation Again, from considering the order of differentiation of y_2, $a_1 + [2, 1] \overline{\gtrless} Y_2 - 1$, $a_2 + [2, 2] \overline{\gtrless} Y_2 - 1$, and so on Since every differentiation gives us a new equation, among the quantities in question we get altogether $n + a_1 + a_2 +$ &c $+ a_n$ equations; hence the difference between this and $Y_1 + Y_2 +$ &c. must be equal to the number of quantities to be assumed in order to solve these equations It is easy to see that the most favourable way in which the differentiations a_1, a_2, &c, can be disposed consistently with the inequalities to be satisfied will give the number stated above as the least number of this difference Now suppose the equations obtained in the Calculus of Variations from making the coefficients of δy_1, δy_2, . . δy_n vanish are denoted by (1), (2), . . . (n), and let the [1] denote the highest differential coefficient of y_1 occurring in the second or in a higher degree in the function to be made synclastic, [2] that of y_2, and so on, then it is easy to see that y_r cannot enter into the equation (p) by fluxions of order higher than $[r]+[p]$, and that y_p will enter into it in the order $2p$ Hence in this case $[p] + [r] = [pr]$, $[r] + [p] = [rp]$, and we have for this case to find the set for which $\Sigma[p, r]$ is greatest Now, since we are to take one index for each equation, in our $\Sigma\{(p) + (r)\}$ we are to take only one term from equation (r), and hence (r) on the right-hand side is to appear only once. Moreover, we are to take only one term from each variable, and therefore we are simply to take $\Sigma\{(p) + \Sigma(r)\}$, where each refers to the values (1), (2), (3), &c., and hence, in all, double the sum of the order of the highest of all the fluxions of each variable in the expression to be made synclastic But this is exactly the number of the conditions supplied by equating the limiting terms of δU to zero in the expression to be integrated, except in the case we are at present discussing (where some of the highest of all the fluxions do not appear in the second degree, but in the first only), and therefore in this case the limiting conditions cannot be satisfied, and the problem becomes, in general, incapable of solution.

V. *On Ellipsoidal Current-Sheets*

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It is a problem of some interest in Electromagnetism to determine the natural modes of decay, and the corresponding persistencies, of free currents in a given conductor. When this has been solved it is an easy matter to find the currents induced by given varying electromotive forces.

The general theory for a system of *linear* circuits is of course well known If the variables $\dot{y}_1, \dot{y}_2, \ldots \dot{y}_n$, which specify the currents, be so chosen that the electrokinetic energy T and the dissipation-function F are both expressed by sums of squares, say

$$2T = L_1\dot{y}_1^2 + L_2\dot{y}_2^2 + \ldots + L_n\dot{y}_n^2,$$
$$2F = R_1\dot{y}_1^2 + R_2\dot{y}_2^2 + \ldots + R_n\dot{y}_n^2,$$

then $y_1, y_2, \ldots y_n$ are for the present purpose the "normal coordinates" of the system; and the equations of motion of electricity are of the form

$$L\ddot{y} + R\dot{y} = E,$$

where E is the external electromotive force of the type in question In the case of free currents, $E = 0$, and consequently

$$\dot{y} = Ae^{-\lambda t},$$

where

$$\lambda = R/L$$

If we put

$$\tau = \lambda^{-1} = L/R,$$

then τ may be called the "modulus of decay," or the "persistency," of free currents of this type.

In considering the effect of varying electromotive forces, it is convenient to suppose these expressed, as regards the time, in a series of simple harmonic terms, each of which may be taken separately. Assuming, then, that $E \propto e^{ipt}$, we have, for the induced current,

$$y = \frac{E}{R + ipL} = \frac{E}{R(1 + ip\tau)} = \frac{E \cos\theta}{R} \cdot e^{-i\theta},$$

if

$$\tan\theta = p\tau.$$

Hence the phase of the currents lags behind that of the inducing electromotive force by an amount arc tan $p\tau$. This remark, obvious as it is, is of some importance in relation to the practically interesting question of the rotation of a conductor about an axis of symmetry in a constant magnetic field. The magnetic potential of any normal type will be proportional to cos $s\omega$ or sin $s\omega$, where ω is the azimuth about the axis of symmetry, and s is integral (or zero). If now, as in MAXWELL's 'Electricity,' § 600, we employ coordinate axes moving with the conductor, the electromotive forces relative to these will vary as $e^{isp t}$, where p is the angular velocity of rotation. On account of the symmetry about the axis, the retardation of phase above spoken of comes to this, that the system of currents of any normal type is, owing to its inertia, displaced relatively to the field through an angle $1/s$ arc tan $sp\tau$, where τ is the modulus of decay proper to the type. (See §§ 7, 16, below.)

For other than linear conductors the problem above stated was first solved by MAXWELL in the case of an infinite plane sheet of uniform conductivity. The cases of solid spherical and cylindric conductors, and of thin spherical and cylindric shells, have been treated by Prof. C. NIVEN,* Lord RAYLEIGH,† and the writer ‡ It is remarkable that, with a certain exception,§ no difference of electric potential, and consequently no surface distribution of electricity, is called into existence during the decay of free currents in conductors of the forms mentioned.

In § 675 of his 'Electricity and Magnetism,' MAXWELL has indicated a certain arrangement of currents over the surface of an ellipsoid, which produces a uniform magnetic field in the interior. I do not know that it has yet been noticed that this arrangement fulfils the conditions for a natural mode of decay of free currents in a thin ellipsoidal film whose conductivity (per unit area) varies as the perpendicular from the centre on the tangent plane; or, say, in a thin shell of uniform material bounded by similar and coaxial ellipsoids. This is proved in Part I. of the following paper, and we thence easily find the currents induced in such a shell when situate in a uniform magnetic field of varying intensity; or, again, the currents induced by rotation of the shell in a uniform and constant field.

I have attempted to generalise these results and to ascertain the remaining normal types of currents in a shell of the kind indicated. In Part II. is given the complete solution of this problem, including the determination of the corresponding persistencies,

* 'Phil. Trans', 1882.

† 'Brit. Assoc. Rep.', 1882.

‡ 'Phil. Trans.', 1883; 'London Math Soc. Proc', vol. 15, pp. 139 and 270

§ That of the currents of the "Second Type" in a spherical conductor Such currents cannot, however, be excited by any electromagnetic operations outside the sphere

for the case where two of the axes of the ellipsoid are equal, when the LAMÉ's functions which naturally present themselves in such an investigation reduce to spherical harmonics, and so can be handled with comparative facility The solution of the problem of *induced* currents can then be obtained in a very simple manner.

Of the special forms which the conducting shell may assume, the most interesting is that in which the third axis (that of symmetry) is infinitesimal, so that we have practically a circular disk whose resistance varies as $\sqrt{(a^2 - r^2)}$, where r is the distance of any point from the centre, and a the radius. In view of the physical interest attaching to the question, it would be desirable to have a solution for the case of a *uniform* circular disk rotating in any magnetic field; but, in the absence of this, the solution for the more special kind of disk here considered may not be uninstructive.

It appears that, except in the case of currents symmetrical about the axis, when the ellipsoid is one of revolution, there is always a surface distribution of electricity in the problems considered in this paper.

I.

1. If u', v', w', be the components of electric current at any point of a thin conducting film, F, G, H, those of electric momentum at any point (x, y, z) of space; the following conditions must be satisfied At all points external to the film we must have

$$\nabla^2 F = 0, \qquad \nabla^2 G = 0, \qquad \nabla^2 H = 0, \quad . \quad . \qquad . \quad . \quad (1)$$

where $\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2$. The functions F, G, H, are everywhere continuous, but their derivatives are discontinuous at the film, viz, we have

$$\frac{dF}{d\nu_0} + \frac{dF}{d\nu_1} = -4\pi u', \qquad \frac{dG}{d\nu_0} + \frac{dG}{d\nu_1} = -4\pi v', \qquad \frac{dH}{d\nu_0} + \frac{dH}{d\nu_1} = -4\pi w', \quad . \quad (2)$$

where $d\nu_0$, $d\nu_1$, are elements of the normal drawn from the film on the two sides. If u', v', w', satisfy the solenoidal condition over the film, these conditions ensure that

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

everywhere. The electric potential ψ satisfies the equation

$$\nabla^2 \psi = 0$$

at all points external to the film; it is everywhere continuous, but its normal derivatives may be discontinuous at either or both of the surfaces of the film.

If ρ' be the resistance of the film, per unit area, the equations of electromotive force are

$$\rho' u' = -\frac{dF}{dt} - \frac{d\psi}{dx}, \qquad \rho' v' = -\frac{dG}{dt} - \frac{d\psi}{dy}, \qquad \rho' w' = -\frac{dH}{dt} - \frac{d\psi}{dz} \quad . \quad (4)$$

In these equations ψ is supposed to have the value appropriate to the space included between the two surfaces of the film, which may differ in form from the values which it has in the external space on either side.

In any natural mode of decay the time occurs through a factor of the form $e^{-\lambda t}$, where λ is real and positive The preceding equations then become

$$\rho' u' = \lambda F - \frac{d\psi}{dx}, \qquad \rho' v' = \lambda G - \frac{d\psi}{dy}, \qquad \rho' w' = \lambda H - \frac{d\psi}{dz} \quad . \quad . \quad . \quad (5)$$

2. Let us apply this to the case of an ellipsoidal shell whose thickness varies as the perpendicular ϖ from the centre on the tangent plane, say it equals $\epsilon\varpi$, where ϵ is a small numerical constant. If ρ be the specific resistance of the material, we then have

$$\rho' = \rho/\epsilon\varpi$$

Let the semi-axes of the shell be a, b, c, and let the axes of coordinates be taken along these. In the most important type of free currents the lines of flow are in planes perpendicular to a principal axis. If this axis be that of z, the current-function over the surface of the ellipsoid is of the form

$$\phi = Cz.$$

The corresponding values of u', v', w', are

$$u' = -\frac{\varpi y}{b^2} C, \qquad v' = \frac{\varpi x}{a^2} C, \qquad w' = 0.$$

The values of F, G, H, in the internal space are

$$\left.\begin{aligned} F &= -2\pi abc\, C \,.\, y \int_0^\infty \frac{dq}{(b^2+q)\,Q} \\ G &= 2\pi abc\, C \quad x \int_0^\infty \frac{dq}{(a^2+q)\,Q} \\ H &= 0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (6)$$

where

$$Q = \{(a^2+q)(b^2+q)(c^2+q)\}^{\frac{1}{2}}.$$

The corresponding values for the external space are obtained by replacing the lower limits of the integrals by the positive root of

$$\frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} + \frac{z^2}{c^2+q} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These values of F, G, H, satisfy (1), they are continuous at the film, and their normal derivatives satisfy (2).*

We now find that the equations (5) are satisfied, provided we assume for the value of the electric potential within the substance of the film

$$\psi = Axy, \quad . \quad . \tag{8}$$

and properly determine A If we write, for shortness,

$$L = 2\pi\, abc \int_0^\infty \frac{dq}{(a^2+q)\,Q}, \quad M = 2\pi\, abc \int_0^\infty \frac{dq}{(b^2+q)\,Q}, \quad N = 2\pi\, abc \int_0^\infty \frac{dq}{(c^2+q)\,Q},$$

the equations in question reduce to

$$-\rho C/\epsilon b^2 = -\lambda MC - A,$$

$$\rho C/\epsilon a^2 = \lambda LC - A,$$

whence for the "persistency" we have

$$\tau = \lambda^{-1} = \frac{\epsilon}{\rho}\frac{L+M}{1/a^2+1/b^2} = \frac{\epsilon}{\rho}(4\pi - N)\frac{a^2b^2}{a^2+b^2} \quad . \quad . \tag{9}$$

Also

$$A = \frac{\rho}{\epsilon}\cdot\frac{La^2 - Mb^2}{(L+M)\,a^2b^2}\,C. \quad . \quad . \quad . \quad . \tag{10}$$

The value (8) of ψ will obtain throughout the internal cavity of the ellipsoid, but in the external space we shall have

$$\psi = A_1 xy \int_q^\infty \frac{dq}{(a^2+q)(b^2+q)\,Q},$$

the lower limit being defined by (7) The continuity of ψ at the outer surface of the film requires

$$A_1 = A \div \int_0^\infty \frac{dq}{(a^2+q)(b^2+q)\,Q} = 2\pi\, abc\,.\frac{a^2-b^2}{M-L}\cdot A.$$

Unless $a = b$, there will be a distribution of electricity over the *outer* surface, the density σ being determined by

$$-\frac{4\pi\sigma}{K} = \frac{d\psi}{d\nu} + \frac{d\psi}{d\nu_1}, \quad . \quad . \quad . \quad . \quad . \quad . \tag{11}$$

where K is the specific inductive capacity of the surrounding medium, and $d\nu$, $d\nu_1$,

* See, for instance, FERRERS' 'Spherical Harmonics,' chap VI.

denote elements of the normal, drawn from the surface on the two sides Since $dq/d\nu_1 = 2\varpi$, we find without difficulty

$$\frac{\sigma}{K} = \frac{\rho}{\epsilon}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\frac{La^2 - Mb^2}{L^2 - M^2}\,\frac{\varpi xy}{a^2b^2}\,C. \qquad (12)$$

3 Some particular cases of the formula (9) may be noticed For a spherical shell we have $L = \frac{4}{3}\pi$, and thence

$$\tau = \frac{4\pi}{3}\cdot\frac{\epsilon a}{\rho}\cdot a,$$

which is right For an ellipsoid of revolution ($a = b$)

$$\tau = \frac{4\pi - N}{2}\,\frac{\epsilon a}{\rho}\cdot a \qquad (13)$$

when the currents are symmetrical round the axis; whilst, in the case of currents in planes parallel to the axis (say $\phi = Cx$),

$$\tau = (4\pi - L)\frac{\epsilon a}{\rho}\cdot\frac{c^2 a}{c^2 + a^2}. \qquad (14)$$

For the prolate form we have

$$\left.\begin{aligned} L = M &= 2\pi\left(\frac{1}{e^2} - \frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right) \\ N &= 4\pi\left(\frac{1}{e^2} - 1\right)\left(\frac{1}{2e}\log\frac{1+e}{1-e} - 1\right)\end{aligned}\right\}, \qquad (15)$$

and for the oblate form

$$\left.\begin{aligned} L = M &= 2\pi\left(\frac{\sqrt{(1-e^2)}}{e^3}\arcsin e - \frac{1-e^2}{e^2}\right) \\ &= 4\pi\left(\frac{1}{e^2} - \frac{\sqrt{(1-e^2)}}{e^3}\arcsin e\right)\end{aligned}\right\}, \qquad (16)^*$$

e denoting in each case the excentricity of the meridian section

Again, if we make $c = \infty$, we get an elliptic "homœoidal" cylinder. We then have

$$L = \frac{4\pi b}{a+b}, \qquad M = \frac{4\pi a}{a+b}, \qquad N = 0; \qquad (17)$$

* Maxwell's 'Electricity,' § 438

so that for the case of currents circulating round the cylinder

$$\tau = \frac{4\pi\epsilon}{\rho} \cdot \frac{a^2b^2}{a^2 + b^2} \quad . \quad . \qquad . \quad (18)$$

The surface density of free electricity is then given by

$$\frac{4\pi\sigma}{\mathrm{K}} = \frac{\rho}{\epsilon} \cdot \frac{\text{[illegible]}}{a^3b^3} \varpi xy \,.\, \mathrm{C} \quad . \quad . \qquad . \quad . \quad (19)$$

For a circular cylinder (18) gives

$$\tau = 2\pi \cdot \frac{\epsilon a}{\rho} \cdot a,$$

which is right

For currents parallel to the axis of the cylinder (say $\phi = \mathrm{C}x$),

$$\tau = \frac{4\pi\epsilon}{\rho} \cdot \frac{ab^2}{a + b} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If in (18) or (20) we make a infinite, we get the case of two uniform parallel plane sheets at a distance $2b$ apart. The persistency of uniform parallel straight currents flowing in opposite directions in the two planes is then

$$\tau = \frac{4\pi b}{\rho'} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

4. Such special results as these may, of course, be obtained more easily by independent processes. Thus for a cylindrical shell of any form, if a current of strength C circulate round each unit length, the magnetic induction in the interior is parallel to the axis and equal to 4πC. Hence, if R be the resistance of unit length to currents circulating round it,

$$\mathrm{RC} = -\frac{d}{dt}(4\pi\mathrm{CS}),$$

where S is the area of the cross-section. This gives

$$\tau = 4\pi\mathrm{S}/\mathrm{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

For a "homœoidal" cylinder we have, if ds be an element of the elliptic contour, and ξ the "excentric angle,"

$$\rho' \, ds = \rho \, ds/\epsilon\varpi = \rho \, ab \, d\xi/\epsilon\varpi^2,$$

whence

$$R = \int \rho' ds = \frac{\pi\rho}{\epsilon} \cdot \frac{a^2 + b^2}{ab},$$

giving, of course, the same value (18) of τ as before.

If we wish to determine, not merely the persistency, but also the distribution of free electricity, we may proceed somewhat as follows. Taking the case of an elliptic cylinder, and resolving parallel to the principal axes, we have, as before,

$$u' = -\frac{\varpi y}{b^2} C, \quad v' = \frac{\varpi x}{a^2} C, \quad w' = 0,$$

and, at the surface,

$$F = -My \cdot C, \quad G = Lx \cdot C, \quad H = 0,$$

where L, M, have the values (17). Resolving in the direction of the current,

$$\begin{aligned}\rho' C &= \lambda\left(F\frac{dx}{ds} + G\frac{dy}{ds}\right) - \frac{d\psi}{ds} \\ &= \frac{4\pi\, ab}{a+b}\lambda\varpi\left(\frac{x^2}{a^3} + \frac{y^2}{b^3}\right)C - \frac{d\psi}{ds}.\end{aligned}$$

The resistance ρ' is here supposed independent of z, but is otherwise unrestricted. Introducing the excentric angle ξ, we have, since $\varpi\, ds = ab\, d\xi$,

$$\frac{d\psi}{d\xi} = -\rho' \frac{ds}{d\xi} C + \frac{4\pi ab}{a+b}\lambda\,(a\sin^2\xi + b\cos^2\xi)\, C. \quad . \quad . \qquad (23)$$

Integrating from $\xi = 0$ to $\xi = 2\pi$,

$$4\pi^2 ab\lambda = \int \rho' ds,$$

which agrees with (22). The value of ψ over the film is found by integration of (23), ρ' being supposed a known function of ξ. We can then find two functions which satisfy $\nabla^2\psi = 0$ and are finite, &c., throughout the interior and exterior spaces respectively, and coincide at the film with value just indicated. Again, within the substance of the film itself, the electromotive force in the direction of the normal must be zero. This gives

$$0 = \frac{\varpi x}{a^2}\lambda F + \frac{\varpi y}{b^2}\lambda G - \frac{d\psi}{d\delta},$$

$$\frac{d\psi}{d\delta} = 4\pi\frac{a-b}{a+b}\lambda C\frac{\varpi xy}{ab}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

where $d\delta$ is an element of the thickness of the film, directed towards the outer surface. If σ_0, σ_1, be the densities of free electricity at the inner and outer surfaces respectively, we then have

$$\left.\begin{aligned} -\frac{4\pi\sigma_0}{K} &= \frac{d\psi}{d\nu_0} + \frac{d\psi}{d\delta} \\ -\frac{4\pi\sigma_1}{K} &= \frac{d\psi}{d\nu_1} - \frac{d\psi}{d\delta} \end{aligned}\right\}, \qquad (25)$$

$d\nu_0$, $d\nu_1$, denoting, as before, elements of the normal, drawn from the film, on the inside and outside respectively.

For the case of a homœoidal cylinder this process leads to the result already obtained. For other laws of thickness there will, in general, be a distribution of electricity on *both* surfaces of the film *

5. Another case of interest is obtained by supposing c infinitesimal, so that the conductor may be taken to be an elliptic *disk* whose resistance per unit area varies according to the law

$$\rho' = \rho'_0 \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$$

In the case of a circular disk the formula (13) is replaced by

$$\tau = (4\pi - N)\frac{a^2}{4\rho'\varpi}, \dagger$$

where we may put

$$\frac{N}{4\pi} = 1 - \frac{\pi}{2}\frac{c}{a},$$

$$\varpi = \frac{ac}{\sqrt{(a^2 - r^2)}},$$

$$\rho' = \rho'_0 \sqrt{\left(1 - \frac{r^2}{a^2}\right)}$$

Hence

$$\tau = \frac{\pi^2 a}{2\rho_0'} \qquad (26)\ddagger$$

* I find, however, that in the case of a film bounded by *confocal* cylinders the *inner* surface alone becomes electrified

† The symbol ρ' here refers to the *disk* Since this is the limit of a double film, its resistance at any point is half that of the corresponding portion of the film on either side

‡ For an elliptic disk

$$N/4\pi = 1 - \frac{E_1(e)}{\sqrt{(1-e^2)}}\frac{c}{a},$$

where a is the semi-major axis, e the excentricity, and E_1 the complete elliptic integral of the second kind. This gives

$$\tau = \frac{2\pi ab}{a^2+b^2}E_1(e).\frac{a}{\rho_0'}.$$

The current at any point is proportional to

$$\frac{r}{\sqrt{(a^2 - r^2)}}.$$

It would be interesting for many reasons to have a solution for the case of a uniform disk, but at all events the above result shows that the time-constant of a disk of radius a and uniform resistance ρ' must be considerably less than $4{\cdot}93\ a/\rho'$. I find, by methods similar to those employed in Lord RAYLEIGH's 'Sound' (§§ 89, 305, &c.), that the true value lies between $\pi a/\rho'$ and $2{\cdot}26\ a/\rho'$, the latter value being probably not far removed from the truth.* For a disk of copper ($\rho = 1600$ C.G.S.) whose radius is a decimetre and thickness a millimetre this lower limit gives ·0014 sec. For disks of different dimensions the result will vary as the radius and the thickness conjointly.

6. Let us next calculate the currents induced in a homœoidal shell when situate in a uniform magnetic field $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ of varying intensity. It is sufficient to consider the case where the lines of force are parallel to a principal axis. Also the expression for the magnetic force may be supposed resolved, as regards the time, into a series of simple harmonic terms, each of which may be taken separately. Putting, then,

$$\bar{\alpha} = 0, \qquad \bar{\beta} = 0, \qquad \bar{\gamma} = \mathrm{I}\, e^{ipt},$$

and denoting by $\overline{\mathrm{F}}, \overline{\mathrm{G}}, \overline{\mathrm{H}}$, the components of vector potential due to the field, we may write

$$\overline{\mathrm{F}} = -\tfrac{1}{2}\mathrm{I}\, y\, e^{ipt}, \qquad \overline{\mathrm{G}} = \tfrac{1}{2}\mathrm{I}\, x\, e^{ipt}, \qquad \overline{\mathrm{H}} = 0.$$

The induced currents will be of the type

$$u' = -\frac{\varpi y}{b^2}\mathrm{C}\, e^{ipt}, \qquad v' = \frac{\varpi x}{a^2}\mathrm{C}\, e^{ipt}, \qquad w' = 0,$$

and the corresponding components of electric momentum at the film will be

$$\mathrm{F} = -\mathrm{MC}\, y\, e^{ipt}, \qquad \mathrm{G} = \mathrm{LC}\, x\, e^{ipt}, \qquad \mathrm{H} = 0$$

Assuming

$$\psi = \mathrm{A}xy\, e^{ipt},$$

and substituting in the equations

$$\rho' u' = -\frac{d\mathrm{F}}{dt} - \frac{d\overline{\mathrm{F}}}{dt} - \frac{d\psi}{dx}, \text{ \&c., \&c.},$$

we find

$$-\rho\mathrm{C}/\epsilon b^2 = \quad ip\,\mathrm{MC} + \tfrac{1}{2}\, ip\,\mathrm{I} - \mathrm{A},$$
$$\rho\mathrm{C}/\epsilon a^2 = -ip\,\mathrm{LC} - \tfrac{1}{2}\, ip\,\mathrm{I} - \mathrm{A},$$

whence

$$\left\{\frac{\rho}{\epsilon}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + (\mathrm{L} + \mathrm{M})\, ip\right\}\mathrm{C} = -ip\,\mathrm{I},$$

* [See 'Roy. Soc. Proc.,' vol. 42, 1887, p. 294.]

or, by (9),

$$C = \frac{-\iota p\tau}{1+\iota p\tau} \cdot \frac{I}{L+M} \quad . \quad . \qquad . \quad (27)$$

The retardation of phase of the induced currents relatively to the electromotive forces of the field is arc tan $p\tau$, as usual. When p is very great in comparison with τ^{-1} this is equal to $\pi/2$. Since the magnetic force in the interior of the shell, due to the currents alone, is given by

$$\frac{dG}{dx} - \frac{dF}{dy} = (L + M)\, C\, e^{\iota pt},$$

we see that in this case the currents just neutralise, in the interior, the magnetic action of the field, in accordance with a well-known principle

7. Take next the case where the shell rotates with constant angular velocity p about a principal axis (z) in a uniform and constant magnetic field. It is shown in MAXWELL'S 'Electricity,' § 600, that the problem is the same if we suppose the shell to be fixed, and the field to rotate in the opposite direction, provided we add to the electric potential the function

$$\psi' = p\,(y\mathrm{F} - x\mathrm{G}). \quad . \qquad . \qquad (28)$$

First let us suppose the lines of force to be perpendicular to the axis of rotation, so that we may write for the components of the field

$$\bar{\alpha} = \mathrm{I} \cos pt, \qquad \bar{\beta} = -\mathrm{I} \sin pt, \qquad \bar{\gamma} = 0\,;$$

whence

$$\bar{\mathrm{F}} = 0, \qquad \bar{\mathrm{G}} = 0, \qquad \bar{\mathrm{H}} = \mathrm{I}\,(x \sin pt + y \cos pt) = \mathrm{I}\,(y - ix)\, e^{ipt}, \quad . \quad (29)$$

if, as usual, we retain in the end only the real parts. Hence the solution of our problem follows by superposition from the results of the preceding section. Omitting the time-factor e^{ipt}, we assume for the current-function

$$\phi = \mathrm{C}x + \mathrm{D}y,$$

which gives

$$\left.\begin{aligned} u' &= \frac{\varpi z}{c^2}\,\mathrm{D} \\ v' &= -\frac{\varpi z}{c^2}\,\mathrm{C} \\ w' &= \frac{\varpi y}{b^2}\,\mathrm{C} - \frac{\varpi x}{a^2}\,\mathrm{D} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The corresponding values of F, G, H, at the film are

$$\left.\begin{aligned} F &= DNz \\ G &= -CNz \\ H &= CMy - DLx \end{aligned}\right\} \qquad (31)$$

If we further assume

$$\psi + \psi' = (Ax + By)z, \qquad (32)$$

and substitute in the equations

$$\rho' u' = -\frac{dF}{dt} - \frac{d\bar{F}}{dt} - \frac{d}{dx}(\psi + \psi'), \text{ \&c., \&c },$$

then, equating coefficients, we obtain the following four equations to determine A, B, C, D :—

$$\left.\begin{aligned} \rho/\epsilon c^2 . D &= -ip N . D - A \\ -\rho/\epsilon c^2 . C &= ip N . C - B \\ -\rho/\epsilon a^2 . D &= ip L . D - pI - A \\ \rho/\epsilon b^2 . C &= -ip M . C - ipI - B \end{aligned}\right\} \qquad (33)$$

Hence

$$\left\{\frac{\rho}{\epsilon}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + ip\,(M + N)\right\}C = -ip\,I,$$

$$\left\{\frac{\rho}{\epsilon}\left(\frac{1}{a^2} + \frac{1}{c^2}\right) + ip\,(L + N)\right\}D = pI.$$

If τ_1, τ_2, denote the persistencies of free currents of the types $\phi = Cx$, $\phi = Dy$, respectively, these equations may be written

$$\left.\begin{aligned} C &= \frac{ip\tau_1}{1 + ip\tau_1} \cdot \frac{I}{M + N} \\ D &= \frac{p\tau_2}{1 + ip\tau_2} \cdot \frac{I}{L + N} \end{aligned}\right\} \qquad (34)$$

The components of magnetic force within the shell, due to the currents alone, are found to be

$$\alpha = (M + N)\,C, \qquad \beta = (L + N)\,D, \qquad \gamma = 0,$$

so that we have for the total magnetic field inside

$$\alpha + \bar{\alpha} = \frac{I}{1 + ip\tau_1}, \qquad \beta + \bar{\beta} = \frac{iI}{1 + ip\tau_2}, \qquad \gamma + \bar{\gamma} = 0. \qquad (35)$$

These diminish indefinitely as p increases.

If we write

$$p\,\tau_1 = \tan\omega_1, \qquad p\,\tau_2 = \tan\omega_2,$$

and restore the time-factor, the expression for the current-function becomes, on discarding the imaginary part,

$$\phi = \frac{\mathrm{I}\sin\omega_1}{\mathrm{M}+\mathrm{N}}\,x\sin(pt-\omega_1) + \frac{\mathrm{I}\sin\omega_2}{\mathrm{L}+\mathrm{N}}\,y\cos(pt-\omega_2) \quad . \quad . \quad . \qquad (36)$$

The currents flow at any instant in a system of ellipses whose planes are parallel to one another and to the axis of rotation. When the ellipsoid is one of revolution about z we have $\mathrm{L} = \mathrm{M}$, $\omega_1 = \omega_2$ The planes of the currents are dragged round, as it were, in the direction of the rotation of the shell, through a constant angle ω_1 from the direction of the magnetic force in the inducing field, in accordance with a general principle pointed out at the beginning of this paper.*

8. When the lines of force are parallel to the axis of rotation there are no induced currents, but only a superficial distribution of electricity The calculation of this distribution involves assumptions which vary with the particular theory of electromagnetism adopted; and even MAXWELL's theory has been differently interpreted in this respect by different writers. It may be well, therefore, to state with some care the view here taken.

Considering, for the sake of simplicity, the case of a solid conductor rotating in a field of uniform intensity γ about an axis (z) parallel to the lines of force, and supposing the axes of x, y, to move with the solid, then, on the hypothesis that there are no currents, we have, throughout the interior,

$$\left.\begin{aligned} 0 &= \gamma\,.\,px - \frac{d\psi}{dx} \\ 0 &= \gamma\,.\,py - \frac{d\psi}{dy} \\ 0 &= \qquad\;\; - \frac{d\psi}{dz} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

whilst in the surrounding dielectric (taken to be sensibly at rest)—

* The case of a *spherical* shell has been discussed by C. NIVEN (*loc. cit.*) and J. LARMOR, 'Phil Mag.,' Jan , 1884

MAXWELL has considered the currents induced by rotation of a *solid* ellipsoid (see STEWART and TAIT, 'Roy Soc. Proc.,' vol. 15, 1867, p 291), leaving out of account, however, the mutual action of the currents themselves This is equivalent to supposing the period of rotation to be long in comparison with the modulus of decay of free currents

$$\left.\begin{aligned}\frac{4\pi f}{\mathrm{K}} &= -\frac{d\psi}{dx}\\ \frac{4\pi g}{\mathrm{K}} &= -\frac{d\psi}{dy}\\ \frac{4\pi h}{\mathrm{K}} &= -\frac{d\psi}{dz}\end{aligned}\right\}, \qquad (38)$$

where f, g, h, are the components of dielectric polarisation, and K the specific inductive capacity. We have also the solenoidal condition

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0. \qquad (39)$$

It is to be carefully borne in mind that nothing is known of the function ψ beyond what is contained in these equations, except that it is everywhere continuous. The familiar *electrostatic* relations of ψ may be deduced from these equations by putting $p=0$. In the present problem we have

$$\psi = \tfrac{1}{2} p\,\gamma\,(x^2+y^2) + \text{const.} \qquad (40)$$

throughout the conductor, whilst in the external space ψ satisfies the equation

$$\nabla^2\psi = 0,$$

with the conditions that its value at the surface shall agree with (40) and its first derivatives vanish at infinity. The surface density σ is then given by

$$\sigma = lf + mg + nh = -\frac{\mathrm{K}}{4\pi}\frac{d\psi}{d\nu_1}, \qquad (41)$$

where l, m, n, are the direction-cosines of the element $d\nu_1$ of the normal drawn outwards

The solution of this problem for an ellipsoidal conductor is obtained by an adaptation of the results given by FERRERS ('Spherical Harmonics,' chapter 6, §§ 29, 30). I do not think it worth while to transcribe these, as the result for the more specially interesting case of an ellipsoid of revolution, and in particular for a circular *disk*, is given below in § 15.

II.

9. The result of § 2 can be generalised, and it can be shown that the different normal types of free currents in a homœoidal shell are obtained by equating the current-function ϕ to the LAMÉ's functions of various orders. But it may be sufficient here to consider the case where two of the axes of the shell are equal, when the functions in question reduce to spherical harmonics.

Taking first the case where the ellipsoid is of the *prolate* form, we transform to elliptic coordinates (ζ, μ, ω) or (η, θ, ω) by writing

$$\left.\begin{aligned} x &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2-1)}\cos\omega &&= k\sin\theta\sinh\eta\cos\omega \\ y &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2-1)}\sin\omega &&= k\sin\theta\sinh\eta\sin\omega \\ z &= k\zeta\mu &&= k\cos\theta\cosh\eta \end{aligned}\right\}, \quad . \qquad (42)$$

the axis of z being that of symmetry. The value of μ may range from -1 to $+1$, that of ζ from 1 to ∞. The surfaces $\zeta = \text{const}$ are confocal ellipsoids of revolution, whose semi-axes are

$$\begin{aligned} a = b &= k\sqrt{(\zeta^2-1)} = k\sinh\eta, \\ c &= k\zeta \qquad\quad = k\cosh\eta, \end{aligned}$$

the distance between the common foci being $2k$ The value of ζ for the surface of the shell will be distinguished, where necessary, by ζ_0. The perpendicular on the tangent plane at any point of the shell is

$$\varpi = k\frac{\zeta_0\sqrt{(\zeta_0^2-1)}}{\sqrt{(\zeta_0^2-\mu^2)}} \quad . \qquad . \quad (43)$$

LAPLACE's equation $\nabla^2 V = 0$ transforms into

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2V}{d\omega^2} = \frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dV}{d\zeta}\right\} + \frac{1}{1-\zeta^2}\frac{d^2V}{d\omega^2}. \quad . \quad . \quad (44)$$

Considered as a function of μ, ω, V may be expanded in a series of spherical harmonics whose coefficients are functions of ζ, and it is easily seen that each term of the expansion must separately satisfy (44) Taking first the case of the zonal harmonic, if we put

$$V = P_n(\mu) . Z,$$

where

$$P_n(\mu) = \frac{1.3.5 \quad (2n-1)}{\lfloor n}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\mu^{n-4} - \ldots\right\}, \quad (45)$$

we find

$$\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dZ}{d\zeta}\right\} + n(n+1)Z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

showing that Z must be a zonal harmonic of order n, of the first or second kind. We thus obtain the solutions

$$\left.\begin{aligned} \mathrm{V} &= \mathrm{P}_n(\mu) \,.\, \mathrm{P}_n(\zeta)^* \\ \mathrm{V} &= \mathrm{P}_n(\mu) \,.\, \mathrm{Q}_n(\zeta) \end{aligned}\right\}, \qquad (47)$$

where Q_n denotes the zonal harmonic of the second kind, viz. —

$$\begin{aligned} \mathrm{Q}_n(\zeta) &= \mathrm{P}_n(\zeta)\int_\zeta^\infty \frac{d\zeta}{\{\mathrm{P}_n(\zeta)\}^2(\zeta^2-1)}, \\ &= \tfrac{1}{2}\mathrm{P}_n(\zeta)\log\frac{\zeta+1}{\zeta-1} - \frac{2n-1}{1\,n}\mathrm{P}_{n-1}(\zeta) - \frac{2n-5}{3(n-1)}\mathrm{P}_{n-3}(\zeta) - \ldots, \\ &= \frac{\lfloor n}{1\,3\,\ldots(2n+1)}\left\{\zeta^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}\zeta^{-n-3}\right. \\ &\qquad \left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2\,4(2n+3)(2n+5)}\zeta^{-n-5} + \ldots\right\} \qquad (48) \end{aligned}$$

The former of the solutions (47) is appropriate to the space inside the shell, the latter to the external space.

In like manner, when V involves ω, we have the solutions

$$\left.\begin{aligned} \mathrm{V} &= (1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{P}_n(\zeta)}{d\zeta^s}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}s\omega \\ \mathrm{V} &= (1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{Q}_n(\zeta)}{d\zeta^s}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}s\omega \end{aligned}\right\} \qquad (49)†$$

10. Proceeding now to the problem of free currents, we shall show that the conditions for a normal type are satisfied whenever ϕ, considered as a function of μ and ω,

* The following are the values of the first four solid harmonics of this type, expressed in terms of x, y, z.—

$$k\mathrm{P}_1(\mu)\,\mathrm{P}_1(\zeta) = z,$$
$$k^2\mathrm{P}_2(\mu)\,\mathrm{P}_2(\zeta) = \tfrac{1}{4}\{6z^2 - 3(x^2+y^2) - 2k^2\},$$
$$k^3\mathrm{P}_3(\mu)\,\mathrm{P}_3(\zeta) = \tfrac{1}{4}z\{10z^2 - 15(x^2+y^2) - 6k^2\},$$
$$k^4\mathrm{P}_4(\mu)\,\mathrm{P}_4(\zeta) = \tfrac{1}{64}\{280z^4 - 840z^2(x^2+y^2) + 105(x^2+y^2)^2 - 240k^2z^2 + 120k^2(x^2+y^2) + 24k^4\}$$

† I have here only recapitulated, for purposes of reference, the principal steps in the deduction of the solutions (47), (49) For details see HEINE, 'Kugelfunctionen,' vol 2, part ii., chap. ii., or FERRERS, 'Spherical Harmonics,' chap vi.

The following are the values of a few of the more important solid harmonics of the form given by the first of equations (49) —

$n=1$,	$s=1$.	x,	y.
$n=2$,	$s=1$:	$9xz$,	$9yz$.
$n=2$,	$s=2$:	$9(x^2-y^2)$,	$18xy$
$n=3$,	$s=1$:	$\frac{3}{2}x\{20z^2-5(x^2+y^2)-4k^2\}$,	$\frac{3}{2}y\{20z^2-5(x^2+y^2)-4k^2\}$.
$n=3$,	$s=2$:	$225z(x^2-y^2)$,	$450xyz$.
$n=3$,	$s=3$:	$225(x^3-3xy^2)$,	$225(3x^2y-y^3)$
$n=4$,	$s=1$:	$\frac{35}{4}xz\{28z^2-21(x^2+y^2)-12k^2\}$,	$\frac{35}{4}yz\{28z^2-21(x^2+y^2)-12k^2\}$

is a zonal, tessaral, or sectorial harmonic of integral order Since any arbitrary value of ϕ can be expanded in a series of such harmonics, the results thus obtained will enable us to represent the decay of any initial distribution of current whatever.

If ds_μ, ds_ω, be linear elements drawn on the surface along a meridian and a parallel of latitude respectively, viz.,

$$\left.\begin{aligned} ds_\mu &= k\frac{\sqrt{(\zeta_0^2-\mu^2)}}{\sqrt{(1-\mu^2)}}\,d\mu \\ ds_\omega &= k\sqrt{(\zeta_0^2-1)}\,\sqrt{(1-\mu^2)}\,d\omega \end{aligned}\right\}, \quad . \quad . \quad . \tag{50}$$

the current may be resolved into the components

$-\dfrac{d\phi}{ds_\omega}$ along the meridian, towards the positive pole; and

$\dfrac{d\phi}{ds_\mu}$ along the parallel, in the direction of ω increasing.

Take, first, the case of the zonal harmonic

$$\phi = \mathrm{C}\,.\,\mathrm{P}_n(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{51}$$

The currents then flow in circles round the axis of z, the strength of the current at any point being

$$\mathrm{C}\,\frac{\varpi}{k^2\zeta_0\sqrt{(\zeta_0^2-1)}}\sqrt{(1-\mu^2)}\,\frac{d\mathrm{P}_n(\mu)}{d\mu}. \quad . \quad . \quad . \quad . \tag{52}$$

If Ω be the magnetic potential due to these currents, we have

$$\nabla^2\,\Omega = 0,$$

with the conditions that at the surface ζ_0

$$\Omega_1 = \Omega_0 + 4\pi\phi,$$

(the suffixes denoting the values on the two sides), whilst the normal derivative is continuous. Assuming

$$\left.\begin{aligned} \Omega_0 &= \mathrm{A}\,\mathrm{P}_n(\mu)\,\mathrm{P}_n(\zeta) \\ \Omega_1 &= \mathrm{B}\,\mathrm{P}_n(\mu)\,\mathrm{Q}_n(\zeta) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \tag{53}$$

the surface conditions give

$$\begin{aligned} \mathrm{B}\,\mathrm{Q}_n(\zeta_0) &= \mathrm{A}\,\mathrm{P}_n(\zeta_0) + 4\pi\mathrm{C}, \\ \mathrm{B}\,\mathrm{Q}_n'(\zeta_0) &= \mathrm{A}\,\mathrm{P}_n'(\zeta_0); \end{aligned}$$

whence, in virtue of the relation

$$\mathrm{P}_n'(\zeta)\,\mathrm{Q}_n(\zeta) - \mathrm{P}_n(\zeta)\,\mathrm{Q}_n'(\zeta) = \frac{1}{\zeta^2-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{54}$$

we find

$$\left.\begin{aligned} A &= 4\pi(\zeta_0^2 - 1)\, Q_n'(\zeta_0)\,.\,C \\ B &= 4\pi(\zeta_0^2 - 1)\, P_n'(\zeta_0)\,.\,C \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (55)^*$$

Owing to the symmetry about the axis, there will be no difference of electric potential, and the electromotive force of induction will be everywhere in the direction of the current The magnetic induction across any element of the surface will be

$$-\frac{d\Omega}{d\nu_1}\, ds_\mu\, ds_\omega = -\frac{\varpi}{k^2\zeta_0}\frac{d\Omega}{d\zeta}\, ds_\mu\, ds_\omega$$

$$= -k(\zeta_0^2 - 1)\frac{d\Omega}{d\zeta}\, d\mu\, d\omega. \qquad . \quad . \quad . \quad . \quad (56)$$

Substituting from (53), and integrating over the portion of the surface bounded by a parallel of latitude and including the positive pole, we find for the total induction through the parallel

$$-8\pi^2 k(\zeta_0^2 - 1)^2\, P_n'(\zeta_0)\, Q_n'(\zeta_0)\int_\mu^1 P_n(\mu)\, d\mu\,.\,C.$$

If the system of currents defined by (51) remain always similar to itself, the electromotive force round a parallel is equal to $-d/dt$ of this. The electromotive force at a point is derived by dividing by $2\pi k\sqrt{(\zeta_0^2 - 1)}\sqrt{(1 - \mu^2)}$. Since

$$\int_\mu^1 P_n(\mu)\, d\mu = \frac{1 - \mu^2}{n(n+1)}\frac{dP_n(\mu)}{d\mu},$$

we obtain in this way

$$\frac{4\pi}{n(n+1)}(\zeta_0^2 - 1)^{\frac{3}{2}}\, P_n'(\zeta_0)\, Q_n'(\zeta_0)\sqrt{(1 - \mu^2)}\,\frac{dP_n(\mu)}{d\mu}\,.\,\frac{dC}{dt}. \qquad . \quad . \quad (57)$$

Equating this to

$$\rho'\frac{d\phi}{ds_\mu},$$

where $d\phi/ds_\mu$ has the value (52), and $\rho' = \rho/\epsilon\varpi$, we find that all the conditions of the problem are satisfied, provided

$$\frac{dC}{dt} + \frac{C}{\tau} = 0,$$

where

$$\tau = -\frac{4\pi k^2}{n(n+1)}\frac{\epsilon}{\rho}\zeta_0(\zeta_0^2 - 1)^2\, P_n'(\zeta_0)\, Q_n'(\zeta_0). \qquad . \quad . \quad . \quad . \quad (58)$$

* We may prove in a similar manner that in the case of an ellipsoidal shell with unequal axes magnetised everywhere in the direction of the normal so that the "strength" ϕ is proportional to a Lamé's function, the potential on either side is everywhere proportional to ϕ.

Since

$$P_1(\zeta) = \zeta,$$
$$Q_1(\zeta) = \frac{\zeta}{2}\log\frac{\zeta+1}{\zeta-1} - 1,$$

this gives, for $n = 1$,

$$\tau = \frac{2\pi k^2\epsilon}{\rho}\zeta_0(\zeta_0^2-1)^2\left\{\frac{\zeta_0}{\zeta_0^2-1} - \tfrac{1}{2}\log\frac{\zeta_0+1}{\zeta_0-1}\right\};$$

or, putting

$$a = k\sqrt{(\zeta_0^2-1)}, \qquad \zeta_0 = 1/e,$$
$$\tau = \frac{2\pi a^2\epsilon}{\rho}\left\{\frac{1}{e^2} - \frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right\}, \quad . \quad . \quad . \qquad (59)$$

which agrees with (13).

For $n = 2$, I find

$$\tau = \frac{2\pi a^2\epsilon}{\rho}\left\{\frac{3-2e^2}{e^4} - \tfrac{3}{2}\frac{1-e^2}{e^5}\log\frac{1+e}{1-e}\right\}. \quad . \quad . \quad . \quad . \qquad (60)$$

The case of a *spherical* shell may be deduced from (58) by making $k = 0$, $\zeta = \infty$, $k\zeta = a$. For large values of ζ we need only retain the first terms of the series in (45), (48). In this way we reproduce the known result

$$\tau = \frac{4\pi a^2\epsilon}{(2n+1)\rho}.$$

11 The simplest plan of dealing with the case where the current-function is a tessaral or sectorial harmonic is to consider the current round any infinitely small circuit bounded by meridians and parallels. If R, S, be the components of electric momentum along a meridian and a parallel respectively, we have

$$\left.\begin{aligned} -\rho'\frac{d\phi}{ds_\omega} &= -\frac{dR}{dt} - \frac{d\psi}{ds_\mu} \\ \rho'\frac{d\phi}{ds_\mu} &= -\frac{dS}{dt} - \frac{d\psi}{ds_\omega} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

or, putting

$$\rho' = \frac{\rho}{\epsilon\varpi} = \frac{\rho}{k\epsilon}\cdot\frac{\sqrt{(\zeta_0^2-\mu^2)}}{\zeta_0\sqrt{(\zeta_0^2-1)}},$$

$$\left.\begin{aligned} -\frac{\rho}{k\epsilon}\cdot\frac{\zeta_0^2-\mu^2}{\zeta_0(\zeta_0^2-1)(1-\mu^2)}\frac{d\phi}{d\omega} &= -\frac{dR}{dt}\frac{ds_\mu}{d\mu} - \frac{d\psi}{d\mu} \\ \frac{\rho}{k\epsilon\zeta_0}\cdot(1-\mu^2)\frac{d\phi}{d\mu} &= -\frac{dS}{dt}\frac{ds_\omega}{d\omega} - \frac{d\psi}{d\omega} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (62)$$

Eliminating ψ,

$$\frac{\rho}{k\epsilon\zeta_0}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\phi}{d\mu}\right\} + \frac{\zeta_0^2-\mu^2}{(\zeta_0^2-1)(1-\mu^2)}\frac{d^2\phi}{d\omega^2}\right]$$
$$= \frac{d}{dt}\left\{\frac{d}{d\omega}\left(R\frac{ds_\mu}{d\mu}\right) - \frac{d}{d\mu}\left(S\frac{ds_\omega}{d\omega}\right)\right\}. \quad . \quad . \qquad (63)$$

If we now assume

$$\phi = \mathrm{C}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\sin s\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

the first member of (63) reduces to

$$-\frac{\rho}{k\epsilon\zeta_0}\left\{n(n+1)+\frac{s^2}{\zeta_0^2-1}\right\}\phi, \quad . \quad . \quad . \quad . \quad . \quad (65)$$

by the differential equation of tessaral harmonics Again

$$\frac{d}{d\omega}\left(\mathrm{R}\frac{ds_\mu}{d\mu}\right)-\frac{d}{d\mu}\left(\mathrm{S}\frac{ds_\omega}{d\omega}\right)=-k(\zeta_0^2-1)\frac{d\Omega}{d\zeta}, \qquad (66)$$

where Ω is the magnetic potential due to the currents (64); for, if we multiply each member by $d\mu\, d\omega$, the left-hand side is equal to the line-integral of the electric momentum round an infinitesimal circuit, and the right-hand side gives the magnetic induction through the circuit. To find Ω, we assume

$$\left.\begin{aligned}\Omega_0 &= \mathrm{A}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{P}_n(\zeta)}{d\zeta^s}\sin s\omega \\ \Omega_1 &= \mathrm{B}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{Q}_n(\zeta)}{d\zeta^s}\sin s\omega\end{aligned}\right\} \quad . \quad . \quad (67)$$

for the spaces inside and outside the shell respectively. If we write, for shortness,

$$\left.\begin{aligned}\mathrm{T}_n^s(\zeta) &= (\zeta^2-1)^{s/2}\frac{d^s\mathrm{P}_n(\zeta)}{d\zeta^s} \\ \mathrm{U}_n^s(\zeta) &= (\zeta^2-1)^{s/2}\frac{d^s\mathrm{Q}_n(\zeta)}{d\zeta^s}\end{aligned}\right\}, \quad . \quad . \quad . \qquad (68)$$

the conditions to be satisfied at the surface of the shell give

$$\begin{aligned}\mathrm{BU}_n^s(\zeta_0) &= \mathrm{AT}_n^s(\zeta_0)+4\pi\mathrm{C}, \\ \mathrm{B}\frac{d\mathrm{U}_n^s(\zeta_0)}{d\zeta_0} &= \mathrm{A}\frac{d\mathrm{T}_n^s(\zeta_0)}{d\zeta_0};\end{aligned}$$

whence

$$\left.\begin{aligned}\mathrm{A} &= (-)^s 4\pi\frac{\lfloor n-s}{\lfloor n+s}(\zeta_0^2-1)\frac{d\mathrm{U}_n^s}{d\zeta_0}\mathrm{C} \\ \mathrm{B} &= (-)^s 4\pi\frac{\lfloor n-s}{\lfloor n+s}(\zeta_0^2-1)\frac{d\mathrm{T}_n^s}{d\zeta_0}\mathrm{C}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (69)$$

the reduction being effected by the formula

$$U_n^s \frac{dT_n^s}{d\zeta} - T_n^s \frac{dU_n^s}{d\zeta} = (-)^s \frac{\underline{|n+s}}{\underline{|n-s}} \frac{1}{\zeta^2 - 1}. \qquad (70)^*$$

Hence the right-hand side of (63)

$$= -k(\zeta_0^2 - 1) \frac{d}{dt} \frac{d\Omega}{d\zeta},$$

$$= -k(\zeta_0^2 - 1) \frac{dA}{dt} \frac{dT_n^s}{d\zeta_0} (1-\mu^2)^{s/2} \frac{d^s P_n(\mu)}{d\mu^s} \sin s\omega,$$

$$= (-)^{s-1} 4\pi k \frac{\underline{|n-s}}{\underline{|n+s}} (\zeta_0^2 - 1)^2 \frac{dT_n^s}{d\zeta_0} \frac{dU_n^s}{d\zeta_0} \cdot \frac{d\phi}{dt} \qquad (71)$$

by (64) and (69). Equating (65) and (71), we see that the assumption (64) satisfies the conditions of a normal type, and that the corresponding modulus of decay is

$$\tau = (-)^{s-1} \frac{4\pi k^2 \epsilon}{\rho} \frac{\underline{|n-s}}{\underline{|n+s}} \frac{\zeta_0(\zeta_0^2 - 1)^3}{n(n+1)(\zeta_0^2 - 1) + s^2} \frac{dT_n^s(\zeta_0)}{d\zeta_0} \frac{dU_n^s(\zeta_0)}{d\zeta_0}. \qquad (72)$$

The accuracy of this result may be tested by putting $k = 0$, $\zeta_0 = \infty$, $k\zeta_0 = a$, when we obtain the correct result

$$\frac{4\pi a^2 \epsilon}{(2n+1)\rho}$$

for the case of a spherical shell.

The case of the *sectorial* harmonic is obtained by putting $s = n$ in (72). When $n = 1$, in which case $\phi \propto y$, we obtain

$$\tau = \frac{2\pi k^2 \epsilon}{\rho} \frac{\zeta_0^2(\zeta_0^2 - 1)^2}{2\zeta^2 - 1} \left\{ \frac{\zeta_0}{2} \log \frac{\zeta_0 + 1}{\zeta_0 - 1} - \frac{\zeta_0^2 - 2}{\zeta_0^2 - 1} \right\},$$

or, writing

$$\zeta_0 = 1/e, \qquad k^2(\zeta_0^2 - 1) = a^2,$$

$$\tau = \frac{2\pi a^2 \epsilon}{\rho} \frac{1}{2 - e^2} \left\{ \frac{1 - e^2}{2e^3} \log \frac{1+e}{1-e} - \frac{1 - 2e^2}{e^2} \right\},$$

which will be found to agree with (14).

The results (58) and (72) were originally obtained by a method more analogous to that of § 2, the currents and the electric momentum being resolved parallel to x, y, z. This method is much longer than that here given, and involves the determination of

* TODHUNTER, 'Functions of LAPLACE, &c.,' § 109.

the electric potential ψ. It may be worth while, however, to record the value of ψ thus obtained for the space included between the two surfaces of the film, viz —

$$\psi = \mathrm{D}\,(\mathfrak{T}_{n+1}^s - \mathfrak{T}_{n-1}^s)\cos s\omega + \mathrm{E}\,(\mathfrak{U}_{n+1}^s - \mathfrak{U}_{n-1}^s)\cos s\omega, \qquad (73)$$

where

$$\mathfrak{T}_n^s = \frac{\underline{|n-s}}{\underline{|n+s}}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{P}_n(\zeta)}{d\zeta^s},$$

$$\mathfrak{U}_n^s = \frac{\underline{|n-s}}{\underline{|n+s}}(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}(\zeta^2-1)^{s/2}\frac{d^s\mathrm{Q}_n(\zeta)}{d\zeta^s},$$

and D, E, are certain constants The first part of (73) corresponds to a distribution of electricity on the *outer* surface of the film, the second to a distribution on the *inner* surface. In the case of the sectorial harmonic, $\mathrm{E} = 0$.

12. When the shell is of the *oblate* form, the elliptic coordinates to be employed are as follows. We write

$$\left.\begin{aligned} x &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2+1)}\cos\omega = k\sin\theta\cosh\eta\cos\omega \\ y &= k\sqrt{(1-\mu^2)}\sqrt{(\zeta^2+1)}\sin\omega = k\sin\theta\cosh\eta\sin\omega \\ z &= k\mu\zeta \qquad\qquad\qquad\qquad\quad\;\; = k\cos\theta\sinh\eta \end{aligned}\right\}, \qquad (74)$$

where ζ may range from 0 to ∞. The surfaces $\zeta = \text{const.}$ are confocal ellipsoids of revolution, the extreme case $\zeta = 0$ being a circular disk of radius k in the plane xy. Comparing these equations with (42), we see that they may be obtained by writing $i\zeta$ for ζ and $-ik$ for k. The equation $\nabla^2\mathrm{V} = 0$ therefore becomes

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\mathrm{V}}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2\mathrm{V}}{d\omega^2} = -\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\mathrm{V}}{d\zeta}\right\} + \frac{1}{1+\zeta^2}\frac{d^2\mathrm{V}}{d\omega^2}. \qquad (75)$$

The type of solutions symmetrical about the axis is

$$\mathrm{V} = \mathrm{P}_n(\mu)\,.\,\mathrm{Z},$$

where

$$\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\mathrm{Z}}{d\zeta}\right\} - n(n+1)\,\mathrm{Z} = 0. \qquad (76)$$

One integral of this is

$$p_n(\zeta) = \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{\zeta^n + \frac{n(n-1)}{2(2n-1)}\zeta^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\zeta^{n-4} + \ldots\right\}; \qquad (77)^*$$

* See FERRERS, 'Spherical Harmonics,' chap. VI.

this becomes infinite for $\zeta = \infty$. The second solution is

$$
\begin{aligned}
q_n(\zeta) &= p_n(\zeta) \int_\zeta^\infty \frac{d\zeta}{\{p_n(\zeta)\}^2 (\zeta^2+1)}, \\
&= (-)^n \left\{ p_n(\zeta) \operatorname{arc\,cot} \zeta - \frac{2n-1}{1 \; n} p_{n-1}(\zeta) + \frac{2n-5}{3\,(n-1)} p_{n-3}(\zeta) - \dots \right\}, \\
&= \frac{\lfloor n}{1 \;\; 3 \;\; . (2n+1)} \left\{ \zeta^{-n-1} - \frac{(n+1)(n+2)}{2\,(2n+3)} \zeta^{-n-3} \right. \\
&\qquad \left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \;\; 4\,(2n+3)(2n+5)} \zeta^{-n-5} - \dots \right\}, \quad . \quad (78)
\end{aligned}
$$

the latter expansion, however, being only convergent when $\zeta > 1$. This function $q_n(\zeta)$ vanishes at infinity.

Hence we have the following solutions of (75)

$$
\left.\begin{aligned}
V &= P_n(\mu) \;\; p_n(\zeta) \\
V &= P_n(\mu) \,.\, q_n(\zeta)
\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)
$$

the former being appropriate to the space inside the ellipsoid, the latter to the external space.*

In like manner, when V involves ω we have the solutions

$$
\left.\begin{aligned}
V &= (1-\mu^2)^{s/2} \frac{d^s P_n(\mu)}{d\mu^s} (\zeta^2+1)^{s/2} \frac{d^s p_n(\zeta)}{d\zeta^s} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} s\omega \\
V &= (1-\mu^2)^{s/2} \frac{d^s P_n(\mu)}{d\mu^s} (\zeta^2+1)^{s/2} \frac{d^s q_n(\zeta)}{d\zeta^s} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} s\omega
\end{aligned}\right\} \quad . \quad . \quad (80)
$$

It seems unnecessary to go through the details of the investigations corresponding to those of §§ 10, 11. The results are, for the zonal harmonic,

$$
\begin{aligned}
\phi &= C \,.\, P_n(\mu), \\
\tau &= -\frac{4\pi k^2}{n(n+1)} \frac{\epsilon}{\rho} \zeta_0 (\zeta_0^2+1)^2 \frac{dp_n(\zeta_0)}{d\zeta_0} \frac{dq_n(\zeta_0)}{d\zeta_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)
\end{aligned}
$$

and, for the tessaral harmonic,

$$
\begin{aligned}
\phi &= C \,.\, (1-\mu^2)^{s/2} \frac{d^s P_n(\mu)}{d\mu^s} \sin s\omega, \\
\tau &= (-)^{s-1} \frac{4\pi k^2 \epsilon}{\rho} \frac{\lfloor n-s}{\lfloor n+s} \frac{\zeta_0 (\zeta_0^2+1)^2}{n(n+1)(\zeta_0^2+1) - s^2} \frac{dt_n^s(\zeta_0)}{d\zeta_0} \cdot \frac{du_n^s(\zeta_0)}{d\zeta_0}, \quad . \quad . \quad . \quad (82)
\end{aligned}
$$

* The second solution is finite even when $\zeta = 0$, but its space derivatives are infinite at the *focal circle* $x^2 + y^2 = k^2$, $z = 0$

where

$$t_n^s(\zeta) = (\zeta^2+1)^{s/2}\frac{d^s p_n(\zeta)}{d\zeta^s},$$

$$u_n^s(\zeta) = (\zeta^2+1)^{s/2}\frac{d^s q_n(\zeta)}{d\zeta^s}$$

When $n = 1$, (81) becomes

$$\tau = \frac{2\pi k^3\epsilon}{\rho}\zeta_0(\zeta_0^2+1)^2\left\{\operatorname{arc}\cot\zeta_0 - \frac{\zeta_0}{\zeta_0^2+1}\right\};$$

or, putting

$$k\sqrt{(\zeta_0^2+1)} = a, \qquad \zeta_0^2+1 = 1/e^2,$$

$$\tau = \frac{2\pi a^2\epsilon}{\rho}\left\{\frac{\sqrt{(1-e^2)}}{e^3}\operatorname{arc}\sin e - \frac{1-e^2}{e^2}\right\}, \quad . \quad . \quad . \quad (83)$$

which agrees with (13).

13. If we make ζ_0 infinitesimal, we get results applicable to a circular disk, of the kind considered in § 5, provided $n - s$ be *odd*. Putting $k = a$, $k\zeta_0 = c$, and

$$\rho/\epsilon = 2\rho'\varpi = 2\rho_0'c,$$

where c is ultimately made to vanish, the formulæ for the symmetrical currents become

$$\phi = \mathrm{C}\,.\,\mathrm{P}_n\left\{\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (84)$$

$$\tau = -\frac{2\pi a}{n(n+1)\rho_0'}\left[p_n'(\zeta)\,q_n'(\zeta)\right]_{\zeta=0}, \quad . \quad . \quad . \quad . \quad . \quad (85)$$

where r denotes the distance of any point of the disk from the centre. For small values of ζ we have, n being odd,

$$p_n'(\zeta) = \frac{1\,.\,3\,.\,.\,n}{2\,.\,4\,.\,.\,(n-1)}, \qquad q_n' = -\frac{\pi}{2}p_n',$$

whence

$$\tau = \frac{n+1}{n}\frac{\pi^2 a}{\rho_0'}\left\{\frac{1\,.\,3\,.\,.\,n}{2\,.\,4\,.\,.\,(n+1)}\right\}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

For $n = 1$, this gives

$$\tau = \pi^2 a/2\rho_0',$$

as in § 5. The persistencies of the successive symmetrical types are as

$$1, \tfrac{3}{8}, \tfrac{15}{64}, \&c.$$

For the tessaral harmonic

$$\phi = \mathrm{C}\frac{r^s}{a^s}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\sin s\omega,* \quad . \quad . \quad . \quad . \quad . \quad (87)$$

where

$$\mu = \sqrt{\left(1 - \frac{r^2}{a^2}\right)},$$

we have

$$\tau = (-)^{s-1}\frac{2\pi a}{\rho_0'}\frac{|\underline{n-s}}{|\underline{n+s}}\frac{1}{n(n+1)-s^2}\left[\frac{dt_n^s}{d\zeta}\cdot\frac{du_n^s}{d\zeta}\right]_{\zeta=0}.$$

From the series (77) we find, without difficulty,

$$\left[\frac{dt_n^s}{d\zeta}\right]_{\zeta=0} = \frac{1 \cdot 3 \ldots (n+s)}{2 \cdot 4 \ldots (n-s-1)},$$

$n - s$ being *odd*. Also from the second line of (78), under the same restriction,

$$\frac{du_n^s}{d\zeta} = (-)^n\frac{\pi}{2}\frac{dt_n^s}{d\zeta},$$

when $\zeta = 0$ Hence

$$\tau = \frac{\pi^2 a}{\{n(n+1)-s^2\}\rho_0'}\cdot\frac{|\underline{n-s}}{|\underline{n+s}}\cdot\left\{\frac{1 \cdot 3 \ldots (n+s)}{2 \cdot 4 \ldots (n-s-1)}\right\}^2. \quad . \quad (88)$$

The most important of the types (87) is that in which $n = 2$, $s = 1$, when

$$\tau = \tfrac{3}{10}\pi^2 a/\rho_0'.$$

14. The methods of §§ 10, 11, might be applied to determine the currents induced by simple harmonic variations of a magnetic field, but it is unnecessary to go through the calculations, as the result can be written down at once from the following considerations.

We must first suppose the magnetic potential ($\overline{\Omega}$, say) due to the field to be expanded, for the space near the conductor, in a series of terms of the forms given by the first lines of (47) and (49)†; or of (79) and (80), as the case may be. Each of these terms will act by itself, and produce a current-function ϕ of the type (51) or (64) Now, in § 11, the equation of free currents of any normal type was brought to the form

$$\mathrm{I}\rho\phi = \mathrm{J}\frac{d\phi}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

* The current lines are the orthogonal projections on the plane xy of the contour lines of spherical harmonics drawn on a sphere of radius a

† F E NEUMANN has shown ('Crelle,' vol 37) how to expand the potential of a single magnetic pole in this way

the left-hand side being obtained as the electromotive force necessary to balance the resistance, and the right-hand side as the electromotive force of induction due to the decay of the currents. If τ be the modulus of decay of the type in question,

$$\tau = -\mathrm{J}/\mathrm{I}\rho.$$

Now let $\bar{\phi}$ represent a fictitious distribution of current over the ellipsoid, which shall have the same magnetic effect in the interior as the actual inducing field. This distribution is found at once from (69). The equation of induced currents will then be

$$\mathrm{I}\rho\phi = \mathrm{J}\left(\frac{d\phi}{dt} + \frac{d\bar{\phi}}{dt}\right)$$

or

$$\phi = -\tau\left(\frac{d\phi}{dt} + \frac{d\bar{\phi}}{dt}\right). \quad \quad (90)$$

When the free currents have died away all our functions will vary as e^{ipt}, where p measures the rapidity of the changes in the field. Substituting in (90), we find

$$\phi = \frac{-ip\tau}{1 + ip\tau}\bar{\phi}. \quad \quad (91)$$

When $p\tau$ is very great this becomes

$$\phi = -\bar{\phi}.$$

The result (91) may be verified in the problem of § 6.

15. Let us next consider the rotation of the shell in a constant field. There will be no currents due to those terms in the expansion of $\overline{\Omega}$ which are *zonal* solid harmonics; the only effect of these being a certain surface electrification. We may complete the investigation of § 8 by finding the density of this electrification in the case of an oblate ellipsoid of revolution rotating in a uniform field about an axis parallel to the lines of force. We have to find a function ψ which shall have the value

$$\psi = \tfrac{1}{2}p\gamma(x^2 + y^2) + \mathrm{C} \quad \quad (92)$$

at the surface of the conductor ($\zeta = \zeta_0$), and shall satisfy $\nabla^2\psi = 0$ throughout the external space. Denoting by a the equatorial radius, (92) may be put in the form

$$\psi = \mathrm{C} + \tfrac{1}{3}p\gamma a^2 - \tfrac{1}{3}p\gamma a^2 . \mathrm{P}_2(\mu) \quad \quad (93)$$

Hence in the external space

$$\psi = (\mathrm{C} + \tfrac{1}{3}p\gamma a^2)\frac{q_0(\zeta)}{q_0(\zeta_0)} - \tfrac{1}{3}p\gamma a^2\mathrm{P}_2(\mu)\frac{q_2(\zeta)}{q_2(\zeta_0)}.$$

But

$$q_0'(\zeta) = -\frac{1}{\zeta^2+1},$$

$$\frac{q_2'(\zeta)}{q_2(\zeta)} = -\frac{1}{p_2(\zeta)\, q_2(\zeta)\, (\zeta^2+1)} + \frac{p_2'(\zeta)}{p_2(\zeta)},$$

and $k^2\, d\zeta/d\nu_1 = \varpi/\zeta_0$. Hence, by (41),

$$\begin{aligned}\frac{4\pi\sigma}{\mathrm{K}} = -\frac{d\psi}{d\nu_1} &= \frac{\varpi}{k^2\zeta_0\,(\zeta_0^2+1)\operatorname{arc}\cot\zeta_0}\,(\mathrm{C}+\tfrac{1}{3}p\gamma a^2)\\ &\quad - \tfrac{1}{3}p\gamma a^2 \mathrm{P}_2(\mu)\left\{\frac{1}{p_2(\zeta_0)\, q_2(\zeta_0)\, \zeta_0\,(\zeta_0^2+1)} - \frac{p_2'(\zeta_0)}{\zeta_0\, p_2(\zeta_0)}\right\}\frac{\varpi}{k^2}.\end{aligned}$$

For the case of a *disk* we have, as in § 5,

$$\varpi = \frac{ac}{\sqrt{(a^2-r^2)}}, \qquad \mu = \sqrt{\left(1-\frac{r^2}{a^2}\right)},$$

and

$$a\zeta_0 = c,$$

where c is ultimately made $= 0$. Also

$$p_2(0) = \tfrac{1}{2}, \qquad q_2(0) = \frac{\pi}{4}, \qquad p_2'(0) = 0,$$

whence

$$\begin{aligned}\frac{4\pi\sigma}{\mathrm{K}} &= \frac{2}{\pi\sqrt{(a^2-r^2)}}\,(\mathrm{C}+\tfrac{1}{3}p\gamma a^2) - \tfrac{4}{3}\,\frac{p\gamma}{\pi\sqrt{(a^2-r^2)}}\,(2a^2-3r^2)\\ &= \frac{2}{\pi\sqrt{(a^2-r^2)}}\,\{\mathrm{C}+p\gamma\,(2r^2-a^2)\}. \quad \ldots \quad (94)\end{aligned}$$

The total charge on both surfaces of the disk is

$$\frac{2a}{\pi}\,(\mathrm{C}+\tfrac{1}{3}p\gamma a^2)\,\mathrm{K}.$$

The constant C is of course to be determined by the other conditions of the problem. If the axis of the disk be uninsulated, we shall have $\mathrm{C} = 0$.

16. The only terms in the value of $\bar{\Omega}$ which give rise to sensible currents in a rotating *disk* are those tessaral solid harmonics for which $n - s$ is odd.

If the value of $\bar{\Omega}$, referred to fixed axes, be

$$\bar{\Omega} = \bar{\mathrm{A}}\,.\,(1-\mu^2)^{s/2}\,\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\,(\zeta^2+1)^{s/2}\,\frac{d^s p_n(\zeta)}{d\zeta^s}\cos s\omega, \quad \ldots \quad (95)$$

the corresponding value of $\bar{\phi}$ will be

$$\bar{\phi} = 2\bar{\mathrm{C}}\,(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\cos s\omega,\text{*} \quad . \quad . \quad . \quad . \quad . \quad (96)$$

where, by the formulæ analogous to (69),

$$\begin{aligned}\bar{\mathrm{A}} &= (-)^s\,4\pi\frac{|\underline{n-s}}{|\underline{n+s}}\left[\frac{du_n^s}{d\zeta_0}\right]_{\zeta=0}\bar{\mathrm{C}},\\ &= -\,2\pi^2\bar{\mathrm{C}}\;\frac{|\underline{n-s}}{|\underline{n+s}}\cdot\frac{1\cdot3\ldots(n+s)}{2\cdot4\ldots(n-s-1)},\\ &= -\,2\pi^2\bar{\mathrm{C}}\;\frac{1\cdot3\ldots(n-s)}{2\cdot4\ldots(n+s-1)} \quad . \quad . \quad . \quad (97)\end{aligned}$$

If now, to use MAXWELL'S artifice, we pass to axes of x, y, moving with the disk, we must write

$$\bar{\phi} = \bar{\mathrm{C}}\,(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\cos s(\omega+pt),$$

where p is the angular velocity of the rotation. For the trigonometrical term at the end we may write $e^{is(\omega+pt)}$ if we retain in the end only the real part. Hence for the induced currents we have, by (91),

$$\phi = \frac{-\,isp\tau}{1+isp\tau}\,\bar{\phi},$$

where τ is the persistency of free currents of the type (n, s).

Putting

$$\eta = \operatorname{arc\,tan} sp\tau,$$

we find, finally, on returning to *fixed* axes of x, y,

$$\phi = \bar{\mathrm{C}}\sin\eta\,(1-\mu^2)^{s/2}\frac{d^s\mathrm{P}_n(\mu)}{d\mu^s}\sin(s\omega-\eta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

The system of currents is stationary in space, but is displaced relatively to the field by a greater or less angle

$$\frac{1}{s}\operatorname{arc\,tan} sp\tau,$$

according to the speed of rotation. The maximum value of this is $\pi/2s$ for a sufficiently rapid rotation.

* This represents a fictitious distribution of currents which would give at all points of the disk the same normal force as the actual field.

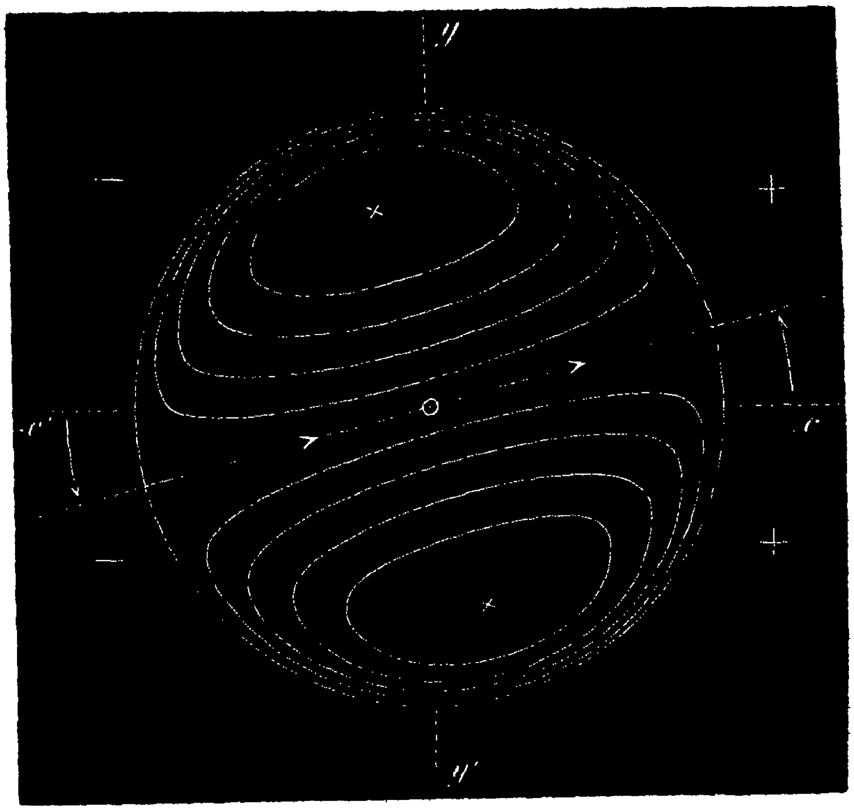
The most important type of induced currents is got by putting $n = 2$, $s = 1$, in (95) In this case

$$\bar{\Omega} \propto xz,$$

so that the lines of force at the disk are normal to it, but the direction of the force is reversed as we cross the axis of y. The current-function relatively to axes displaced through the proper angle η varies as

$$y \sqrt{\left(1 - \frac{r^2}{a^2}\right)} \text{*}$$

The current-lines for this case are shown in the figure. The signs + and − indicate where the normal force due to the field is towards or from the spectator



In the next type we have $n = 3$, $s = 2$, so that

$$\Omega \propto z\,(x^2 - y^2),$$

and the current-function (relatively to displaced axes as before) varies as

$$xy \sqrt{\left(1 - \frac{r^2}{a^2}\right)}.$$

* [It may be shown that, referred to the same axes, the potential ψ varies as

$$-\rho_0\, x\,(1 - 2r^2/a^2).$$

The equipotential lines are not orthogonal to the current-lines, except in the case of the circle $r = a/\sqrt{2}$. —Note added June 30, 1887]

VI. *On the Practical Measurement of Temperature. Experiments made at the Cavendish Laboratory, Cambridge.*

By H. L. Callendar, *B.A., Scholar of Trinity College, Cambridge.*

Communicated by J J Thomson, *F.R.S., Cavendish Professor of Experimental Physics in the University of Cambridge.*

Received June 9,—Read June 10, 1886.*

[Plates 11–13.]

General Account of the Experiments, and Discussion of the Results.

In view of the enormous discrepancies at present existing in estimates of high temperatures, it is exceedingly desirable that strictly comparable thermometric standards should be issued by some recognised authority .

Professor J J. Thomson, in the course of a conversation which I had with him towards the close of 1885, suggested that such standards could be issued in the form of platinum wire, the change of electrical resistance with temperature being determined by comparison for each specimen before issuing. The object of the present investigation was to test whether, in spite of the B. A. report on the Siemens pyrometer (1874), pure platinum wire might not be possessed of the necessary qualifications for such a standard.

Let us first consider what qualifications are necessary.

(I.) Such a standard should always give the same indication at the same temperature, should be free from secular change of zero, and should possess the widest possible range.

(II.) It should be portable and readily copied, so that standards might be multiplied, and, if the original were lost, it might be replaced by means of its copies. The copies should be capable of accurate comparison and verification.

(III.) It would be convenient if it agreed very approximately with any standards already in use, and if, failing this, the relation between the new and pre-existing standards could be accurately ascertained.

(IV.) It would be a great additional advantage if, besides being useful as a standard, it could also be applied directly to all kinds of practical investigations.

With regard to the first point (I.); the self-consistency of the platinum thermometer

* The portions enclosed in square brackets have been added or altered subsequently, June, 1887.

has been abundantly verified in all the experiments undertaken with it. If the wire is *pure* to start with, and care is taken not to alloy it with silicon, carbon, tin, or other impurities, and not to subject it to strain (such as increase of length), its resistance is always the same at the same temperature; at least, this was the case with the specimens of platinum wire used in these experiments. For the proof of these assertions the reader is referred to the experimental details (especially pp. 192, 205 *et passim*).

The wire used here is obtained from the well-known firm, Messrs. JOHNSON, MATTHEY, and Co., and is probably very pure and well adapted for the purpose (see p. 182).

Platinum wire evidently fulfils the conditions laid down in II., since it is essentially a standard of electrical resistance.

The method of comparing and verifying copies of the standard is of fundamental importance and requires a more detailed explanation. The great advantage of the proposed platinum standard over all others is the ease and accuracy with which this may be effected, a degree of accuracy greatly exceeding that of all other thermometric comparisons, and almost independent of the distribution of temperature. The wires are wound side by side on non-conducting material and symmetrically disposed, so that their mean temperatures are always the same and simultaneous values of their resistances are observed. Observations are also taken in melting ice and in steam at atmospheric pressure for the purpose of verification and to test for changes of zero. Details of the method and of the experimental comparison of wires of different metals are given in a subsequent section. An accuracy of 1 in 10,000 is attainable. The general conclusions from these experiments are as follows:—

[If R be the resistance of a given wire at the temperature t Cent., and R_1, R_0, the values of its resistance at 100° and 0° C. respectively, $(R_1/R_0 - 1)$ is the increase of resistance between 0° and 100° C., and is found to vary considerably for different specimens of commercial wire of the same metal. (*E.g.*, for platinum, it varies from 0·25 to 0·35.) But the curves of resistance variation are similar, so that the values of the function $(R/R_0 - 1)/(R_1/R_0 - 1)$ are nearly identical for different wires of the same metal through a range of 600° C., though they differ very widely for different metals. The variations are mainly due to slight chemical impurities, which suffice materially to reduce the temperature-coefficient and make the wire much less suited for thermometry.]

Pure platinum wire, when once annealed, is little liable to alteration. If it is pure, different lengths from the same bobbin do not behave differently, as some observers, perhaps using less pure wire, have stated. Its resistance is not liable to permanent change by mere heating and cooling, provided the wire is not strained or chemically altered. Different specimens of *pure* platinum wire agree very approximately throughout the scale. This is very convenient, but not essential; for, if they did not, they could be accurately compared and the differences tabulated. It

therefore possesses in an eminent degree the qualifications necessary for a scientific standard There seems no reason to expect that the temperature coefficient should be liable to secular change, seeing that violent treatment of these specimens for some months has not produced any measurable alteration.

III. As it is a very important question to determine the relation between temperature by platinum wire and temperature by air thermometer, considerable space is devoted to the discussion of experiments bearing on the question The present standard of thermometry is REGNAULT's normal air thermometer, this, while possessing several advantages as an ultimate standard, is, nevertheless, so difficult to use that the discrepancies at present existing are bewildering, and it cannot be used at all in ordinary work for most purposes; all measurements have, therefore, to be referred to it indirectly, which is a great disadvantage. But until the thermodynamic thermometer shall have assumed a more practical form, the air thermometer gives the best approximation to absolute temperature It happens that the platinum thermometer may be more accurately compared with this standard than any other.

The practical difficulty of all thermometric comparisons is that of maintaining an enclosure at a constant and uniform temperature This was to a great extent avoided in the present case by enclosing the spiral of fine platinum wire inside the bulb of the air thermometer itself There are, of course, some mechanical difficulties in the way, and it was found necessary to devise a modified form of air thermometer for the purpose Full details of the experiments and methods are given in subsequent sections The modified form of air thermometer is also minutely described, as it may be found useful in other and similar investigations. Various sources of error are also discussed, which may have considerable effect in experiments at high temperatures. The investigation of these is still incomplete, but does not present any difficulties which may not be overcome with the aid of the platinum thermometer. Among these we may mention surface actions between the gas and its enclosure; the question of the possibility of eliminating the capricious changes of volume of the envelope; and of determining the amount of dissociation to which gases are subject at high temperatures The experiments are, therefore, necessarily incomplete, they are published chiefly on account of the great practical importance of settling on a universal standard for the more accurate comparison of different measurements of temperature

[The results of the comparison between the platinum and air thermometers are best represented graphically by drawing the curve of difference of temperature. If t stand for the temperature Centigrade by air thermometer, and pt for the temperature Centigrade by platinum wire, that is to say, for the function $100(R/R_0-1)/(R_1/R_0-1)$; the observations (Plate 13) are seen to agree fairly well with the parabola

$$d = t - pt = 1{\cdot}57\{(t/100)^2 - (t/100)\}. \quad \ldots \ldots \quad (d)$$

But the comparisons of different platinum wires show that this formula cannot be absolutely accurate.]

It is very convenient for practical purposes that the platinum thermometer agrees so closely with the absolute thermo-dynamic scale ; but, of course, such agreement does not much affect its value as a standard. We at present require a practical standard to which all measurements may be *directly* referred ; the question of its exact relation to the absolute thermo-dynamic scale may be matter for subsequent investigation, and can be best solved when we have arrived at a more accurate and extended knowledge of the phenomena of heat, of the dependence of radiation, conduction, &c , on temperature measured by a scale, which may be to some extent arbitrary, but which will have the incalculable advantage of uniformity, so that the results of different experimentalists will be accurately comparable. To show how imperfectly the air thermometer fulfils this condition, we may refer to the article "Pyrometer" in the 'Encyclopædia Britannica,' 9th edition, 1885.

IV. In comparing the platinum resistance thermometer with other instruments, it will be seen that it is essentially *practical.* The steam pressure thermometers advocated by Sir WILLIAM THOMSON in his article on Heat, in the 'Encyclopædia Britannica,' are doubtless theoretically more interesting, and could be made of great utility in attaining known constant temperatures when the relation between steam pressure and temperature had once been determined for each case. But the most accurate way of comparing them with each other, and with the standard air thermometer, is by the intermediation of the platinum wire. Other instruments for measuring temperature have either a very small range or lay no claim to accuracy; for this reason it is obviously unnecessary to discuss the mercury thermometer or the thermopile, especially as their defects are so well known, and their applicability so limited, although each possesses distinct advantages for certain purposes.

Platinum wire is well fitted for use as a temperature standard, it is still more admirably fitted for practical use in any experiments where accuracy is required. This arises chiefly from its adaptability. A length of wire is cut off and disposed in any way which may happen to be most convenient for the desired purpose The observations are simple and rapid, the corrections small and easily applied, the sources of error less than with any other kind of thermometer and more easily avoided. For radiation experiments it may be made more sensitive than the thermopile (cf. LANGLEY'S Bolometer); owing to its great sensitiveness, it may be used to measure rapidly varying temperature without inaccuracy; this is often desirable, as in calorimetry by method of cooling. There is, in fact, hardly any experimental investigation in which the measurement of temperature is necessary, which may not be more accurately and simply effected by means of a suitable platinum wire thermometer.

[The results of the investigation considered as a research on the resistance-variation of metals tend to prove the existence of minute irregularities which

cannot be well represented in an empirical formula. The simple parabolic formula (p), $R/R_0 = 1 + at + \beta t^2$, corresponds exactly with (d) (p 163), namely

$$t - pt = \delta\{(t/100)^2 - (t/100)\},$$

where $\delta = 10{,}000\beta/(a + 100\beta)$, and gives an accuracy of the 1°/₀ order through a range of 600° C. Other formulæ may readily be found which give nearer approximations in special cases, and through small ranges. For instance, the exponential formula $\log R = at/(1 + \beta t)$ corresponds to the dotted line, Plate 13, fig 11, and exactly smooths out the characteristic inequalities in the platinum-iron comparison-curve, agreeing with it far more closely than the parabola; but in most other cases the exponential formula has the disadvantage. BENOIT and MATTHIESSEN use the parabolic formula, but MATTHIESSEN expressed his results as variations of conductivity (R_0/R), so that his formulæ are quite inapplicable except between 0° and 100° C.]

I have endeavoured to give the fullest data in each case for the verification of my results, and to make the calculations in each case accurate within the limits of probable error, but among so large a mass of figures I can hardly hope to have escaped an occasional slip.

My most sincere thanks are due to the authorities at the Cavendish Laboratory for the facilities which they gave me for the prosecution of these experiments, and especially to Professor THOMSON for the kind interest he has taken in the subject, and for valuable suggestions and assistance.

In overcoming the mechanical difficulties which are always the most important part of an investigation like the present, I received great help from Professor THOMSON's assistant, Mr. D S. SINCLAIR, though, owing to the many calls upon his time, he was unable actually to make much of the apparatus I had intended to make many other experiments, but my progress has been much retarded by my own want of mechanical skill, and the necessity of making all my own apparatus.

I am at present intending to pursue these experiments to higher temperatures, and have already worked out the experimental details of the application of the platinum thermometer to questions in the theory of heat where I consider greater accuracy may be obtained by its use

Authorities consulted.

On the variation of the electrical resistance of metals with temperature. All experiments almost up to date are collected in the first volume of the last edition of 'Die Lehre von der Elektricitat,' by G. WIEDEMANN. Original papers by BENOIT in the 'Comptes Rendus,' and MATTHIESSEN in the 'Phil. Trans.,' 1862, are of the chief importance, but nearly everything has been collated.

In Thermometry the best general information up to date is to be found in the 'Enc. Brit.,' 9th edit., Articles "Heat" and "Pyrometers," already referred to. I have also studied with great care the whole of the classical researches of REGNAULT in the 'Mémoires de l'Institut,' vols. 21, 26, &c.

On Pyrometry there is a valuable paper by WEINHOLD in 'Poggendorff, Annalen,' 1873, who gives a fairly complete list of original memoirs, nearly all of which have been referred to.

On the air thermometer there is a most valuable paper by BALFOUR STEWART in the 'Phil. Trans.,' containing perhaps the most accurate experiments yet made on the dilatation between 0° and 100°. For high temperature experiments, papers by DEVILLE and TROOST, and E. BECQUEREL, are most important.

SIEMENS suggested the platinum pyrometer, but I cannot find any account, worth mentioning, of his experiments

Determination of the Increase of Resistance with Temperature of the Standard Platinum Wire by comparison with the Air Thermometer.

As stated above, the direct comparison is preferable to the method adopted by BENOIT and others of heating the wire in vapour baths of substances whose boiling points have to be assumed from other experiments, a spiral of fine bright platinum wire is a bad radiator, and is exceedingly sensitive to slight changes in the temperature of the air with which it is in contact; if, therefore, it be fixed inside the bulb of an air thermometer, the mean temperature of the spiral will be always very nearly the same as that of the air, and this is measured directly on the absolute scale.

The Air Thermometer.

Since the ordinary form of the instrument is in several ways inconvenient, it was found necessary to devise a modified form for the purposes of this investigation, a detailed description of which will be given. The sources of error, and the corrections to which it is liable, will also be incidentally considered.

The general plan of the instrument will be easily understood on reference to the accompanying diagram (Plate 11, fig 1). [An improved form is described in the Appendix.]

A bulb A is connected by means of a capillary tube to a U-gauge of small bore (about 2 millims. diameter) containing pure sulphuric acid, which serves to confine a constant mass of air at nearly constant volume; the gauge carries a bulb B on its other limb, which is connected by an indiarubber tube to an adjustable mercury manometer M with wide tubes Attached to the limb of the U-gauge is a millimetre scale by which its reading can be recorded.

The volume of the bulb A is determined by calibration with water or mercury in the usual way, or with air by the method of the volumenometer.*

The coefficient of cubical expansion of the bulb enters as a small correction, amounting to about 1 per cent.; it is best determined at low temperatures by using the bulb as a mercury thermometer, and assuming the absolute expansion of mercury.

* 'Practical Physics,' GLAZEBROOK and SHAW, p 160.

If the bulb itself cannot be used, a mercury thermometer is made from the same tube as the bulb.

Since, however, this method ceases to be applicable above the boiling point of mercury, and is moreover exceedingly liable to inaccuracy, except between 0° and 100° C., it was decided to measure the variations of the linear coefficient and assume that the cubical coefficient varied in the same way.

The method adopted was exceedingly simple, and is, shortly, as follows :—

Linear Expansion of Hard Glass.

A glass tube is heated in a thick iron tube in a long gas furnace, the variations of its length are observed by means of reading microscopes, and the mean temperature of the portion heated is given by observing the resistance of a standard platinum wire extending down the axis of the tube, assuming that when the temperature is steady the mean temperature of the wire is the same as that of the portion of the tube it occupies.

Experiments by this method were made on a piece of hard glass tube, the same as that used in making the air thermometers No. 3 and No 5.

The ends of the tube were drawn down to 5 millims. diameter, to diminish the conduction of heat, and to ensure that the parts exposed should be nearly at the temperature of the atmosphere, copper leads were fused on to the ends of a piece of the standard platinum wire, 69·6 centims. long, which was then inserted down the axis of the glass tube

The glass tube was heated by being placed in an iron gas-pipe, 67 centims long, in a gas furnace of the same length, supported on a board with levelling screws for focussing The copper platinum junctions were exposed to the air temperature at equal distances, about 1 cm. beyond the ends of the iron tube, and, being screened from the heat by bright tinned plate, no appreciable thermo-electric effect was produced, as the junctions were very slightly, and nearly equally, warmed.

A length of about 63 cm. of the glass tube was left in its original state; the drawn-down ends projected about 16 cm. beyond the iron tube, and were practically at the air temperature for their whole length beyond the copper platinum junctions; the variations in length of these portions of the tube may be neglected in comparison with other possible errors, it is, therefore, assumed that the mean temperature of the platinum wire is the mean temperature of the part of the tube it occupies, whose length is variable. If the coefficient of expansion of glass were accurately constant, or if the temperature of the whole portion heated were uniform, this assumption would give mathematically correct results. Since both these conditions are nearly satisfied, the corrections to be applied for the variations at either end are negligible. We take, therefore, the length of the part of the glass tube, whose variations we are measuring, to be the distance between the ends of the platinum wire, and its mean temperature to be that of the platinum wire.

Crosses were made, with a writing diamond, on the extremities of the tube, at a distance of about 15 cm. from the gas furnace, and were observed with reading microscopes.

The reading microscopes were made by the Cambridge Scientific Instrument Society, and are all but perfect. The microscope is fixed in a carriage, which is moveable horizontally by means of a micrometer screw reading to ·00002 of an inch.

They were mounted on solid stone blocks, resting on a thick stone slab which supported the gas furnace, so screened with bright tinned plate that the microscopes and their supports, and the stone slab, were not appreciably heated

After the apparatus has been adjusted the microscopes are placed in position and focussed on the crosses, and are not subsequently touched, except with the micrometer screws, in taking readings.

Since the success of the method depends on the distance between the stands of the reading microscopes remaining invariable, this was verified, from time to time, by means of an auxiliary glass tube, 1 metre long, with diamond crosses, which could be adjusted in position beneath the microscopes without disturbing them. No variations of the distance could be detected, and they were probably less than ·0001 of an inch. Several independent readings seem to show that the probable error of a single reading due to *all* causes is about of the magnitude of 0002 inch.

Readings were also taken of the temperature of the air, and of the air between the gas furnace and the stone slab; the variations were usually small enough to be neglected. The crosses on the glass tube could be very accurately brought into focus beneath the microscopes by adjusting the levelling-screws on which the gas furnace was supported; this is a most important provision, as the furnace expands with heat and puts the tube out of position.

With the above apparatus observations were taken on different occasions. The method employed was to light the gas furnace and wait until the temperature became steady, and then take several readings of resistance and of the microscopes on the plan shown in Table I.

It was soon found that the length of the glass tube was liable to permanent alteration owing to change of structure. For this reason the micrometer-screws were usually read only to the nearest $\frac{1}{10000}$ of an inch, and the other approximations were made in the calculation, which would have been inadmissible if the phenomenon observed were accurately self-consistent.

Table I shows the irregularities produced by these structural changes. Table II., however, contains two series of observations strictly comparable with each other, the highest temperature observation being taken first to avoid the permanent contraction. For the sake of comparison, the third column gives the expansion calculated from the formula $l/l_0 = 1 + \cdot0000068\,pt + 2\cdot7 \times 10^{-9}\,pt^2$, which very nearly represents the observations, and which gives for the cubical expansion

$$v/v_0 = 1 + \cdot0000204\,pt + 8\cdot1 \times 10^{-9}\,pt^2,$$

pt being the temperature Centigrade by platinum wire.

TABLE I-G.—Showing Method of recording Observations and Progressive Contraction of Glass Tube at 500° C.

Observed resistance.	Resistance			Auxiliary thermometers.	Time.	Reading microscopes.			Mean sum	Total expansion	Expansion in percentage of length	Temperature, platinum
	Mean.	Corrected	Reduced to 1 at 0°			Left.	Right.	Sum				
					hrs min							°C
9·37 Bal	9 360	9 342	2 700	S T 17 0 R 15 0	3 9	3308	3279	6587	65905	5605	408	491
9 35 Bal.					3 13	3310	3284	6594				
9 39 Bal	9 386	9 368	2 710	Air. 16 2	3 23	3274	3325	6599	66005	5505	401	494
9·387					3 25	3274	3326	6600				
9·386					3 28	3274	3327	6601				
9 38 Bal					3 30	3276	3326	6602				

The temperature is calculated by the formula (*a*), see page 195.

$$\frac{R}{R_0}=1+\cdot003460pt, \text{ or } pt=\frac{R-R_0}{R_1-R_0}100.$$

Mercury thermometers were used to check changes of temperature in various parts of the apparatus, their readings are recorded in the column headed "Thermometers." S T gives the temperature on the stone slab under the gas furnace, this, being carefully screened, was not found to rise appreciably, even after the furnace had been full on for an hour. R gives the temperature indicated by a thermometer placed inside the resistance box The temperature of the air was taken because variations in the temperature of the reading microscopes or their supports would introduce small errors, as the variations of the air temperature were very small, these errors could be neglected.

TABLE II, giving Results of Observations on Contraction of Hard Glass Tube, the Temperature being given by Platinum Wire

	Temperature, pt	Expansion expressed in percentage of length.	Expansion calculated by formula of interpolation.	Mean coefficient of expansion by formula.	
	° C.				
First series . .	494	401	403	·814	March 30.
	165	·122	·120	·725	
	136	098	·097	717	
	98	·068	·069	·706	
	60	043	042	·696	
	17	·012	012	·685	
Second series . . .	481	391	389	810	March 31.
	330	254	·254	·769	
	222	165	·164	·740	
	102	070	072	706	
	19	·013	013	·685	

On the first occasion the permanent contraction of length which the glass tube underwent amounted to ·052 per cent. of its length, and on the second to ·026 per cent., although this glass does not begin to soften till about 700° C., and was subject to no strain whatever: with other materials (*e.g.*, porcelain) these changes would probably be more serious. They are a most important source of error, and place a limit to the accuracy attainable with an air thermometer at such temperatures, unless an arrangement is adopted to measure them.

The volume of the bulb is also liable to change, owing to the variation of pressure to which it is subjected. At ordinary temperatures these changes are usually less than $\frac{1}{10000}$ per atmosphere, and may be neglected; but at high temperatures near the softening point of the glass they might become sensible, and the bulb might even undergo a permanent deformation. As the phenomenon in question cannot be readily measured at high temperatures, it is best to avoid the difficulty by so adjusting the mass of air enclosed that its pressure may amount to an atmosphere at the highest temperature to be reached, and the glass may thus be entirely freed from the uncertain effects of a strain at the high temperature Such alterations of volume are

about twenty times as important in a mercury thermometer, where they produce notable changes of zero. One great advantage of the air thermometer is that they can usually be neglected.

The bulb is connected with the U gauge by a tube of capillary bore, part of which is heated with the bulb, and part of which is at the known atmospheric temperature, a length of about 5 centims. being at intermediate and unknown temperatures. If the bore be between three and five-tenths of a millimetre, and the capacity of the bulb 50 cub. centims, the uncertain error will be much less than $\frac{1}{10000}$, and may be neglected.

Some kinds of glass are liable to be acted on chemically by the gas at high temperatures. For instance, hydrogen or coal gas exerts a powerful reducing action on lead glass at a dull red heat, and the action proceeds more slowly at much lower temperatures Perhaps nitrogen is, for this reason, the most suitable thermometric material. Water vapour and carbon dioxide also are well known to exert a disintegrating action, and are for other reasons also especially to be avoided.

The Sulphuric Acid Gauge.

It has usually been the custom to confine the air to constant volume in the air thermometer by a column of mercury whose level is adjusted up to a fixed mark before taking an observation. The use of mercury necessitates a very wide tube, otherwise the effects of capillarity produce large errors. the volume of the air unheated must, in consequence, be large and also variable with the shape of the meniscus. Moreover, for accurate work, it is necessary not only to thoroughly dry and clean the bulb of the air thermometer, but also the tubes and mercury of the manometer which is in direct connexion with it This it is practically impossible to do perfectly. Again, it is in most observations exceedingly inconvenient to have a mercury manometer rigidly connected with the bulb of the air thermometer, and in such close proximity with the source of heat that the mercury becomes unequally heated. The chief advantage of mercury is that its vapour tension is so small (about ·001 centim at ordinary temperatures) that it is negligible. Pure H_2SO_4, however, possesses this same qualification, and has none of the above disadvantages. Since it wets the tube, the correction for capillarity is quite constant. If the bore of the tube be about 0 2 centim., the capillary elevation is about 0·5 centim. With a gauge tube of this bore the correction volume may be made quite small, and may be very accurately known, the shape of the meniscus being constant. This enables us to use bulbs of smaller capacity, which are, for many purposes, more convenient, without unduly increasing the magnitude of this correction. The use of sulphuric acid in the gauge tube is also attended with several other advantages. The air thermometer is thus rendered very sensitive, the slightest change of temperature is at once indicated, and it is very easy to see when the temperature is steady. The mass of air under

observation is perfectly confined, and the same mass of air may be kept under observation for a lengthened period, and is maintained perfectly dry

Since the bore of the H_2SO_4 gauge tube is small compared with that of the mercury manometer, a change of pressure of air in the bulb first takes effect in the small-bore U gauge, and only alters the reading of the wide tube mercury manometer very slightly: this enables the instrument to be used when the temperature is not absolutely constant, as the reading of the U gauge can be instantly recorded, and the mercury manometer read at leisure. It might be thought that errors would be introduced by capillarity and viscosity of the acid in so small a tube, but it can be easily shown by direct experiment that this is not the case to an appreciable extent. If an open gauge be connected to a mercury manometer, the acid exactly and perfectly follows every readjustment of the manometer, and its capillarity and density may thus be easily determined in terms of mercury.

It has been suggested that the acid might have some action on the confined mass of air, either absorbing it or giving off SO_2: it would be difficult to prove that this is not the case. On the other hand there does not seem much reason to believe that it is.*

One great advantage of the H_2SO_4 gauge is that, when the instrument is at constant temperature, such as that of melting ice, by altering the adjustment of the mercury manometer, the reading of the H_2SO_4 gauge is altered consistently, and a great number of quite independent observations may be taken, which ought, of course, on reduction, to lead to almost identical results, the mean of which will be to a great extent freed from scale and personal errors

The H_2SO_4 gauge is provided with a millimetre scale, and is carefully calibrated; it is difficult to make an error of ·02 centim in reading the scale, and all errors are divided by about 7 in the reduction to mercury, so that they may practically be neglected as far as the pressure is concerned, they become more important in applying the volume correction, especially if the volume of the bulb be small.

The extremely simple sort of gauge shown in the diagram may be used with great advantage, but it is liable to some slight inconveniences.

Supposing the air in the bulb to be at atmospheric pressure and temperature, a variation of 1° C. in its temperature alters the reading of the H_2SO_4 gauge by nearly 2 centims. If the limb of the gauge be only 10 centims. long, it has thus a range of only about 5° without readjustment of the mercury manometer, so that, when the temperature is changing, the manometer requires to be continually readjusted.

More elaborate forms were, therefore, actually used in most of the experiments, one of which is shown in fig 2, Plate 11.

The gauge tube carries a bulb B which is calibrated, and whose volume is about

* [illegible] complete answer to this and other objections to the use of sulphuric acid will be found in [illegible] Sir WILLIAM THOMSON's article on "Heat," in the ninth edition, 1880, of the 'Encyclopædia Britannica,' vol. 11, p. 561.

equal to that of the bulb of the air thermometer · a little below the bulb the gauge tube widens out into a cylindrical reservoir C, from which a side tube communicates with the three-way tap T, which also serves to close the entrance to the large cylindrical bulb D A side tube *o*, near the lower extremity of this limb of the V gauge, serves for the introduction of the requisite sulphuric acid, and can be sealed off when required.

The bulb B serves to determine the volume of the bulb of the air thermometer at any time, on the principle of the volumenometer, and so to check any alteration in the volume of the bulb without the necessity of dismounting and filling it afresh ; it may also be used for several other purposes.

When it is desired to fill the bulb of the air thermometer with pure dry air by repeated exhaustion and re-admission at a high temperature, the whole of the acid (if the gauge is already full) is first drawn back into the bulb D, which it completely fills, and the tap T is turned thus ($\perp$) · the contained air is then exhausted, and the dry air re-admitted repeatedly by way of the tube CeT When this has been satisfactorily effected the tap T is turned thus ($\top$), and the acid allowed to flow down into B, completely cutting off the mass of dry air in the bulb.

The mass of air enclosed may also be easily adjusted by means of the manometer, to give any desired pressure at any desired temperature

The volume of the bulb B gives the necessary range to the gauge in heating or cooling ; if the tap T be turned thus ($\dashv$), so as to put C and D in communication and cut off the manometer, the instrument may be left to itself to cool or heat through a large range

When the air thermometer has been thus left to itself to cool or heat, it often happens that the pressure is only roughly known, so that, if communication were opened direct between the gauge and the mercury manometer, a sudden and disastrous movement of the acid in either direction might take place if the manometer did not happen to be exactly adjusted. It is convenient, therefore, to introduce the small auxiliary piece of apparatus ST′ on the way between the H_2SO_4 gauge and the mercury manometer.

The U tube S is of small bore, about 1 mm , and is used as a kind of safety indicator to show when the pressure is the same on both sides of the three-way tap T′; for this purpose it is half filled with mercury.

The pressure of the air in D is always known to within 20 cm. or so of mercury, so that the manometer can be approximately adjusted, the tap T is then turned thus ($\top$) ; the tube S indicates the residual difference of pressure on either side of T′; the manometer is adjusted till this difference vanishes, and is then put into direct communication with the H_2SO_4 gauge. If these precautions were not taken, the acid might sometimes get sucked back into the bulb or the pressure tubes, either of which events would ruin the series of observations.

Comparison of the Coefficients of Expansion of Gases.

The sulphuric acid gauge is specially adapted for effecting this comparison by a differential method.

Similar thermometers, filled with different gases at approximately the same pressure, are exposed to the same source of heat and connected to the same mercury manometer, so that the external circumstances are precisely the same for both, and the difference of their pressures at any time is indicated by the sulphuric acid in the gauge.

In a comparison of air and CO_2 between 0° and 100° the difference amounts to about 40 mm. of acid, a very measurable quantity.

The Mercury Manometer.

In the first experiments the manometer belonging to a "JOLLY" air thermometer was used. It had a glass mirror scale of millimetres divided with considerable accuracy, but it was found that the tubes were of too small diameter, and the mercury was apt to stick appreciably.

Another manometer was therefore constructed, with wider tubes, about 1·7 cm. internal diameter, which was found to be exceedingly satisfactory; the same mirror scale was used, and the same adjusting apparatus.

The scale is only divided to millimetres, but it is possible, with practice, to estimate with considerable accuracy to the tenth of a millimetre. It appears from the many observations made with the apparatus that the probable error of a single observation is of about this magnitude It would have been theoretically preferable to take the readings with a kathetometer, but this would have involved great loss of time, and there was not one to spare.

To avoid errors of parallax, &c., readings were taken as follows:—The mirror scale was fixed vertically in a vertical plane, with the aid of a plumb-line. A telescope capable of sliding up and down, and being clamped at any height on a vertical axis, was levelled by focussing it on its own image in the mirror at the distance of about a metre; this gave a magnifying power of about three diameters as compared with the naked eye at a distance of 10 inches.

To eliminate personal errors in estimating the tenths of a millimetre, a series of independent observations at different points of the scale were taken at each constant temperature. The close agreement of these shows that the apparatus performed satisfactorily.

The mirror scale was tested for scale errors by comparing it with the standard metre, with the aid of reading microscopes; the errors were found in all cases to be less than ·01 cm., and are therefore neglected.

In taking observations, the reading of the top of the meniscus in each limb is recorded and the difference set down in an adjacent column; this is then corrected for temperature and reduced to zero, allowing for the linear expansion of the glass scale.

The temperature of the mercury is repeatedly taken in the course of the observations by the aid of a delicate thermometer immersed in the mercury of the open limb It almost always differs slightly from that of the air.

The correction for temperature is only applied to the nearest tenth-millimetre, partly because the readings are only accurate to this order, and partly because, since the temperature is varying and is not the same throughout, the correction itself is liable to a slight uncertainty. To avoid inaccuracy, the calculations were usually carried to the next figure, to which, however, but little weight can be attached.

The Barometer.

Readings of the atmospheric pressure were taken with the standard Fortin barometer. Each reading is subject to the following corrections —

1. The most important is the correction for temperature. The scale is of brass, graduated in English inches, and is correct at a temperature of 62° Fahr. The scale errors are negligible

The density of the mercury is normal at 0°.

Thus for a height of 76·0 cm. the correction is —·024 cm. at 0° Cent., since the brass scale is incorrect, and at 62° Fahr., when the brass scale is correct, the mercury correction is —·228 cm, and the change in the correction is ·0123 cm. per degree Centigrade.

The correction is uncertain, because it does not follow that the thermometer indicates the mean temperature of the mercury and scale.

2 The correction for capillary depression and vapour tension of mercury amounts altogether to about +·005 cm. at 15° Cent.

3. The corrections for gravitation and height above sea-level need not be applied, since absolute measures of pressure are not required (except for determining the temperature of saturated steam at atmospheric pressure).

4. The level of the cistern of the barometer is at a height of 4 metres above the manometer. A constant correction of + 04 cm. is applied to allow for this.

5. *Sources of Error.*—The atmospheric pressure varies incessantly and capriciously, so that the reading of the barometer cannot be relied on beyond the nearest tenth-millimetre * (REGNAULT), even if all the mechanical adjustments could be perfectly effected and the temperature correction were certain. The barometer is, therefore, read several times in the course of each series of observations, and the corrections are applied to each reading to the nearest tenth-millimetre.

6. The constant errors of the attached thermometer are sufficiently small to be negligible.

In reading the barometer, all precautions were taken to avoid inaccuracy in the mechanical adjustments, the probable error from this source being only about ·001 in

* 'Paris, Mémoires de l'Institut,' tom. 21, p. 69

Formulæ *

The formulæ applicable to this form of air-thermometer are, of course, very similar to those of the ordinary form Let m be the mass of air confined. Let V_0 be the volume at 0° C. of the bulb, together with that portion of the capillary tube which is heated with it, and let v_0 be the volume of the capillary and gauge tubes from this point to the division x_0 of the gauge tube Let p_0 be the pressure at 0° C. when the temperature of the gauge tube is θ' and the acid stands at the division x_0 of the scale. Let θ_0 be the absolute temperature corresponding to 0° C. Let v be the correction volume for the portion unheated at any other time when the temperature, pressure, and volume of the air in the bulb are θ, p, V, respectively, and when θ'' is the temperature of the gauge tube, and x the scale reading thereof. Then, by the law of a perfect gas, we have the equation

$$p\left\{\frac{V}{\theta}+\frac{v}{\theta''}\right\} = mk = p_0\left\{\frac{V_0}{\theta_0}+\frac{v_0}{\theta'}\right\}, \quad \ldots \quad (1)$$

whence, to a first approximation,

$$\theta = \frac{p\theta_0}{p_0}.$$

Let γ be the mean cubical coefficient of expansion of the glass of the bulb between θ_0 and θ.

Then

$$V = V_0\{1+\gamma(\theta-\theta_0)\}.$$

Let u be the volume per centim. of the gauge tube.

Then

$$v = v_0+u(x-x_0).$$

Let

$$\frac{v}{V} = \beta. \quad \frac{v_0}{V_0} = \beta_0.$$

Making these substitutions in equation (1), and substituting for θ the first approximation in the small terms, we obtain at once

$$\theta = p\frac{\theta_0}{p_0}\left\{1-\left(\gamma\theta_0+\beta_0\frac{\theta_0}{\theta'}\right)+\frac{p}{p_0}\left(\gamma\theta_0+\beta\frac{\theta_0}{\theta''}\right)\right\}. \quad \ldots \quad (2)$$

If we write

$$\gamma\theta_0+\beta_0\frac{\theta_0}{\theta'} = y_0 \text{ and } \gamma\theta_0+\beta\frac{\theta_0}{\theta''} = y,$$

this becomes

$$\theta = p\frac{\theta_0}{p_0}\left\{1-y_0+\frac{p}{p_0}y\right\}.$$

* For better method with aid of slide-rule, see Appendix.

Let

$$\frac{p}{p_0}y - y_0 = \delta, \text{ and write } p(1+\delta) = \mathrm{P}$$

Then, finally,

$$\theta = \mathrm{P} \cdot \frac{\theta_0}{p_0} \qquad . \qquad (3)$$

It is, in general, useless to proceed to a third approximation, because the uncertainty of the corrections is at least of the same order, as a rule.

The value of P, which may be called the "corrected" pressure, is obtained directly from the observations by applying the small corrections in the manner detailed below

P is proportional to the absolute temperature The ratio θ_0/p_0 is theoretically constant for each filling, though, as a matter of fact, it is variable with the "zero-errors" of the air thermometer. The value of θ is calculated from P by the aid of logarithms, the logarithm of θ_0/p_0 being tabulated once for all for each series of observations. This is the only part of the reductions where logarithms are required The correction terms being, from their very nature, uncertain, often to the extent of nearly 1 per cent, it is quite sufficient to perform all the multiplications they involve by the aid of a small slide-rule giving results correct to 1 per 1000. This method saves trouble, paper, time, and mistakes

Reduction of the Pressure (p).

The correction of the mercury columns for temperature is most easily applied by a graphic method by ruling two series of straight lines, giving the correction to be applied to any length of mercury column less than 1 metre for each degree of temperature Centigrade, both for the glass millimetre scale and for the English standard barometer with the brass scale. The barometer lines need not extend through the whole range, since its reading never varies far from 76 centims

Let H be the height of the barometer, corrected for temperature, &c, as above described Let M_R, M_L, be the readings of the right and left limbs of the mercury manometer; and let $M_R - M_L$ be corrected for temperature and added to H. Let x' be the reading of the H_2SO_4 gauge when the pressure is the same on the acid in both limbs. Let x be the recorded reading in centims. Let

$$\frac{(x-x') \text{ density of } H_2SO_4}{\text{density of mercury}} = q.$$

Then q is the pressure due to the acid, expressed in terms of mercury. q is usually so small that no temperature correction need be applied. Both x' and the reduction factor are determined by previous experiment with the acid and gauge used.

Then, finally,

$$p = \mathrm{H} + [\mathrm{M}_{\mathrm{R}} - \mathrm{M}_{\mathrm{L}}]_0 + q.$$

For reasons previously given, p is correct only to the nearest tenth of a millimetre.

Reduction of the Observations.

This is a formidable piece of arithmetic, with the formulæ as usually given for the ordinary air thermometer it would be still more formidable with the sulphuric acid gauge. However, by the aid of graphic methods and a slide rule, it may be effected with great rapidity and accuracy

The value of $y = \gamma\theta_0 + \frac{v_0 + u(x-x_0)}{\mathrm{V}} \frac{\theta_0}{\theta''}$ is calculated for any two values of x and for temperatures 15°+20° C. Two straight lines are ruled on logarithm paper joining these points, giving the value of y for any value of x at these two temperatures. A variation of one degree of temperature produces a variation of about $\frac{1}{300}$ in the value of y, so that the lines very nearly coincide.

Similarly the value of q is calculated for two values of x, and the ruling of a straight line gives it for all other values; this line is ruled on the same piece of logarithmic paper, with the same abscissa scale. A correction diagram of this kind has to be constructed from the calibration and other details once for all for each thermometer The values of y and q, corresponding to the recorded value of x, are taken from the correction diagram and copied into adjacent columns headed y and q; p is evaluated by the simplest addition and subtraction; the multiplication yp/p_0 is performed in one operation on a slide-rule, and δ is found by subtracting y_0; the product δp is given at once by the slide-rule and added to p, giving P, which is proportioned to the absolute temperature θ: thus the whole of the calculation involved in finding P may be exhibited in a single line and worked out on the spot in the note-book used for recording the observations. This gives a great advantage in point of compactness, and the ease with which mistakes may be discovered and rectified. Where it is advisable, from the magnitude of the volume correction, to go to a third approximation, this may be taken as

$$\theta = \frac{\theta_0}{p_0} p\{1 + \delta + \delta^2\} \text{ by neglecting } \gamma\delta\theta_0.$$

δ^2 is at once given by the slide-rule and added to δ, so that this involves very little extra trouble.

But the greatest advantage of this method of reduction is that it avoids the accumulation of superfluous and meaningless figures, and thereby escapes that fictitious accuracy which is so seductive and treacherous in all physical investigations.

Corrections to be applied to the Air Thermometer.

Air does not follow exactly the equation of elasticity of a perfect gas $pv/\theta = mk$. Since, however, it follows this law very approximately, its true equation of elasticity may be written

$$\frac{pv}{\theta} = mk(1-\phi),$$

where ϕ is an unknown function of the variables which must be very small. The experiments bearing on this deviation are those of REGNAULT on the compressibility of gases at different temperatures and the comparison of air thermometers, and, in particular, the experiment of THOMSON and JOULE ('Phil. Trans.,' 1862) "On some Thermal Effects of Fluids in Motion."

From all these experiments RANKINE deduced as the approximate equation of elasticity $v = \mathrm{R}\frac{\mathrm{T}}{p} - \frac{1}{3}\mathrm{AC}p\left(\frac{a}{\mathrm{T}}\right)^2$.*

It may, of course, be written *in our notation* in the form

$$\frac{pv}{\theta} = mk(1-\phi), \quad \ldots \quad \text{(E)}$$

in which for air

$$\phi = +\cdot 001173\frac{p}{p_0}\cdot\left(\frac{\theta_0}{\theta}\right)^3.$$

TABLE. Φ.

The following Table gives approximately the values of this correction at different points for the standard air-thermometer :—

Formulæ	At constant density, $t=\frac{p-p_0}{p_1-p_0}100$		At constant pressure, $t=\frac{v-v_0}{v_1-v_0}100$	
True temperature Centigrade	Values of ϕ. p_0=76 centims	Correction to true temperature	Values of ϕ p=76 centims.	Correction to true temperature.
0	001173	+ 00	001173	+ ·00
100	000626	+·00	·000457	+ ·00
200	000393	+·04	·000225	+ ·084
300	000267	+ 09	·000127	+ ·20
500	000147	+ 23	·000052	+ 47
1000	·000054	+·62	000012	+1 19

* This is the form and notation given by CLAUSIUS, 'Mechanical Theory of Heat,' p. 227

A somewhat similar Table is given by THOMSON and JOULE ('Phil. Trans,' 1854), calculated from a more complicated, empirical formula, which does not represent the results of experiment nearly so well. It will be seen that the corrections are so small that it is not worth while to apply them, except in investigations of a higher degree of accuracy than any yet attempted

They are rather larger for a CO_2 air thermometer, but still so small that a CO_2 air thermometer is practically correct up to 300°* (REGNAULT), that is, the corrections are smaller than the uncertainty due to the expansion of the envelope or other causes; the magnitude of this correction varies directly as the initial pressure p_0.

Assuming equation (E), we ought to have for the compressibility of air at 0° C.

$$1 - \frac{pv}{p_0 v_0} = \phi - \phi_0 = \cdot 001173\left(\frac{p}{p_0} - 1\right),$$

or, at 5° C,

$$1 - \frac{pv}{p_0 v_0} = \cdot 00111\left(\frac{p}{p_0} - 1\right) \text{ approximately}$$

From REGNAULT's formula,† if $p = 2$ metres and $p_0 = 1$ metre of mercury, we find, for air at 4·75° C,

$$1 - \frac{pv}{p_0 v_0} = \cdot 00109.$$

But the equation (E) does not agree so well with his results throughout the whole range of his experiments, because it is only a rough first approximation to the departure of air from the law of a perfect gas, and is only applicable within moderate limits of pressure and temperature. At high temperatures dissociation would occur, and the formula would no longer apply In the case of CO_2 the dissociation is quite measurable, even at 1000° C

The Sources of Error to which the Air Thermometer is liable

The first and most important source of error, for which no correction can be applied, is impurity in the dry air. Water vapour and CO_2 are most to be avoided, and the bulb of the air thermometer must be very thoroughly cleaned. The apparatus used for supplying pure dry air, used in the earlier experiments, appeared perfect, but has since been proved faulty. See Appendix, p. 222.

The air thermometers were dried by repeated exhaustion and re-admission of dry air at temperatures between 300° and 500° C

* 'Mémoires de l'Institut,' tom. 21, p 187.

† 'Mémoires de l'Institut,' tom 21, p. 421.

*Changes of Zero Pressure.**

In spite of all precautions, slight changes of zero pressure were observed with all the air thermometers, amounting in some cases to upwards of a millimetre of mercury in a week, these were probably due to surface condensation. In support of this it may be remarked that the vortex atom theory of gases requires the existence of an air film of swollen vortices sticking to the sides of the containing vessel; the phenomenon is also well attested by many experiments, in particular, we may mention that of Professor BUNSEN "on the condensation of CO_2 on the surface of fine-spun glass."† It is still necessary, however, to determine in what way the air film varies with the temperature. I am investigating this, but the experiments are not yet completed. For the present we may assume that it exists and see what kind of error it will produce.

Let m be the mass of air enclosed in grammes; p, v, θ, its pressure, volume, and absolute temperature, k the difference of the specific heats of unit mass of air; σ the surface density of the air film in grammes per square centim.; and S the surface of the air thermometer in square centims. σ is a function of the pressure and temperature, and also apparently of the nature of the gas, and of the surface, of its past history, and of the time.

We shall have the equation

$$k\mathrm{S}\sigma+\frac{pv}{\theta}=mk=k\mathrm{S}\sigma_0+\frac{p_0 v_0}{\theta_0} \quad . \quad . \quad . \quad . \quad (\Sigma)$$

First approximation, $\theta=\frac{p\theta_0}{p_0}$. Solving this to a second approximation, σ being small, we have

$$\theta=p\frac{\theta_0}{p_0}\left\{1+\frac{\mathrm{S}'}{m}(\sigma-\sigma_0)\right\},$$

neglecting other corrections.

Let p_1, p_0, be the pressures corresponding to 100° and 0° C. respectively, then, according to REGNAULT,‡ $p_1/p_0=1\cdot3665$, according to BALFOUR STEWART,§ $p_1/p_0=1\cdot3673$. In REGNAULT's experiments $\mathrm{S}/m=400$ roughly. In BALFOUR STEWART's case $\mathrm{S}/m=1060$, so that, if we assume $\sigma_1-\sigma_0=\cdot000001$ gramme per square centim., we shall derive, both from REGNAULT and BALFOUR STEWART, the value $p_1/p_0=1\cdot3659$, corrected for surface condensation; also, in the case where $\mathrm{S}/m=2400$, as in my air thermometer (No. III), we should expect to find the value

* Similar changes were observed by REGNAULT.

† 'Phil. Mag.,' Series 5, vol. 17, p. 161.

‡ *Vide* 'Mém. Inst.,' tom. 21.

§ *Vide* 'Phil. Trans.,' 1863.

of p_1/p_0, uncorrected for surface condensation, to be 1 3693, which very nearly agrees with that actually observed (see Table A, III.) [For another explanation see Appendix, p 222]

It appears, however, that when the air film has been very completely removed by exhaustion at a high temperature it increases gradually for some time before it has recovered its normal state

To illustrate this, the following Table gives the changes of zero pressure observed with two of my air thermometers.

TABLE. Σ.

Date.	Zero pressure. Oxygen thermometer	Zero pressure. Nitrogen thermometer	Remarks.
1886 Jan 21 . .	69·31	69 54	Four days after filling by exhaustion at 300° C.
„ 22 .	69 29	69·51	
„ 23 .	69 29	69·49	After heating to 100° C.
„ 25 . .	69 20	69 38	
„ 25	69·22	69·41	After heating to 100° C.

Such changes of zero pressure are a most important source of uncertainty, especially if the instrument be used at high temperatures.

We have said a great deal about the air thermometer, because the chief experimental difficulties arose from it; the other essential part of the apparatus is the platinum spiral, which we will now proceed to consider.

The Platinum Spiral.

The spirals used in this investigation were all made from the same reel of pure platinum wire supplied by Messrs. JOHNSON, MATTHEY, and Co. Its purity is attested by the following facts: (1) the high temperature coefficient of its resistance, ·00346, instead of about ·0030, as with ordinary platinum; (2) it was very infusible and unalterable: after being kept for about two hours at a temperature of about 1200° C., its resistance had not altered by 1 in 5000 at 0° C.; (3) no impurities could be detected by chemical analysis.

Details of the wire.—Diameter, about ·017 centim.; resistance, about 5 ohms per metre; mass, ·00488 gramme per centim. About 1 metre of the wire is measured off and its ends fused on to tails of thicker platinum wire (diameter, ·073 centim.) in the oxy-hydrogen blowpipe. (It is less trouble to solder with copper, but the spiral cannot then be boiled with acids in the bulb of the air thermometer.) The wire is then annealed by passing it slowly through the solid BUNSEN flame of a FLETCHER

furnace; this renders the wire very soft. (If the wire is not annealed, its resistance will be slightly diminished on the first heating, owing to a change of structure) It is then wound on a glass tube and annealed again, to make it preserve a perfect spiral form.

Measurement of the Resistance.

If the resistance be too small, chance errors, such as change in the contacts, are unduly magnified To make the resistance large involves either inconveniently fine wire or an awkwardly long spiral. In the actual experiments, therefore, spirals were used with resistances varying from 5 to 20 ohms. The most rapid and accurate way of measuring such a resistance is by the WHEATSTONE bridge method with an ordinary post-office box, and a sensitive THOMSON mirror galvanometer with lamp and scale. Resistances of 10 and 1000 are taken out in the arms of the balance, so that the adjustable resistance is 100 times that of the spiral when the balance is obtained This gives the value direct to four figures, and a fifth may be obtained by accurately observing the galvanometer throws and interpolating.

The box used was a very accurate box of B.A units by Messrs ELLIOTT On the extraordinary accuracy of the boxes the reader may be referred to the paper by GLAZEBROOK, "On the Determination of the Ohm in Absolute Measure" (see 'Phil. Trans,' 1883, p. 262), whence it appears that an uncertainty of 1° C in the temperature of the box will give an error of about ·0004 in the absolute value of a resistance, but that ratios of resistances, if the temperature of the box be uniform may be determined with an error of only about ·00002

The sensitive galvanometer employed had a resistance of about 14½ ohms, being approximately that most suited to the given arrangement of resistances. It was arranged so as to be extremely sensitive. A deflection of one scale division corresponded to a difference of ·0001 ohm when the resistance to be measured was 10 ohms It was also nearly dead-beat to facilitate the reading of deflections for interpolation.

The battery consisted of from four to six LECLANCHÉ elements in series, as the circuit had a resistance of about 500 ohms, they worked steadily.

Sources of Error.

Thermo-electric effects were made as small as possible by careful arrangement of the junctions; they were then completely eliminated by using a rocking mercury cup commutator in the battery circuit, for rapidly reversing the current and observing when the reversal had no effect on the galvanometer; it is important, however, to make the thermo-electric effect small, because, if it is large and variable, it greatly interferes with accurate observation of the resistance by making the galvanometer unsteady.

There were only two screw contacts; these were between the thick copper leads

and the resistance box They were always very tight and clean, and cannot have introduced appreciable errors The plugs of the resistance box were very large and truly ground, and their contacts good

It was suggested that the conductivity of the glass at high temperatures might prove a source of error in the resistance measurements. A rough experiment was therefore tried. A cubic millimetre of the hard glass at a dull red heat had a resistance of about a megohm, and about ten times less at a bright red heat This is less than 1 in 10,000, and may be neglected. Moreover, since only one of the platinum terminals was actually fused through the glass in this case, the bad contact further diminished any possible error from this source. Current heating of the wire would produce a small and nearly constant error, since, however, the current was never greater than ·01 ampère, the effect must have been almost negligible. From experiments on the cooling by air contact of this particular kind of platinum wire, I am in a position to estimate the excess of temperature due to a current of ·01 ampère at about 016° Centigrade.

Reduction of the Resistance.

There are some small corrections to be applied.

The resistance of the copper wires connecting the coil to the resistance box is generally about ·01 ohm, and varies very slightly with the temperature. The whole correction amounts to about 1°/₀, and may be measured on each occasion by means of double electrodes.

The temperature of the resistance box varies slightly. A special experiment gave ·00040 as the value of the temperature coefficient. The box is correct in B.A units at 14·2° Centigrade The temperature is therefore taken by a mercury thermometer kept inside the box, and the observed resistance reduced to 14·2° Centigrade. This correction is also very small, and can be applied with tolerable certainty.

Finally, to render the results comparable with each other, each observed resistance of the platinum coil is reduced to what it would have been for a coil having the resistance of a unit at 0° Centigrade, by dividing by the observed value of the resistance of the coil at 0° Centigrade.

Explanation of the Tables of Comparison of Platinum and Air Thermometers.

The letters at the head of each column have the same meaning as explained in section entitled Formulæ (see page 176). The column headed Thermometers gives the observed temperatures B of the barometer, v of the air near the U gauge, Hg of the mercury in the right limb of the manometer, R of the air in the resistance box, all these being taken with the same mercury thermometer.* Each line contains the

* [This is bad and may introduce small errors. latterly a separate thermometer was set apart for each purpose.]

whole work of the reduction of a single complete observation. The method of observing is as follows:—When the temperature is nearly steady, the approximate resistance having been ascertained, the galvanometer circuit is closed and the moment observed when reversing the battery current with the rocking commutator does not affect the galvanometer; the simultaneous reading x of the U gauge and the time are recorded, and the readings of the manometer and barometer are taken.

The column θ gives the absolute temperature corresponding to the mean corrected pressure P; t the temperature Centigrade $=\theta-\theta_0$; R *observed* the corrected value of the resistance reduced to 1 at 0^c C. by dividing by the value of R_0; R *calculated* the value of R calculated from the empirical formulæ

$$\frac{R}{R_0} = 1+\alpha t-\beta t^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

(see page 194), and

$$\frac{R}{R_0} = e^{\frac{\alpha t}{1+\beta t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e)$$

Observations.—Series I.

I constructed an experimental instrument in November, 1885, of lead glass at the Cavendish Laboratory. It was necessarily imperfect from want of skill in glass working, and lack of the requisite assortment of glass tubing, However, it was well suited to test the capacities of the method, and the observations taken with it agree as nearly as may be expected with subsequent and improved apparatus. Details.—$V_0 = 32{\cdot}25$ cubic centims., $\beta_0 = {\cdot}0378$, $\gamma = {\cdot}0000263$.

The observations are not all given, as they are less accurate for several reasons. Most of them were taken with the temperature varying rapidly, and the mercury manometer belonging to a "Jolly" air thermometer was used, the tubes of which were too narrow for purposes of accuracy. Nearly five centims. of the fine wire was outside the bulb, and the resistance measurements were rendered less accurate by some small thermo-electric effects and a slight uncertainty in the resistance of the connecting wires, and by neglecting to take the temperature of the resistance box.

The capillary tube was also of the same bore as the tube of the H_2SO_4 gauge, as there was none finer to be had on the spot, this of course renders the volume correction more uncertain.

Table A–I. contains all the observations taken when the temperature was approximately steady. The column R *calculated* shows how very nearly they agree with the formulæ. Some of the observations, however, taken when the temperature was varying rapidly about 250° C., are nearly two degrees out.

TABLE A–I.—Containing some of the Results of Observation with the experimental Air Thermometer No. 1.

Number of observation	Temperature by air thermometer.	Resistance observed, reduced	Resistance calculated by formula (a).	Resistance calculated by exponential formula
1	88 0	1·304	1 304	1 304
2	96 9	1·335	1 335	1·335
3	49 9	1 173	1 173	1 172
4	205 5	1 707	1 711	1 708
5	210 2	1 723	1·727	1 724
6	280 3	1 960	1·970	1 958
7	138 7	1·480	1·481	1 481
8	179 1	1 620	1 620	1 619

Observations.—Series II.

Owing to the success of the experimental instrument, a slightly improved form was devised and ordered of a London glass-worker about the beginning of December, 1885, for which I wound a platinum spiral of the same wire at the Cavendish Laboratory; the first spiral was broken and another sent, and other casualties intervened, so that the instrument did not arrive until February, 1886, and then, owing probably to some want of clearness in the specification, I had to alter it in some essential details before it could be used. It was cleaned and calibrated in the usual manner, and finally, on February 13, it was filled with dry air, at a temperature of about 350°, and a series of observations, recorded in Table A–II., were taken with it in a large mercury vapour bath; the fixed points being determined with the aid of a hypsometer and melting ice.

On February 17 another series of observations was begun, but the zero pressure had increased in four days from 34·10 cm. to 36·09 cm., owing to leakage, the platinum wires not having been fused air-tight through the glass. In an attempt to remedy this somewhat serious defect, the instrument was unfortunately broken. This was much to be regretted, as less reliance can be placed on observations taken while the temperature is varying; all the observations are given, however, as they show how perfectly the method works. The time observations show the rate of variation of the temperature. A mercury thermometer in the same enclosure often differed from 20 to 50 degrees, in its indication, from the air thermometer.

Values of R are calculated by both formulæ (*b*) and (*e*) for comparison. The agreement with the formula (*e*) is in some cases so close that it must be fortuitous to some extent; it is, however, satisfactory to see that single observations are so consistent and give so smooth a curve.

Details of Air Thermometer No. II.

Volume of bulb $V_0=77\ 01$ cub. centims.
Total volume of capillary tube$=\cdot 0840$ cub. centim.; diameter, $\cdot 03$ centim
The U gauge tube was of very uniform bore · $u=\cdot 0442$ cub. centim. per centimetre.

$$\left.\begin{array}{llll} x=1, v_0=\cdot 122 \text{ cc}, & \beta_0=\frac{v_0}{V_0}=\cdot 001585, & y_0=\cdot 0084 \\ x=10, \quad v=\cdot 520, & \beta=\cdot 00675, & y=\cdot 0133 \end{array}\right\} \theta'=\theta''=15^\circ \text{ Cent}$$

Mean coefficient of expansion of glass between 0° and 100° Cent, $y=\cdot 0000251$, by mercury weight thermometer.

From the observations in ice and steam, on February 13, we have

$$\theta_0=\frac{99\ 97\ \ 34\ 102}{12\ 620}=270\cdot 12 \qquad\qquad \log\frac{\theta_0}{p_0}=\cdot 89881$$

To find the other temperatures, $\theta=\frac{\theta_0}{p_0}\text{P}$.

For the platinum spiral the resistance of the connexions was, at 0° Cent., $\cdot 0139$, at 353° Cent., $\cdot 0221$. The value of this small correction at intermediate temperatures was found by ruling a straight line on logarithm paper with sufficient accuracy Applying corrections, we find $R_0=5\ 0845$ B.A. ohms, colog $R_0=9\cdot 29375$.

As the U gauge tube had been made unnecessarily long, the 'q' correction was large and negative. By direct comparison with the mercury of the manometer, the ratio of the density of acid to mercury at 15° was determined to be $\cdot 1382$ and the capillarity $\cdot 45$ centim

Thus, when $x=0$, $q=-3\ 12$ centims.; and, when $x=10$, $q=-1\ 75$ centim.

From these details the "correction diagram" is constructed which is shown in Plate XI., fig. 5, for the purpose of illustrating the method of reducing the observations. The ordinate scale, on one side, gives values of y, on the other, values of q, for the same abscissa scale of x A difference of $+5°$ in θ'' makes $y=\cdot 0132$ instead of $\cdot 0133$ when $x=10$, so that variations of θ'' may almost be neglected. Only one line is therefore drawn, corresponding to $\theta''=15°$ C. (The lines of the section paper are only indicated, every tenth line being reproduced in the diagram.)

[When the apparatus was cooling the lower part of the bath was some 50° hotter than the upper. The platinum spiral was hung vertically in this instrument, and was deformed by its own weight, so as to be massed chiefly in the lower and hotter portion of the bulb. The observations taken during cooling are, therefore, affected by a *constant error* amounting to two or three degrees, the mean temperature of the spiral being higher than that of the air.]

TABLE A–II.—Showing Comparison of Platinum and Air Thermometeıs

Thermometers	Manometer readings: Left.	Manometer readings: Right	Difference	Temperature correction.	Sulphuric acid, U gauge: x	Sulphuric acid, U gauge: y	Sulphuric acid, U gauge: q	Time	Resistance balance observed	Corrected for connexions and temperature	Manometer reduced and corrected.	Barometer reduced and corrected	Pressure, p	$\frac{py}{p_0}$	δ	$p\delta$	P	Mean P	No of observation	θ absolute	t Centigrade	R observed	R calculated (b)	R calculated, e)	Difference
R 14 1. . .	51 37	55·77	+ 4 40	— 01	2·63	93	−2 75	h m 10 56	11 153		+1 68	75 96	77 59	212	128	99	78 68								
			4 39		2·77	94	2 73		11 160		1 65		77 61	214	130	100	78 61								
	51·38	55·76	4 38		2 50	92	2 77	11 20	11 150	Mean 11 120 corrected	1 60		77 56	210	126	98	78 54	78 480							
	51·38	55·75	4·37		2 25	91	2 81		11 140		1 55		77 51	207	123	95	78 46			623 7	353 6	2 1872	2 1902	2 1952	
			4·36		1·95	89	2 85	11 38	11 130		1 50	75 96	77 46	203	120	93	78 39			Glyce	rine b	ubble in	pressur	e tube	
	51·39	55 74	4 35		1 67	87	2 89	11 36	11 120		1 45		77 41	200	115	89	78 30					0 1 cm	long		
	51 09	54 02	+ 2 93	— 01	7 15	117	2 14	11 39	11 10	11 078	+0 79		76 75	262	178	136	78 11		(4)	618 7	348 6	2 1788	2 1768	2 1793	+ 0005
	51 12	54·00	2 88		5·98	111	2 30	11 42	11 06	11 038	0 57		76 53	248	164	125	77 78		(5)	616 1	346 0	2 1709	2 1686	2 1711	+ 0002
	51 15	53·97	2 82		4 25	102	2 54	11 44	11 00	10 978	0 27		76 23	227	143	109	77 32		(6)	612 5	342 4	2 1591	2 1570	2 595	+ 0004
	51·18	53·93	2 75		2 80	94	2 73	11 45½	10 95	10 928	+0 01		76 97	208	124	94	76 91		(7)	609 2	339 1	2 1493	2 1466	2 1492	— 0001
	51 21	53 90	2 69		1 45	86	2 92	11 47	10 90	10 878	−0·24	75 96	75 72	191	107	81	76 53		(8)	606 2	336 1	2 1394	2 1369	2 1395	+ 0001
	50·72	51·00	+ 0·28	00	9 50	130	1 82	11 49	10 82	10 799	−1 54		74 42	283	199	148	75 90		(9)	601 2	331 1	2 1239	2 1209	2 1235	— 0004
R 14 3 .	50 78	50 93	0 15		6 05	112	2 29	11 52	10 70	10 679	2 14		73 82	241	157	116	74 98		(10)	593 9	323 8	2 1003	2 0971	2 0998	— 0005
	50·82	50 87	0 05		3 20	96	2 68	11 54	10 60	10 579	2 63		73 33	206	122	89	74 22		(11)	587 9	317 8	2 0806	2 0780	2 0807	+ 0001
	52·09	49 16	— 2·93	+ 01	6 65	114	2 20	12 0	10 30	10·279	5 12		70·84	235	151	107	71 91		(12)	569 6	299 5	2 0217	2 0183	2 0218	— 0004
	52·14	49 11	3 03		3 70	98	2 61	12 3	10·20	10 179	5 63	75 96	70 33	202	118	83	71·16		(13)	563 7	293 6	2 0020	1 9990	2 0020	± 0000
	51·90	47 22	— 4 68	+ 01	4 60	103	2 49	12 8	10 00	9 980	7 16		68 80	207	123	84	69 64		(14)	551 6	281 5	1 9629	1 9594	1 9625	— 0004
	51 93	47 19	4 74		2 90	94	2 72	12 9	9 95	9 930	7 45		68 51	188	104	71	69 22		(15)	548 3	278 2	1 9530	1 9484	1 9515	— 0005
v 16 0 .	51·55	44 35	— 7·20	+ 02	5 90	111	2 81	12 15	9 70	9 680	9 49		66·47	216	132	88	67 35		(16)	538 5	263 4	1 9038	1 9001	1 9028	— 0010
	51·60	44 30	7·30		2 95	95	2 71	12 17	9 60	9 580	9 99		65 97	184	100	66	66 68		(17)	527 8	257 7	1 8842	1 8807	1 8834	— 0008
	51·17	41 15	—10 02	+·02	5·40	106	2 88	12 25	9 30	9·281	12 38		63 68	201	117	74	64 32		(18)	509 5	239 4	1 8254	1 8193	1 8224	— 0030
	51·19	41 12	10 07		3 90	100	2 58	12 27	9·25	9·231	12 63		63 33	186	102	64	63 97		(19)	506 7	236 6	1 8156	1 8104	1 8135	— 0021
	51 21	41·10	10 11		2 35	91	2 79	12 28	9·20	9 181	12 88	75 96	63 08	168	84	53	63 61		(20)	503 9	233 8	1 8057	1 8011	1 8042	— 0015
R 14 5 . .	50 66	37·46	—13·20	+·03	6 85	116	2 19	12 37	8 90	8 881	15 36		60 60	205	121	73	61 33		(21)	485 8	215 7	1 7467	1 7403	1 7430	— 0037
	50·69	37·42	13 27		3 60	98	2 63	12 40	8 80	8 781	15 87		60 09	172	88	53	60 62		(22)	480 2	210 1	1 7270	1 7215	1 7242	— 0028
	50 71	37 40	13 31		2 00	89	2 84	12 42	8 75	8 731	16 12		59 84	156	72	42	60·26		(23)	477 3	207 2	1 7172	1 7118	1 7145	— 0027
v 14 8 . .	63 86	24 41	39·45	+ 10	4·21	101	2 54	3 40			41 89	75 93	34·04	101	17	6	34 10								
Bar 12·2 . .	63·65	24 32	39 33		3 50	97	2 64	3 44			41 87		34 06	97	13	4	34 10								
	63·63	24 35	39·28		3 10	95	2 69	3 49			41 87		34 06	95	11	4	34 10								
Hg 15·0 . .	63 49	24 31	39 18		2 60	93	2 76	3 53			41 84		34 09	93	9	3	34 12					1 0000			
R 14 6 . .	64 58	25 20	39 38		3 78	99	2 60	3 57	5 0985	5 0845	41 88	75 92	34 04	99	15	5	34 09	34 102	2	θ_0=270 12	0° Centigrade		In melting ice washed and pounded		
	64 90	25 36	39 54		4 87	106	2 45	4 0			41·80		34 03	105	21	7	34 10								
Bar 12·5 .	65 79	26 18	39 61		5 48	108	2 37	4 10			41 88	75 91	34 03	108	24	8	34 11								
	66·84	27 74	39 10		2 12	90	2 83	4 12			41 83		34 08	90	6	2	34 10								
v 16·0 .	55·77	28·75	27 02	+ 07	4 60	103	2·49	5 21			29 44	75 90	46 46	140	56	26	46 72								
	55 78	28·76	27·02		4 60	103	2 49	5 26			29 44		46 46	140	56	26	46 72								
Hg 16·5 .	55 76	28 76	27 02		4 60	103	2·49	5 35			29 44		46 46	140	56	26	46 72								
Bar 13·0 .	55·16	28·53	26 63		2 27	91	2 80	5 40	6 8594	6 8432	29 36	75 90	46 54	124	40	19	46 73	46 722	3	370·09	99 97	1 34600=R_1/R_0	In hypsometer, steam		
	54·99	28·41	26 58		1 95	89	2 85	5 47			29 36		46 54	121	37	18	46 72								
R 14 8 .	55·64	28 65	26 99		4 32	102	2 53	5 53			29 45		46 45	139	55	26	46 71								
	55·84	28 38	26 46		1 35	86	2 93	6 8			29 32		46 58	117	33	15	46 73								

Observations.—Series III

Since lead-glass begins to soften at about 450° C., it was considered advisable to make an instrument of harder glass. Air thermometer No. 3 was therefore constructed of combustion tubing. Since hard glass cannot well be joined on to other tubing, the capillary tube was formed by drawing out a piece of combustion tubing, which formed the bulb, the platinum spiral was then inserted and the other end drawn out, and the whole cleaned and calibrated

To avoid leakage, one of the electrodes passed out through the whole length of the capillary tube, and the fine wire itself was sealed through the glass at the other end, a length of 1 centim. being perfectly coated with glass. A capillary tube formed in this way is of course exceedingly fragile, and some delicacy of manipulation is required, but it appears that the instrument cannot be otherwise made of hard glass without a glass-furnace in which a bulb can be blown

This thermometer was suitably mounted and filled with dry air at a temperature of 470° C., and the series of observations recorded in Table A–III. were taken with it. It was heated in a thick iron tube constructed to fit it, and the temperature maintained steady by regulating the gas supply.

It showed no tendency to leak, till at the conclusion of observation (10) it was unfortunately cracked by too sudden exposure to the air after heating in the steam bath.

The chief objection to this thermometer was the small volume of the bulb, and the large volume correction, which necessitated a third approximation; the value of $\frac{S}{m}$ (see equation Σ, p. 181) being large, all the uncertain effects of surface condensation are unduly magnified. For instance, for observation (7) we obtain for the value of the temperature 397·6 or 398·3, according as we use the first or second value of $\frac{\theta_0}{p_0}$. The mean deviation of observations (1), (6), (7), from the curve (*e*) is 2°·5.

For this reason, I made a larger and better instrument on the same plan, but it was unfortunately broken just as it was completed. In spite of this accident, I hoped to make another series of experiments with a better thermometer, but a most unfortunate accident has completely disabled my right hand for the last month and put a stop to my experiments.

Details of Air Thermometer No. 3.

Volume of bulb. $V_0 = 15{\cdot}02$ cub. centims.

Capillary and gauge tube.—$u = {\cdot}0442$ cub. centim. per centim.

$$\left.\begin{array}{l} x = 1 \text{ centim., } v = {\cdot}270 \text{ cub. centim., } \beta = {\cdot}0178 \\ x = 10 \text{ centims., } v = {\cdot}667 \text{ cub. centim., } \beta = {\cdot}0442 \end{array}\right\} \theta' = \theta'' = 288^\circ.$$

By observations on linear expansion we deduce the mean cubical coefficien from 0° to t.

$$\gamma = \cdot 0000204 + 8\cdot 1 \times 10^{-9} t.$$

Pressure correction :—

$$x=3\cdot 12, \qquad q=0\ 0\ ;$$
$$x=0, \qquad q=-\cdot 425\ ;$$
$$x=7, \qquad q=+\cdot 54.$$

Two pairs of observations were taken in ice and steam to determine the fixec points.

From observations (2) and (3), taken on February 23, we have

$$\theta_0 = \frac{100\ 30 \times 37\ 435}{13\ 885} = 270\cdot 42 \,.\, \log \frac{\theta_0}{p_0} = \cdot 85875.$$

From (4) and (5), taken on February 24,

$$\theta_0 = \frac{100\ 18 \times 37\ 400}{13\ 872} = 270\cdot 10 \,.\, \log \frac{\theta_0}{p_0} = \cdot 85864.$$

On February 25, at the beginning and end of the series, we have at 100° 20 C., observation (6),

$$P = 51\cdot 25\ ; \quad \text{and, observation (10),} \quad P = 51\cdot 22 \text{ in steam}$$

For the platinum spiral the resistance of the connexions at 0° was ·0501, and at 470° was ·0635, only a small portion being heated.

For the platinum spiral.—Resistance at 0°, corrected,

$$R_0 = 6\cdot 2374, \quad \log R_0 = \ 79500$$

In the values of the resistance calculated for comparison—

(*b*) refers to the formula

$$\frac{R}{R_0} = 1 + \alpha t - \beta t^3\ ;$$

(*e*) to the exponential formula

$$\frac{R}{R_0} = e^{\frac{00342591}{1+\cdot 001529t}},$$

which last agrees best with the observations.

TABLE A–III.—Comparison of Platinum and Air Thermometers.

Date	No of observation.	Thermometers.	Mercury Manometer L	Mercury Manometer R	Difference R–L.	Correction for temperature.	Sulphuric acid, U gauge, x	Sulphuric acid, U gauge, y	Sulphuric acid, U gauge, q	Time h m	Resistance balance observed	Corrected for connexions and temperature	Manometer reduced and corrected	Barometer reduced and corrected.	Pressure reduced.	$\frac{p_N}{p_0}$	δ	$\delta+\delta^2$	$p\delta'$	P	Mean P	θ absolute temperature	t temperature Centigrade	R observed	R calculated.
Feb. 23	1	B 12 5	46 47	65·22	+18 75	− 05	4 75	339	+ ·23	1 18	1 5950	1 5887	+18 93		95 91	870	635	675	642	102 33	102 35	739 3	468 9	2 5472	(b) 2·5479 (c) 2 5488
		Hg 15·2	46 82	68 05	21 23		1 87	259	− 17	1 25	1 597		21 01	76 98	97 99	679	444	463	454	102 58					
		R. 14 2 .	46 56	65 79	19 23		4 00	318	+ 12	1 33	1 593		19 30		96 28	818	583	617	593	102 21					
		v. 16·1 .																							
Feb 23	2	B 13 3 .	64·72	38 56	−26 16	+ 07	2·40	273	− 10	4 31	8 4561	8 1013	−26 19		50 63	371	136	138	70	51 33	51 32	370 72	100 30	1 3470	1 3470
		R. 13 3	64 70	38 53	26 17		2 32	271	− 11	4 36			26 21		50 61	369	134	136	69	51 30					
		v. 15·1 . .	64 78	38 86	25 92		1·62	252	−·20	4 45			26 05	76 82	50 77	343	108	109	55	51 32					
		Hg. 15·0 .	64·08	35 93	28 15		8 62	448	+ 66	4 52			27 42		49 40	610	375	389	193	51 83					
Feb 23	3	B 13 4 . .	67 83	28 63	−39·20	+ 11	1·40	246	− 24	5 23	6 290*		39 33		37 48	246	11	11	4	37 52			0° Centigrade		
		R. 13 2 . .	67 84	28 63	39 21		1 20	241	−·26	5 25	6·289*		39 36	76 81	37 45	241	6	6	2	37 47					
		v 15·8 .	67·30	27 17	40 13		4·00	343	+ 25	5 27	6·2883	6 2364	39 77		37 04	343	108	109	41	·37 45	37 435	270 42		99983	1 0000
		Hg 14 9	67·74	28 44	39 30		1 37	245	−·24	5 37	6 2881		39 48		37 38	245	10	10	4	37 42					
			66 77	27 40	39 37		1 76	256	−·19	5 43	6 2881		39 45	76 80	37 35	256	21	21	8	37 43					
			68 60	29 24	39 36		1 70	254	−·18	5 49	6·2881		39 43		37 37	254	19	19	7	37 44					
Feb 24	4	Hg 15·2	62 10	36 47	−25 63	+ 07	1 60	252	− 21	4 24	8 4565	8 4004	25 77		50 73	343	108	109	55	51 28	51·272	370 28	100 18	1 3468	1 3467
		v. 15·4	62·06	36 28	25·78		2 00	262	15	4 28	8 4567		25 86	76 50	50 64	357	122	124	63	51 27					
		B 12 8 .	62 10	36 58	25 52		1 82	244	25	4 35	8·4570		25 70		50 80	332	97	98	49	51 29					
		R. 12 8 . .	62·72	37·20	25 52		1 20	241	·26	4 41	8 4569		25 71		50 79	329	94	95	48	51 27					
			63 90	38 22	25 68		1 72	255	19	4 47	8 4568		25 80	76 49	50 69	346	111	112	57	51 26					
			65 58	39 46	26 12		3 10	293	02	4 54	8 4569		26 07		50 42	398	163	166	84	51·26					
Feb 24	5	B 12 9 .	69 29	30 37	38 92	+ 10	1 10	238	−·27	5 42	6 2888*		39 09		37 42	238	3	3	1	37 43			0° Centigrade		
		R. 13 9 .	69·11	29 88	39 23		2·28	270	11	5 45	6 2880		39 24	76 51	37 27	270	35	35	13	37 40					
		v. 16 0 . .	68 67	29 70	38 97		1 20	241	26	5 53	6·2881		39 13		37 38	241	6	6	2	37 40					
		Hg 16·2 .	68 56	29 39	39 17		1 93	261	16	5 58	6 2881	6·2374	39 23		37 28	261	26	26	10	37 38	37 400	270 10		1 0000	1 0000
			69·98	30 43	39·55		3 58	306	+·07	6 12	6 2879		39 38	76 52	37 14	306	71	72	27	37 41					
			70 78	31 52	39 26		2 06	264	− 14	6 21	6 2880		39·22		37 30	264	29	29	11	37 41					
Feb. 25	6	v. 12 0 . .	61 68	35 61	−26 09	+ 05	2 75	282	− 04	9 2	8 4604		26 06		50 50	380	145	147	74	51 24					
		Hg. 11 4 .	61 86	36 35	25 51		1 14	239	·27	9 7	8 4608		25 73	76 56	50 83	322	87	88	44	51·27					
		R. 11·1 .	62·27	36 45	25 82		1 96	261	16	9 9	8 4612	8 4004	25·93		50 63	352	117	118	60	51 28	51 252	370 3	100·20	1·3468	1 3467
		B. 10·0 . .	62 35	36 80	25 55		1·20	241	·26	9 15	8 4613		25 76		50 80	325	90	91	46	51 26					
			63·52	37 52	26 00		2 55	278	08	9 28	8 4611		26 03	76 57	50 54	373	188	140	71	51 25					
			62 87	36·73	26 14		2 83	286	·04	9 33	8 4610		26 13		50 44	385	150	152	82	51 26					
Feb 25	7	Hg 12·5 .	58·20	69 38	+11·18	− 03	3 92	310	+ 11	10 20	14 510		11 26		87 84	743	508	534	470	92 54					
		v 14·3 . .	58·21	69 37	11·16		4 10	341	14	10 22	14 530		11 27		87 85	753	518	545	479	92 64					
		R. 12·2 . .	58·20	69·37	11·17		4 18	323	·15	10 25	14 550		11 29		87 87	760	525	553	486	92 73					
		B 11·2 . .	58·20	69 37	11·17		4 25	325	16	10 28	14 560		11 30	76 58	87 88	765	530	558	490	92 78					
			56·20	69 01	12 81		1 80	257	−·18	10 38	14 560	14·183	12 60		89 18	610	375	389	347	92 65	92 65	660 1	398 0	2 3222	(b) 2 3325 (c) 2 3332
			56 20	69·00	12 80		1 70	255	19	10 41	14 557		12 58		89 16	603	368	382	340	92 56					
			56 20	68 99	+12 79		1 73	256	−·19	10 46	14 558		12 57		89 15	605	370	384	342	92 57					
			56·21	68·97	12 77		2 00	263	15	10 53	14 590		12 58		89 16	623	388	403	360	92 76					
Feb 25	8	Hg 14 0	38 70	72·00	+33 30	− 08	3 10	293	00	11 30	17 94		33 22		109 81	863	628	668	737	117 18					
		v 15 5 .	38 71	71·98	33 27		3 35	300	+ 80	11 35	17 98		33 22		109 81	883	648	690	760	117 41					
		Bar 11·7 .	38·72	71·97	33·25		3 80	318	10	11 42	18 06	17 947	33 27	76 59	109 86	925	690	738	812	117 98	117 68	849 8	579 7	2 8775	(b) 2 8636 (c) 2 8657
		R. 13·6 . .	38 72	71 96	33·24		4 00	318	12	11 48	18 10		33 28		109 87	935	700	749	826	118 13					
			38·73	71·90	33 17		3 17	295	10	11 55	18 00*		33 11		109 70	870	635	675	744	117 14					
Feb. 25	9	Hg. 14·0	55·24	51·05	− 4·19	+ 01	2 38	273	− 10	12 19	12 02		4 28		72 28	526	291	290	216	74 44					
		v 15 6 .	55·23	51 05	4 18		2 20	268	12	12 23	12 01	11 962	4 20		72 27	516	281	289	209	74 46	74 35	536 9	266 8	1 9163	(b) 1 9109 (c) 1 9140
		R 14·1 . .	55·00	51·01	3 99		1·80	257	18	12 28	12 01		4 16	76 56	72 40	496	261	268	194	74 34					
		Bar. 12 3 .	55 00	51·01	3 99		1 70	254	19	12 32	11 998		4 17		72 39	488	253	259	187	74 26					
		The following observations were subsequently taken —																							
Feb. 25	10	R. 14·0 . .								1 0	8 4004			76 55						51 22		100·20			Instrument cracked
Feb. 25	11	R. 14 4 . .								4 0	8 3998			76 53						52 53		100 19			
Feb. 25	12	R. 14·2 . .								5 0	6·2873									37·07					

* Observations marked as doubtful and not included in the means

GENERAL Table of Results IV —Observations at fixed points 0° and 100°.

A —*In melting ice.*

Resistance observed and corrected.	Reduced to 1 at 0°.	Temperature	Date.	No of observation
5 0845	1 0000	0° C	Feb 13	II. 2
6 2364	99983	(Ice not washed)	Feb 23	III 3
6 2374	1 0000	(Ice washed)	Feb 24	III 5
6 2373	1 0000	After long heating	Feb 25	III 12

B.—*In hypsometer steam*

Resistance observed.	Reduced to 1 at 0°.	Temperature by steam	Resistance calculated (*a*).	Date.	No. of observation.	
6 8432	1·3459	99 97	1 3459	Feb 13	II 3	
8·4013	1 3470	100 30	1 3470	Feb. 23	III 2	
8 4004	1 3468	100 18	1 3467	Feb. 24	III. 4	
8 4004	1·3468	100 20	1 3467	Feb. 25	III 6	9 A M.
8 4004	1 3468	100 20	1·3467	Feb. 25	III. 10	2 P M.
8 3996	1 3467	100 19	1 3467	Feb. 25	III 11	4 P.M.

This Table illustrates well the absence of zero error and the constancy of the coefficient of increase of resistance Between observations III. 6 and III. 10 the instrument was kept for some time at a red heat It will also be seen that the two different spirals II. and III. agree perfectly in their temperature coefficient.

*Observations at Steady Temperatures.**

TABLE V.

Number of series and observation	Temperature by air thermometer.	Resistance observed and reduced.	Resistance calculated by empirical formulæ			Resistance calculated by exponential formula (*e*)
			(*a*)	(*b*)	(*c*)	
I 3	49 9	1·173	1 173	1·173	1 174	1 172
I. 1	88·0	1·304	1 304	1 304	1·304	1·304
I. 2	96 9	1·335	1 335	1 335	1·335	1 335
I. 7	138·7	1 480	1 481			1·481
I. 8	179·1	1·620	1·620			1 619
I. 4	205·5	1·707	1·711	1 706		1 708
I. 5	210 2	1 723	1 727	1·722	1·717	1·724
III. 9	266 8	1·9163	1·9232	1·9109	1 9029	1 9140
I. 6	280·3	1·960	1 970	1·955	1·947	1 958
II. 1	353·6 (?)	2·1872	2·2235	2·1902	2 1830	2·1952
III. 7	398·0	2·3222	2·3805	2 3358	2·3266	2 3332
III. 1	468·9	2·5472	2·6224	2·5479	2 5452	2·5488
III. 8	579·7	2·8775	3·0057	2·8636	2·8804	2 8657

* For correction of these observations, see Appendix, p 222

Note Observation II 1 is marked doubtful in my book of notes, as there was a small glycerine bubble in the air tube connecting the mercury manometer with the sulphuric acid gauge ; moreover, it disagrees with observations taken a few minutes after with the same instrument, when the bubble had been dislodged (see II 4, 5, &c , next Table). [Nevertheless, as the temperature was steady, it is not liable to the constant error (see p 187, end)]

Observations with Temperature varying

Table VI.—Thermometer No 2.

No of observation	Temperature Centigrade	Resistance observed	Resistance calculated by empirical formulæ		Resistance calculated by exponential formula (e)	Difference
			(a)	(b)		
23	207 2	1 7172	1 717	1 712	1 7145	— 0027
22	210 1	1 7270	1 727	1 722	1 7242	— 0028
21	215 7	1 7467	1 746	1 740	1 7430	— 0037
20	233 8	1 8057	1 809	1 801	1 8042	—·0015
19	236 6	1 8156	1 819	1 810	1 8135	— 0021
18	239 4	1 8254	1 828	1 819	1 8224	— 0030
17	257 7	1 8842	1 891	1 881	1 8834	— 0008
16	263 4	1 9038	1 911	1 900	1 9028	— 0010
15	278 2	1 9530	1 963	1 948	1 9515	— 0005
14	281 5	1 9629		1 959	1 9625	—·0004
13	293 6	2 0020		1 999	2 0020	+·0000
12	299 6	2 0217		2 018	2 0213	— 0004
11	317 8	2 0806		2 078	2 0807	+ 0001
10	323 8	2·1003		2 097	2 0998	— 0005
9	331 1	2 1239		2 121	2 1235	— 0004
8	336 1	2 1394		2 137	2 1395	+ 0001
7	339 1	2 1493		2 147	2 1492	— 0001
6	342 4	2 1591		2 157	2 1595	+·0004
5	346 0	2 1709		2 169	2 1711	+ 0002
4	348 6	2 1788		2 177	2 1793	+ 0005

The above are all single observations, yet they are remarkably consistent with each other, and give a very smooth curve. They are given to show the capabilities of the method, and the great sensitiveness of a fine platinum spiral

Empirical Formulæ

It is usual to apply the method of least squares to calculate the empirical formula which best suits the observations. This has the advantage of giving a great appearance of accuracy and completeness. In the present case, however, the gain, if any, would not compensate the trouble, and the simpler method of calculating from selected numbers gives equally good results.

If we assume as an empirical formula

$$R/R_0 = 1 + \alpha t + \beta t^2 + \gamma t^3,$$

and, taking data from the curve, calculate the values of the constants α, β, γ, we shall obtain a more or less unsatisfactory formula of interpolation To show how much the values of α, β, γ, vary for slightly different data, the following Table may be interesting —

Data			Formula.	Values of constants		
100°	300°	500°		α	β	γ
1·3460	2 024	2 640	(b')	0034508	+ 00000020	– 00000000108
1 3460	2 020	2 640	(b)	0034675	0	– 00000000075
1 3460	2·010	2 640	(c)	003505	– 00000045	0

The formula (b') represents the observations fairly well within the experimental range, but is quite inadmissible for extrapolation, for it has a maximum $R/R_0 = 3{\cdot}52$ at about 1090° C., whereas I have observed values of R/R_0 exceeding 5 0. The second formula,

$$R/R_0 = 1 + \alpha t - \beta t^3, \qquad \ldots \qquad (b)$$

has only two constants, and represents the curve nearly as well, but is subject to the same objection The third formula (c) is approximately the nearest parabola, but does not represent the observations so satisfactorily at low temperatures.*

SCHLEIERMACHER,† who has incidentally investigated the question, unfortunately gives no numbers, but represented his results graphically by drawing a series of straight lines for each interval of 100°, using the curve thus obtained to give the temperature in terms of the resistance. His observations extended to 1000°, but he does not publish the curve.

In 1871 SIEMENS suggested that the true law of change of resistance with temperature was $R = \alpha\theta^{\frac{1}{2}} + \beta\theta + \gamma$; where α, β, γ, are constants and θ the absolute temperature

The nearest empirical formula of the SIEMENS type calculated from the above data is

$$R = {\cdot}03771\theta^{\frac{1}{2}} + 0025200\theta - {\cdot}2450. \quad \ldots \quad (s)$$

This differs radically from the other empirical formulæ. It represents the observations about as well as the parabola (c), but at high temperatures $dR/d\theta$ approaches

* For correction of this formula, see Appendix, p. 220

† WIEDEMANN, 'Annalen,' vol 26, p 287, 1885

the limit ·00252, and does not vanish or give a maximum value for R. SIEMENS' formula makes R negative when $\theta = 0$

SIEMENS found for platinum

$$R = \cdot 039369\theta^{\frac{1}{2}} + \cdot 00216407\theta - \cdot 2413,$$

which happens to agree nearly with (*s*), except that (*s*) is straighter and steeper

It has been usual to use a formula of the type

$$R/R_0 = 1 + \alpha t + \beta t^2$$

for extrapolation, which corresponds to Formula (*d*), pp. 163, 165.

For instance, BENOIT gives $R/R_0 = 1 + \cdot 002454t - \cdot 000000594t^2$, and recently HUMPIDGE * $R/R_0 = 1 + \cdot 0027t - \cdot 00000019t^2$, with these we may compare my formula.

From these we deduce, as on p. 165 ·

BENOIT . .	$\delta = 2{\cdot}48$	$R_1/R_0 = 1{\cdot}24$ (roughly)
HUMPIDGE .	$\delta = 0{\cdot}71$	$R_1/R_0 = 1{\cdot}27$,,
CALLENDAR .	$\delta = 1{\cdot}57$	$R_1/R_0 = 1{\cdot}34$,,

BENOIT took the boiling point of cadmium to be 860° C., instead of about 770° C. But the discrepancies in the values of δ seem too large to be entirely attributable to errors of observation

Formulæ of extrapolation, unless very accurately verified through a wide range, are unsatisfactory, since they disagree so widely at high temperatures, and a very small change in the data from which they are calculated makes a great difference in the value of the coefficients , whereas, if instead a formula of the type $R/R_0 = 1 + \alpha t$ were used, it seems probable that different platinum wires would give almost identical results

In support of this, the reader is referred to the "*Comparison of platinum wires*" (p 200). For the value of α in this formula we substitute the mean coefficient between 0° and 100° C For instance, in the present paper we use the formula (*a*),

$$pt = \frac{R - R_0}{R_1 - R_0} 100, \quad \ldots \quad (a)$$

and call the temperature so calculated "*temperature by platinum wire*," just as we speak of "*temperature by mercurial thermometer*," although mercurial thermometers differ much more widely and irregularly from each other † Temperatures so calculated will be strictly comparable, and the actual relation between platinum and absolute temperature may be subsequently determined by a more enlightened and skilful

* 'Roy Soc. Proc.,' vol. 39, No 239, Feb. 1885.

† See REGNAULT, 'Mémoires de l'Institut,' tome 21

generation: meanwhile, the wire may be used as a standard, and preserve the record for comparison with future determinations. And, since the range of the platinum thermometer is very great, and different wires may be accurately compared throughout their range, we may expect that by adopting such a standard we shall at any rate secure consistency in place of the discrepancies at present prevailing

[An exponential formula of the type $\log R/R_0 = \alpha t/(1+\beta t)$* was found to represent the comparison of platinum and iron very approximately (see p. 202) If we assume from Series A.–III the data $t = 100°$, $R/R_0 = 1{\cdot}3460$ $t = 500°$, $R/R_0 = 2\ 6400$, and calculate the values of the constants for platinum, we obtain $\alpha = {\cdot}0034259$, $\beta = {\cdot}0015290$. By an odd coincidence it happens that this formula satisfies almost exactly the observations at varying temperature in Series A –II. Values of R calculated by this formula (*e*) are therefore given in Tables A.–II. and A.–III. for comparison.]

Comparison of Different Wires.

Different specimens of wire have very different coefficients; when, however, the resistance-variation of one specimen of wire has been determined by direct comparison with an air thermometer as above described the resistance-variations of other wires may be compared with the standard by a method as accurate as the comparison of resistances, and practically independent of all knowledge of temperature. It is only necessary that the temperature should be approximately uniform and steady enough to allow observations to be taken. The method consists in winding the wires to be compared side by side on a cylinder of non-conducting material, such as clay, porcelain, or asbestos. By this arrangement we may be sure that their mean temperatures are the same, and we have only to measure their resistances several times in rapid succession when the temperature is steady, or oscillating slightly about a maximum, and take the mean of the observations. The resistance of the standard wire gives the mean temperature, which is the same for both coils. If, however, the resistance-variation of the standard wire be not considered known with sufficient accuracy, this does not affect the accuracy of the comparison as such; the resistance-variation of the others is simply referred to temperature on the scale of the standard wire.

The simultaneous resistances are measured by precisely the same apparatus and the same method as before. They are therefore liable to the same errors and corrections; that is to say, they are corrected for the resistance of the connecting wires and the temperature of the resistance-box: the latter is less material, since it affects both wires proportionately. The resistances are reduced to a unit at 0° C.; thermo-electric effects are especially to be avoided, and are, of course, eliminated as before.

[illegible] changes of structure may not occur, producing permanent changes in the [illegible] the wire must be first annealed. To prevent chemical changes such as

* 'Mémoires de l'Institut,' tom. 21, p. 619.

oxidation, which would produce the same effect, the wires are sealed up in a hard glass or porcelain tube nearly exhausted, from which the air has been removed by substituting CO_2 or N_2

The insulation resistance from one wire to the other can also be measured with the same bridge if less than a megohm. It may be neglected if it is greater than 100,000 ohms. This method affords also a fairly accurate means of measuring the temperature variation of the resistance of non-conductors, which begin to conduct electrolytically when heated. The chief difficulty is polarisation, but this may be avoided in the usual way.

It would, of course, be possible by this method of comparison to draw up very accurate Tables of the resistance variation of metals and other conductors referred to platinum as the standard. Such Tables would, however, be of little practical use, apart from the particular specimens of wire for which the comparisons were made, except in so far as they tended to elucidate the phenomenon itself. A Table of empirical formulæ of the type $R/R_0 = 1+\alpha t+\beta t^2$ has been published by BENOIT for most common metals.

The method of comparison was not adopted, as far as I can discover, in spite of its obvious advantages, and as he used different specimens of wire his results do not agree very closely with mine. It is, however, from a thermometric point of view that this method of comparison is of the greatest possible importance. For this reason we proceed to give full details of a comparison between two platinum wires Since the same apparatus was used for all the comparisons it will be described once for all.

The Diagram of Connections (Plate 11, fig 3) shows the double Wheatstone bridge arrangement adopted

The arm of the balance, AB, is divided in the ratio $\frac{BC}{AC}=\frac{10}{1000}$; the adjustable resistances, AD, AE, are taken from post-office boxes, and may have any value up to 10,000. The sensitive galvanometer may be put in circuit between D and C, or C and E, or D and E, so that the resistances of either DB or BE, or their ratio, may be observed without the necessity of constantly altering the resistances in the arms AE, AD, except in so far as is necessary to follow the slow change of temperature. A complete observation of both resistances can be thus made in a few seconds. The actual arrangement of the apparatus is shown in Plan (fig 4), which is lettered to correspond with the diagram. The ends of the wires to be compared are fused on to copper tails, and the junctions arranged close side by side at J to avoid the production of thermo-electric currents. F represents the rocking commutator in the battery circuit to eliminate any residual effects. The copper tails are in turn soldered to thick copper rods, passing, air-tight, through an indiarubber cork which fits the porcelain tube in which the wires are heated. The copper rods are amalgamated, and dip into mercury cups, numbered 1, 2, 3, 4, 5, on the plan. Cup No. 2 is connected by a thick copper rod to the binding-screw, B, of the resistance-box. Cups 2 and 3

are connected by a thick copper piece. Thick copper wires pass from D and E to Cups 1 and 5, C, D, and E, are connected to mercury cups d, e, c, into which the galvanometer terminals may be dipped. The porcelain tube and the apparatus for exhausting and heating it are not shown The method of interwinding the wires is roughly indicated by the plain and dotted lines. To observe the resistance of the connexions, Cups 4 and 5 are short-circuited, and the resistance B3J4D is measured The copper tails corresponding to Cups 3 and 4 are fused together at their extremities, J, where they join the platinum wire; by symmetry, therefore, this resistance represents that of the connexions, which may be taken to be the same for both coils, and is observed on each occasion, since it varies slightly with the temperature

To test the insulation, the copper piece connecting 2, 3, is removed, and the resistance between the coils observed as a matter of precaution. The insulation was, however, found, in all cases, to be amply sufficient The more nearly the wires are alike, the more accurate the comparison, since all sources of error affect both more equally. This is illustrated by the following comparison between two platinum wires, one of which was the standard wire, described previously, and the other was sent from Messrs JOHNSON, MATTHEY, and Co, in reply to the request for the purest possible platinum wire As was expected, their temperature coefficients nearly agreed. The ratio of their resistances was, in consequence, very nearly constant. Let R, R′, denote the simultaneous resistances of the wires, and r their ratio, then we have, evidently,

$$\frac{R}{R_0}-\frac{R'}{R'_0}=\frac{R}{R_0}\left(1-\frac{r}{r_0}\right)$$

Since r is very nearly constant, the term $\left(1-\frac{r}{r_0}\right)$ is always very small; consequently, if r be observed accurately, an approximate value of R will suffice to give the differences of the temperature coefficients with all the accuracy of which the resistance-boxes are capable. It is evidently unnecessary to correct the value of r for changes of temperature in the resistance-boxes Errors will, however, be introduced by any want of uniformity in their temperatures, or in the material of which the various resistance coils are composed. Precautions were therefore taken to maintain the temperature of the boxes as uniform as possible

The correction to be applied to the observed value of the ratio r to allow for the resistance of the connexions is taken as $(r-1)\frac{z}{R}$, where z is the observed resistance of the connexions; this is sufficiently approximate, since z is small

The great advantage of the method of ratio is that the observation affects both coils simultaneously; moreover, when the ratio is nearly constant, the reduction of observations is much more simple, and may be effected with sufficient accuracy by the use of a small slide-rule, since it depends on small differences.

The following Table is given to show the method of recording observations, and practically explains itself. The letters M, E, refer to the resistance-boxes used. The galvanometer throws are given in scale-divisions, and could be read with great accuracy

for the purpose of interpolation The close agreement of the values of the ratio r, obtained from different resistances and at considerable intervals, shows the accuracy of the resistance-boxes and the reliability of the method Observations taken at lower temperatures were even more concordant The porcelain tube was enclosed in an iron tube to render the temperature more uniform and steady.

TABLE IX

Time, May 25	Quantity observed	Resistance from box	Galvanometer throws	Deduced value	Remarks.
h m					
4 15	R	E 14 38	Balance	14 380 E	Iron tube full red hot
	$r=\left(\frac{R'}{R}1000\right)$	M 640	15 left	64026 $\frac{M}{E}$	Temperature nearly steady
		641	43 right		
		640	15 left		
4 21	R	E 14 39	Balance	14 390 E	Temperature of air inside box E, 18° 5 C., constant throughout this hour
	$\left(\frac{R'}{R}14\ 39\right)$	M 921	11 left	19 2138 M	Ratio deduced, 64029 $\frac{M}{E}$
		922	18 right		
	Resistance of connexions {2, 3, 2}		150 right 105 left 148 right	0259 E	
	Insulation resistance	$\frac{10000}{10}$2250		225000 E	
4 33	R	E 14 43	Balance	14 430 E	
	$\left(\frac{R'}{R}\ 1443\right)$	M 924	Balance	9 240 M	Ratio deduced, 64033 $\frac{M}{E}$

The following Table X. shows the reduction of some of the observations The first pair of observed resistances are reduced from the observations recorded in the previous Table The difference of reduced resistances is found by the simplest possible reduction from the corrected ratio by means of the formula

$$\left(\frac{R}{R_0}-\frac{R'}{R_0'}\right)=\frac{R}{R_0}\left(1-\frac{r}{r_0}\right).$$

The value of R/R_0 is only approximate, and the difference of the mean coefficients $(\alpha-\alpha')$ is found by dividing $\left(\frac{R}{R_0}-\frac{R'}{R_0'}\right)$ by the approximate value of the temperature. The mean coefficient of the standard wire is assumed to be constant, and the relative variations of the other tabulated, the column of temperatures is calculated by assuming for each wire a constant coefficient equal to the mean value between 0° and 100°; the difference column shows how very closely the two platinum wires agree: although their temperature coefficients differ about 3 per cent., yet the temperatures deduced by using either wire agree to nearly 1 in 1000 Several other observations were taken at different times with the same comparison coils; they all agree within the limits of error of the resistance-boxes. No zero variation of either wire could be with certainty detected; and observations taken with the temperature varying at the rate of 10° per minute agree to 1 in 5000 with the best.

TABLE P -I —Showing Reduction of Observations by Ratio Method.*

Resistance pairs observed.	Ratio observed $\frac{R'}{R}$, $\frac{M}{E}$	Resistance of connexions observed. z.	Correction to ratio for z $(r-1)\frac{z}{R}$	Ratio r corrected.	Ratio difference $\left(1-\frac{r}{r_0}\right)$	Difference of resistances. $\left(\frac{R}{R_0}-\frac{R'}{R_0'}\right)$	Resistance pairs reduced $\left(\frac{R}{R_0}, \frac{R'}{R_0'}\right)$	Mean coefficients.	Difference $(a-a')$	Values of temperature by Formula (a), p 195 pt_1, pt_2.	Difference (pt_1-pt_2)	Temperature by air thermometer, approximate
R. 14·400	·64030	0259	— 00065	63965	01740	0510	2 9300	003460	000091	557 80°	—0 76	596
R'. 9 2203							2 8790	003369		558 56		
R. 10·6400	64259	·0225	— 00076	64183	01410	0305	2 1700	003460	000091	338 15	—0·59	348
R' 6 8372							2 1395	·003369		338 74		
R. 5 5000	·65013	·0174	—·00111	64902	00304	0034	1 1200	003460	000098	34 68	+0 02	34 7
R'. 3·5757							1 1166	003362		34 66		
R 5 2300	·65099	·0170	— 00114	64985	00176	0019	1 0650	003460	000100	18 78	+0 02	18 8
R' 3 4047							1 0631	003360		18 76		
R 6 5931	64744	0192	—·00103	64638	00711	0095	1 3429	003460	000096	99 15	0 00	99 15
R'. 4 2686							1 3334	·003364				
R 4 9100	65220	0169	— 00120	65100	00000	0000	1 0000					
R'. 3 2023							1 0000					

* For improved method of reduction, see Appendix, p 225

Comparison of Platinum and Iron.

The comparison of platinum and iron is interesting because iron behaves so peculiarly A considerable number of comparisons were made by different methods; the most accurate is the last series, VI, for which the same apparatus was used as in the comparison of two platinum wires. All the series, however, agree with each other as nearly as can be expected, considering the different modes of treatment, and that ordinary commercial soft iron wire was used.

The first experiments were made at the beginning of November, 1885, with unannealed iron wire wound side by side with platinum on a strip of asbestos board, and heated in an air bath up to 200° C., a mercury thermometer being used to show when the temperature was steady.

The following simultaneous values of the resistances were observed :—

Iron	2 254	1 890	1 449	1 075	Series I
Platinum	1 700	1 536	1 294	1 055	
Temperature, *pt*	202 0	0154 7	85 1	16 0	

Afterwards the wire was partially annealed, and the same coil was heated in paraffin in the same air bath, with the following results —

Iron	2 250	1 941	1 278	1 143	1 000	Series II.
Platinum	1 701	1 561	1 194	1 105	1 000	
Temperature, *pt*	202 2	161 8	56 0	30 1	In ice	

In order to command higher temperatures, a new comparison coil was made and enclosed in an exhausted tube, to prevent oxidation of the iron, which was first carbonised by annealing in coal gas; the whole was heated in an iron tube by a regulated gas furnace. The following observations were taken :—

Iron observed	6 015	3·704	1 5170	2 3840	3 2414	Series III.
Platinum	2 8264	2 225	1 3388	1 7713	2·0807	
Temperature, *pt*	528 0	354 1	98 0	223 2	312 1	

To investigate more particularly the effect of annealing, another comparison coil was wound with the iron wire unannealed, and observations were taken with it up to 100° C., with the following results .—

Order of observation	2	3	4	5	1	
Resistance of iron	1 5330	1 4114	1·2486	1 0704	1·0688	Series IV.
Resistance of platinum	1 3463	1 2732	1·1721	1·0507	1·0495	
Temperature, *pt*	100 09	79 0	49·6	15 0	14 6	

Before annealing, the resistances at 0° C. were :—Resistance of iron, 5·001 ohms; resistance of platinum, 9·887. After annealing, resistance of iron, 5·557 ohms;

platinum, 9·886. So that the resistance of the iron was considerably increased, while that of the platinum remained practically unaltered. The following observations were then taken :—

Resistance, iron observed	1·0803	1·0905	1·5256	1·3135	3·5718	2·5960	Series V
Platinum	1·0582	1·0673	1·3462	1·2162	2·1927	1·8517	
Temperature, pt	17·0	19·4	100·06	62·5	345·0	246·4	

They show how slightly the temperature coefficient was affected by annealing.

Curve I in the diagram, Plate 13, fig. 11, is constructed from the following observations, which were taken by heating platinum iron comparison coil in a vacuum in a porcelain tube which was continuously exhausted to remove the steam which is given off by the clay when first heated, and which vitiated the previous series.

Iron	1·5263	6·2667	7·0566	4·8396	3·6715	2·7692	2·0036	1·1732	Series VI.
Platinum .	1·3453	2·8840	3·0314	2·5548	2·2283	1·9220	1·5990	1·1225	
Temperature, pt ..	99·79	544·5	587·1	449·4	355·0	266·5	173·2	35·4	

Many other observations were taken with the same apparatus at different times. The resistance of the iron, when once annealed, was not found to be much permanently altered if due precautions were taken.

The curve representing these and other observations belonging to the same series at once suggests an exponential formula. If we assume that the resistance variations of platinum and iron may both be represented by formulæ of the type

$$R = e^{\frac{\alpha t}{1+\beta t}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

and if R_1, R_2, be any two observed values of the resistance of the platinum coil, and R_1', R_2', those of the iron coil observed simultaneously, we shall obtain at once, by equating values of the temperature for each observation, the conditions

$$\frac{\alpha}{\log R_1} - \frac{\alpha'}{\log R_1'} = \frac{\beta - \beta'}{\log e} = \frac{\alpha}{\log R_2} - \frac{\alpha'}{\log R_2'}.$$

Whence the ratio

$$\left(\frac{1}{\log R_1} - \frac{1}{\log R_2}\right) \Big/ \left(\frac{1}{\log R_1'} - \frac{1}{\log R_2'}\right) = \frac{\alpha'}{\alpha} = \text{constant}.$$

By taking for R_1, R_2, the extreme values at either end of the series

$$\left.\begin{array}{l} R_1 = 1·3453, \\ R_1' = 1·5263, \end{array}\right\} \qquad \left.\begin{array}{l} R_2 = 3·0314, \\ R_2' = 7·0566, \end{array}\right\}$$

we obtain the mean value of the ratio $\frac{\alpha}{\alpha'}$ for the whole series to be

$$\frac{\alpha}{\alpha'} = ·75037.*$$

* By some oversight I used the value ·75034 in the calculations.

Testing the observations in pairs by this condition, the value of the ratio is found nearly constant ; a small deviation from the curve produces a large variation in the ratio, which is a ratio of small differences, and is of course immediately deducible from the differential equation corresponding to Formula (e)

The comparison gives the values of the ratios α α' $(\beta-\beta')$, but not the absolute values of the coefficients Its accuracy is not therefore affected by that of the particular values assumed for the standard wire constants α and β.

If we take the observations on the comparison of two platinum wires and treat them in the same way, we shall find, from observations (1) and (5), the mean value of the ratio

$$\frac{\alpha'}{\alpha} = \cdot 97295.$$

So that, if we assume $\alpha = \cdot 0034259$,

$$\alpha' = \cdot 0033332.$$

From this we obtain

$$\beta - \beta' = \cdot 00000343,$$

so that, if we take the value of β previously found, namely, $\beta = \cdot 0015290$, we shall find

$$\beta' = \cdot 0015256$$

So that β' is nearly equal to β.

If $\beta = \beta'$, we have evidently $\frac{\log R}{\log R'} = \frac{\alpha}{\alpha'}$, R and R$'$ being the values of the resistances at the same temperature.

For iron, assuming the same values of α and β, we shall find

$$\alpha' = \cdot 0045657, \qquad \beta' = \cdot 0007767.$$

The chief difficulty of the comparison is that of protecting the iron wire from alteration when it is maintained at a red heat for some time. In Table XI. the observations given in Series VI are compared with the above exponential formula (e). The observations were mostly taken on different days, and the resistances observed at the air temperature in the intervals show the direction in which the correction for zero variation should be applied, but not its amount. It is noteworthy that in all cases the correction for this would tend to reduce the small differences between the formula (e) and the observations. As it is, the mean difference is only $0^{\circ}\cdot 1$ C.

TABLE XI.—Comparison of Platinum and Iron.

Temperature Centigrade			Resistances reduced to 1 at 0° C			Differences between observed and calculated.	
By air thermometer Formula (e) $\frac{1}{t}=\frac{a \log e}{\log R}-\beta$.	By platinum wire Formula (a) $pt=\frac{R-R_0}{R_1-R_0}100$	By iron wire Formula (a′) $fe=\frac{R'-R_0'}{R_1'-R_0}100$	Simultaneous pairs observed. Platinum	Simultaneous pairs observed. Iron.	Iron calculated by Eqn. (e′) $\log R'=\frac{a' t \log e}{1+\beta t}$.	Expressed in percentage of R′	Expressed in degrees of temperature Cent
	°	°				%	°
0	0	0	1	1	1	0	0
35 57	35 41	32 8	1 1225	1 1732	1 1724	0 07+	0 14+
99 79	99 79	99 79	1 3453	1 5263	1 5263	0	0
173 3	173 2	190 3	1 5990	2 0036	2 0057	0 10−	0·15−
269·2	266 5	335 4	1 9220	2 7692	2 7667	0 09+	0 11+
364·1	355 0	506 4	2 2283	3·6715	3 6636	0 21+	0 20+
470 9	449 4	726 0	2·5548	4 8396	4 8342	0 11+	0 09+
586 3	544 5	998 4	2 8840	6 2667	6 2775	0·18−	0 14−
640 9	587 1	1148·2	3 0314	7 0566	7·0576	0·01−	0 01−

It does not appear that any equally simple and convenient empirical formula could be found to represent the observations so closely. [For the explanation of the small deviations, see Appendix, p. 228.]

Comparison of Platinum and Tin.

Pure tin wire, diameter ·025 centim.
Coil wound on cylinder of asbestos in paraffin in a glass tube
Resistance of copper connexions ·0150 B.A.
Resistance in melting ice of platinum 7·8113, of tin 3·2425.
In hypsometer steam at 100·34, platinum 10·520, tin 4·7268.
Insulation resistance at 200° C., 170,000 B.A.
Insulation resistance at 0° C., 300,000 B.A.

TABLE of Results; Means reduced.

Tin observed . .	1·4599	1·4052	1·3086	1·2440	1·0711	2·168
Tin calculated . .	1 4599	1·4057	1·3092	1·2438	1 0714	2 000
Platinum . . .	1·3474	1·3082	1 2376	1·1888	1·0576	1·710
Temperature . .	100·34	89·1	68·7	54·6	16·5	205·3

The temperature coefficient of this platinum coil, taken from the observations in ice and steam, comes out ·003462: almost, within the limits of error, the same as for other specimens from the same reel. The chief difficulty encountered arose from the softness of the tin near its melting point. The last observation is rather uncertain, as the metal gave way before the temperature was sufficiently steady. The coil was

mended, but the wire gave way again below 100° C. It was, therefore, decided to try by a slightly different method, and in particular to protect the wire from strain. Some difficulty was found at first in fusing the fine tin wire on to the copper connecting wires. Solder was evidently inadmissible if the experiments were to be pushed up to the melting point of the tin.

The tin resistances, calculated from the formula

$$R/R_0 = 1 + \cdot 004323t + \cdot 00000260t^2, \quad \text{whence } R_1/R_0 = 1\cdot 4583,$$

for purposes of comparison, show that, as might be expected, the observations between 0° and 100° are very consistent and continuous. It is probable, however, that, as the last observation seems to indicate, the simple formula will only represent the observations well between narrow limits.

Other formulæ for tin are:—

$$\text{Benoit.—}R/R_0 = 1 + \cdot 004028t + \cdot 000005826t^2, \text{ whence } R_1/R_0 = 1\cdot 4611.$$
$$\text{Matthiessen —}R_0/R = 1 - \cdot 0036029t + \cdot 000006136t^2, \text{ whence } R_1/R_0 = 1\cdot 4262.$$

Effect of heating Wire in a Bunsen Flame.

A small spiral of the standard platinum wire was made, having a resistance of about 1 ohm, and its ends fused on to short lengths of stouter platinum, and these on to thick copper leads.

Resistance of the whole, cold, at 16°·0 C, 1·0496 B.A.

Resistance of copper connexions, about ·0140 B.A.

The fine coil was raised to a bright orange (in full daylight) by being immersed in a solid FLETCHER gas flame; its resistance varied slightly on either side of 4·81 with varying draughts of air, half the thick platinum was heated red-hot. This was continued for about an hour. When the whole had cooled again to the temperature of the air, which had risen to 16°·2 C, the resistance was measured, and found to be 1·0502 B.A. The increase is almost exactly accounted for by the higher temperature. Readings were then taken for some time with the coil in various parts of the flame, and with different states of flame, with the following approximate results:—

Description of flame, and part employed.		Resistance observed	Temperature calculated by empirical formulæ			
			pt	(*s*)	(*c*)	(*e*)
Flame burning feebly, solid just enough to cover the coil		4·77–4·78	1085	1190	1280	1503
Gas full on, flame very large and solid	Edge of flame	4·86–4·87	1110	1230	1320	1571
	Centre of flame	4·65–4·67	1050	1160	1240	1435

In this experiment the wire is radiating freely, so that it attains, *not* the temperature of the flame, but a point at which the loss of heat by radiation balances the heat received from the impact of the burning gas. The more rapid the current, the higher the temperature reached, this accounts partly for the higher temperature to which the spiral was raised in the edge of the flame, where the current is much stronger as well as the supply of air better By suitably screening the wire it would probably be possible to raise it to a higher temperature. The temperature can only be roughly estimated by the radiation, because the latter is selective. If the wire be blackened (platinised) it appears much brighter, because it radiates better, but it is for that very reason not so hot, as the measurement of its resistance shows.

Appendix.

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Since the foregoing paper was communicated to the Royal Society, so many more extended and accurate experiments have been made that it was thought desirable to delay the publication of the paper for a few months, in order that some account of them might be included It was felt that this would render the paper more useful without adding excessively to its length. The observations with the air thermometer in particular have been rendered more accurate, so that the temperature may be determined by aid of the platinum wire with a probable error of less than a degree at 600° C. A much improved method of representing the results graphically has also been adopted (see Plate 13), which shows at a glance the kind of accuracy attained. More accurate and extended comparisons tend to confirm the general conclusions from the earlier experiments that the curves of resistance-variations of different wires of the same metal are similar, but not coincident; that the usual parabolic formulæ give an accuracy, of the order of 1 per cent. only, through a range 0°–600° C.; that the deviations therefrom are not readily expressible by any simple formulæ.

Considerable improvements have been made in the primitive apparatus. An improved form of gauge was devised and ordered about the middle of June 1886, which was subsequently found to work well and to possess many advantages over the primitive kind; the principle was exactly the same as that described on page 172, but the mechanical provisions were more complete and perfect. They will be readily understood on reference to the diagram, fig. 6, Plate 12. The sulphuric acid U-gauge proper is represented by the tube FGHJK; it has a three-way tap T_1, which may be closed in the middle of the acid, so that it cannot move either way, whatever the variations of pressure; this is a most important practical improvement.

The requisite amount of acid is introduced through the side-tube P_1, and the scale

FG should be adjusted so that the acid stands at zero when the pressure is the same on either limb of the gauge at F and K.

In addition to the U-gauge there is another connection between the air thermometer C and the manometer W by way of the bulbs A, M, N, which is filled with pure dry mercury, and is used for observations at constant pressure, for calibrating the bulb with air on the method of the volumenometer, and as an open way through which the bulb may be exhausted and refilled with dry air. It communicates by a three-way tap T_2 with the manometer and P_2; and by T_3 with P_3, and through a long rubber tube with a mercury reservoir Q, which may be adjusted at any height, and which communicates with the outer air through a drying tube R containing $CaCl_2$ and glass wool to keep out damp and dust. Mercury may be introduced or withdrawn *to any extent* by aid of the reservoir Q, whose height is adjusted to suit the pressure, or in *weighed quantities* by the tubes P_2, P_3, which dip into weighed beakers of mercury.

The bulbs M, N, H, J, in the figure are convenient for taking up leakage, and other practical details.

To determine the pressure correction in terms of mercury corresponding to each reading of the scale FG, the way AMN is cleared by drawing off the mercury into the reservoir Q, the tap T_3 is turned off, and T_2 turned thus (**T**), and P_2 is connected with the open air through the drying tubes, so that the atmospheric pressure is communicated by way of P_2 to the F-limb of the U-gauge. Readings of the manometer and the gauge FG are taken at various adjustments, from which the probable error of a single pressure observation and the pressure correction are easily deduced; capillarity is thus eliminated.

To dry and fill the air thermometer, the bulb is maintained at a temperature between 400° and 500° C, W is connected to the exhaust, and P_2 with the dry-air supply By means of the three-way tap T_2 the bulb is connected alternately with either. Mercury is re-admitted to fill the bulb A, when the operation is considered complete.

To take observations at constant pressure, mercury is allowed to run out from the bulb A into the weighed beaker at P_3, till the sulphuric acid gauge indicates that the pressure is approximately that of the air. The scale FG gives the small variations of pressure, and the weight of the mercury the auxiliary volume. The temperature of the bulb A is carefully observed, and is kept steady by being surrounded with a water reservoir. Total volume of Bulb A from S to $S' = 48\ 162$ c.c. at 15° C.

The gauge above described arrived about the beginning of August, 1886. A new thermometer, No. 5, was at once made of hard glass, similar to No. 3, the expansion of which was known.

There are some difficulties in mounting an instrument of this description. The mounting of No. 5 is diagrammatically illustrated in fig. 7, Plate 12. The bulb is fitted with asbestos board in a brass tube AD, into which is screwed a smaller tube DC which carries and protects the electrodes of the spiral and the delicate capillary

tube of the thermometer. The tube DC is soldered through a biscuit tin, which is filled with water, renewed periodically, to keep the tube cool and the conditions uniform. Another brass tube with a cap at A slides over AD, so that the glass bulb is completely protected. The electrodes of the spiral go to mercury cups 1, 2, 3, 4, into any of which the conducting wires from the resistance-boxes may be dipped. R_{12} or R_{13} gives the resistance of the spiral and connecting wires. R_{24} gives the resistance of the double electrodes, which is by symmetry twice that of the connecting wires. The resistance of the thick copper wires between the mercury cups and the boxes was measured. In Series IV. it was ·0077 ohm, in Series V ·0164 ohm. The apparatus was heated in a very thick iron tube in a Fletcher tube furnace. The gas could be adjusted by a screw tap with a graduated head, and might be used with an air blast for obtaining higher temperatures

For heating in steam a special pattern hypsometer was employed with a double jacket, through which a copious supply of steam was made to circulate from a boiler. The brass case AD of the thermometer should be made steam-tight, so that the steam cannot penetrate into the tube CD.

Air thermometer No. 5 was an improvement on No. 3 in several respects. The volume of the bulb was four times as large ($V = 63·3$ c.c.). A double electrode of fine wire was sealed through the glass, so that no part of the spiral itself was outside the bulb.

The spiral was made of a different wire, diameter ·013 cm., length 2 metres, resistance 20 ohms. It was therefore long and delicate. This proved a misfortune; for some one, in inspecting the instrument in my absence, after it was made, held it in a vertical position and deformed the platinum spiral completely, so that it was not uniformly distributed throughout the length of the bulb. This could not be cured without making the whole instrument afresh, and was probably a serious source of error whenever the temperature was at all unsteady (see Observations (11), (13), Series V.). The observations were taken precisely as before, perhaps with greater care and skill, acquired by experience. The electrode which passed out through the capillary tube consisted of 6 in. of Pt wire, diam. ·036 cm., fused on to 12 in. of Cu wire, No. 26 B.W.G., dipping into mercury cup 1 (fig. 7). It was of necessity single. A double electrode was made of the same wires, exactly to imitate it, and coated with the same hard glass for insulation, and placed in close proximity to the capillary tube, inside the small brass tube CD (fig. 7) which carried the thermometer. Its position must be carefully adjusted, for any want of symmetry or misplacement of the imitation electrode may introduce apparent errors in the resistance measurements. The other double electrode consisted of a short length (10 cm.) of fine Pt wire, to the middle point of which the end of the spiral was fused: the fine wire was bent double and fused through the glass at A (fig. 7), and its two ends projecting were fused on to thicker (·036 cm.) Pt wires, coated with hard glass for insulation, and extending as far as the Cu-Pt junction of the other electrode inside the brass tube CD (fig. 7), where they were fused on to No. 20 Cu wire leading to the mercury cups 2, 3. By

this arrangement of the junctions thermo-electric effects were rendered almost inappreciable in spite of the great sensitiveness of the apparatus and the low resistance of the galvanometer. The correction for resistance of connections, owing to the use of so much fine wire, amounted to nearly 3 per cent., but it was measured on each occasion, and could be very accurately applied.

With the instrument above described upwards of 130 independent observations in several sets, at various steady temperatures, were taken during the days August 7 to 12, 1886. They were worked out on much the same plan as that explained below on page 218

To save space, only the observations at steady temperatures are given in Table A.–IV. The experiments were interrupted on August 12 by the cracking of one of the glass joints of the gauge; an attempt to mend it only made matters worse. It was sent to the maker (MULLER, of 89, Whitfield-street) for repairs; and meanwhile other experiments were undertaken. The breaking of the gauge was particularly unfortunate, as I had so far taken no observations at constant pressure, for which the gauge had been specially designed. A great many observations were taken with the temperature varying to test the working of the apparatus; these on reduction were never found to differ by more than a degree from the curve, and often happen to fall exactly on it. To insert them would only confuse the diagram. The earlier observations of Series IV., which were taken just after filling, before the apparatus had reached a steady state, give results which are too high by two or three degrees. This is an error of the air thermometer, and not due to change of the resistance of the platinum; for the observations in ice and steam show that the platinum spiral suffered no change during this series of observations; but the value of the constant *mk* fell slightly, probably owing to surface absorption. The air thermometer was filled with dry air on the 9th August, at a temperature of 350° C., after repeated exhaustion; and it seemed probable that there would be a tendency for the air pressure to fall rather fast at first, owing to surface condensation; between observations (3) and (4) the gas was left unaltered for $3\frac{1}{2}$ hours. The change in the difference column from 7 9 to 7 1 is in the direction of absorption, but condensation probably took place most rapidly as the instrument cooled at lower temperatures. The historical element seems to enter here: the dates and times of the observations are therefore given, heating gradually dissipates the air-film, which may take some time to condense again, so that an observation (*e.g*, No. 12) taken at a lower after a higher temperature may give results which are too high; but this requires further investigation.

The later experiments seem to throw doubt on the suggestion offered a year ago (p. 181) as to the value of $\sigma_1 - \sigma_0$ explaining discrepancies in the values of the coefficient of dilatation according to different observers: the apparatus is not particularly suited for determining the absolute value of the constant θ_0, which depends on the *difference* of pressures in ice and steam, on which the probable error with my apparatus may amount to a quantity of the order of one in a thousand.

[To exclude errors of this nature, and to render the observations of different series strictly comparable, the value of θ_0 assumed must be the same throughout, since the value of t deduced varies directly as that of θ_0 assumed. All the observations entered on Plate 3, fig. 1, are therefore reduced, assuming the same value of θ_0, namely, 272·90° C, which corresponds to a coefficient of dilatation for dry air of ·0036642 per 1° C. This agrees very closely with REGNAULT, and is deduced from the pair of observations (16), (17), Table A.-v. An inspection of the values of mk, calculated with this coefficient from the observations in ice and steam, and at the air temperature, from day to day, will show that it satisfies all the observations practically within the limits of error of the pressure measurements. The largest discrepancy between observations *on the same day* (Table A.-v., March 8, observations (25), (26)) amounts to nearly 1 in 2000.]

The variations of mk *from day to day* are sometimes larger, and more unaccountable, they are generally in the direction of absorption, but I have so far been unable to reduce them to rule. The difficulty is to a great extent avoided by observing mk frequently and using the appropriate value in the reduction of the day's observations: this method must, in part at least, eliminate the error, to *whatever* cause it is due.

TABLE A.-IV.—Comparison of Platinum and Air Thermometers.

Time.	Auxiliary thermometers.	Resistances observed: Spiral	Resistances observed: Connection.	H_2SO_4 gauge: x	Mercury manometer: M_L	Mercury manometer: M_R	Barometer and attached thermometer	Number of observation	Results.
Aug 7	R 19·7°	20 437	1·062	Before filling air thermometer				(1)	$R_0 = 19\ 945$
Aug 9 11–12 A.M.	A. 20·5° R 18 8 M. 21·5	43 435 .. Mean of	2 021 .. several	3 51 .. observations.	$M_R - M_L =$ ..	+7·721	30 085″ 70 2° F.	(2)	$pt = 333\ 5°$ $t = 350\ 5°$ $d = 17\ 0°$
12.8 12.13 12.17 12 22	R. 19·3 A. 20·7 M. 21·4 ..	36·64 36·51 36·42 36·30	. 1 730	3 70 2 60 1 87 3·32	48·09 48 11 48·10 45·23	41·55 41·52 41 51 38 07	30 075″ 70 7° F	(3)	$t = 240\ 4°$ $pt = 232\ 5°$ $d = 7\ 9°$
3.30 3.41 3.48 3.52	A. 20·6 M. 20 9 R. 19·3 ..	36·79 36·75 36·73 36·76	 1 756 . ..	3 40 3 20 1 05 1·25	34 87 34 86 36 26 36 26	28 56 28 54 30 42 30·41	30 052″ 71·7° F.	(4)	$t = 243·7°$ $pt = 236·6°$ $d = 7\ 1°$

TABLE A.–IV.—Comparison of Platinum and Air Thermometers—(*continued*).

Time	Auxiliary thermo-meters.	Resistances observed		H_2SO_4 gauge	Mercury manometer		Barometer and attached thermo-meter	Number of observation	Results
		Spiral	Connec-tion	x	M_L	M_R			
6 25 6 30 6 36 6 42	° A 20 0 R 19 2 M 20 1 ..	22 38 22 32 22 27 22 23	1 155 ..	3 17 2 40 3 29 2 08	50 88 50 89 51 43 51 22	15 57 15 57 15 87 15 78	30 024″ 70 2° F	(5)	$pt = 26\ 89°$ $t = 26\ 58°$ $mk = 8\ 747$
Aug. 10 11 6	R. 18 5 A. 19 4	21 758 .	1 134 ..	1·58 ..	54 07 M 19	18 54 4°	29·722″ 68 6° F	(6)	$pt = 18\ 93°$ $t = 18\ 66°$ $mk = 8\ 747$
12 34 12 38 12 43 12 48 12 52 12 56	A 19 5 R 18 7 M 19 6 A 19 5 R 18 7	20 451 20 451 20 450 20 450 20 449 20 449	. . 1 075	2 50 1 30 4 97 1 98 4 46 2 41	55 33 54 95 56 23 55 21 56 07 55 39	17 17 16 95 17 67 17 12 17 58 17 22	29 700″ 69 1° F	(7)	$R_0 = 19\ 944$ $t = 0°$ $mk = 8\ 745$
1 45 1 53 1 58	A 19 3 R 18 8 M 19 5	20 449 20 449 20 450	. 1 076 ..	2 55 1 00 5 80	55 40 54 83 56 57	17 23 16 93 17 89	29 695″ 68 6° F	(8)	Same
4 12 4 16 4 22 4 28	A 20 0 M 20 1 R. 20 5	27 27 27 28 27 29 27 28	1 405 ..	3 17 3 35 3 57 1 20	51 12 51 12 51 11 50 40	26 18 26·18 26 16 25 85	29 716″ 66 6° F	(9)	$pt = 99\ 0°$ $t = 99\ 0°$ $mk = 8\ 746$
Aug 11 11 56 12 4 12 7 12 12 12 14	.. A 20 3 R 18 1 M 23 0	46·81 46·79 46 76 46 78 46·79 46 81	. 2 129 .. .. 2·136	2 50 2·70 2 73 2 90 2 82 3 00	34 39 36 10 36 17 36 17 37 79 37 78	49 36 50 98 50 97 50 97 52 60 52 59	29·985″ 69 4° F	(10)	$t = 401\ 1°$ $pt = 382\ 2°$ $d = 18\ 9°$
1 5 1.7 1 9 1 12	A 20 6 R 18 9 M 22 2 ..	52 18 52 32 52·46 52·56	2 350 . 2 358 ..	3 40 4 30 3·00 3 65	28 98 28 97 28 00 28 00	55 53 55 55 55 30 55 30	29 987″ 68·7° F	(11)	$t = 495\ 7°$ $pt = 463\ 5°$ $d = 32·2°$
2 47 2 51 3 0	A 19 6 R. 18 8 M 20 8 ..	40 57 40·53 40 54 40 56	.. 1·895 ..	4 37 3 12 2 20 1·90	41 84 41 84 41 22 43 08	43·20 43 18 43 03 45 02	29·978″ 68 1° F.	(12)	$t = 301\ 9°$ $pt = 291\ 6°$ $d = 10\ 3°$

TABLE A.–IV.—Comparison of Platinum and Air Thermometers—(*continued*).

Time.	Auxiliary thermometers	Resistances observed		H_2SO_4 gauge	Mercury manometer		Barometer and attached thermometer	Number of observation	Results
		Spiral	Connection	x	M_L	M_R			
Aug 12	A 18 8	21 702	..	2 60	55 66	19 06	30 016″	(13)	pt = 18 02° C
10 2	R 17 6	21 704	1 130	3 80	56 12	19 33	67 4° F		t = 17 79° C
10 10	M 20 0	21 707	..	2 80	55 76	19 15			mk = 8 741
11 10	R 18 0	20 460		2 55	59 11	20 13	30 010″		
11 17	A 19 6	20 458	1 076	2 18	59 10	20 18	68 7° F.		*In Ice*
11 22	A. 19 9	20 456	..	3 13	61 57	22 50			
11 30	R 18 3	20 455	..	3 00	61 56	22·52		(14)	R_0 = 19 945 B A
11 36	M 21·0	20 454		2 33	63 15	24 19	30 008″		mk = 8 743
12 3	M 20 0	20 453	1 076	2 75	63 18	24·19	69 3° F		
5 25	A. 19 5	27 347		3 05	52·94	27 62	29 920″		
5 27	R 19 3	27 348		2 60	54 39	29 15	66 0° F		*In Steam.*
5 35	M 19 9	27 349	1 397	6 45	53 50	27 58			
5 40	A 20 0	27 348	.	1 90	52·12	27 03	29 918″	(15)	R_1/R_0 = 1 3391
5 42	R 19 5	27 348	1 396	4 50	52 89	27 32	65 8° F		mk = 8 740
5 47	M. 20·1	27 348		2 00	52 11	27 01			

About the middle of February, 1887, the whole apparatus was thoroughly revised, improved in several minor details, and set up afresh. The gauge had meanwhile been mended and improved. A slight error in the position of one of the comparison electrodes was rectified, and the spiral made more uniform by cutting off a short portion and refixing. it was partially re-arranged by means of a wire introduced through the open end.

In mending the gauge, the tube DG had been made of larger bore (·0127 c.c. per cm.), so that it was necessary to affix a scale S and record the reading x'. The bore of the new gauge tube was ·0232 c.c. per cm. instead of ·0407, as in Series IV., and the scale read 2·80 when the pressure was the same on both limbs of the gauge. In Series IV. it had been adjusted to read zero, but it was not considered worth while to adjust it afresh. A new legal ohm-box was used to measure the resistance in place of the B. A. box used in Series IV. The observations in the following Table (A.–V.) were then taken in the order in which they are given :—

TABLE A.–v.—Comparison of Platinum and Air Thermometers.

Date and time.	Auxiliary thermometers	Observed resistances		H_2SO_4 U-Gauge		Mercury manometer		Barometer.	Number	Remarks.	Results.
		Connection	Spiral.	x	x'	M_L	M_R				
Feb 27 5 10 5 15 5 20	° A. 17 8 R. 16 3 A. 18 0	R_{24} 2 477 R_{34} ·745	54 41 54 39 54 40	4 7 4 65 4 9	9 5 " "	8 21 8 19 8 21	59·86 59 83 59·78	30 647" 58 0° F.	(1)a	Just after filling	$t = 539\ 9°$ $pt = 500\ 9°$ $d = 39\ 0°$
5.43 . . 5·48 .	M 19 1 ..	R_{34} 2 491 .	53 73 53 84	9 0 9 7	" "	11 22 11·22	60·19 60·19		(1)b		$t = 529\ 9°$ $pt = 491\ 5°$ $d = 38\ 4°$
Feb 28 . 12.38	M. 19 7 A 18 0 R. 15 2	R_{24} 1 126 ..	21 415 21 422	5 07 2 30	8 5 8 4	51 92 51·04	21 70 24·28	30·634" 74 7° F.	(2)	At air temp	$pt = 17\ 27°$ $t = 17\ 02°$ $mk = 10\ 578$
1 32 1 40 1 53	M 18 5 A 17 9 R. 15 9	R_{24} 2 519 R_{34} 760	54 30 54 52 54 70	9 5 6 1 8 0	9 6 " "	12 88 10 98 10·95	63 10 62·70 62 65	30·578" 62 0° F	(3)	Gas 105	$t = 541\ 5°$ $pt = 502\ 2°$ $d = 39\ 3°$
2 38 2 50	M 18 3 A. 18 1 R 16 6	R_{24} 2 520 R_{34} 756	54 80 54 74 54 71	4 00 6 45 6 30	9 4 " "	11 30 11 18 11 18	64 50 63 56 63 53	30 576" 62 0° F	(4)	Gas 105	$t = 546\ 9°$ $pt = 505\ 8°$ $d = 41\ 1°$
3 10 3 15 3 19 3 30 .	A 18 2 R 16 9 A 18 2 A 18 3	R_{24} 2 516 R_{34} 750 .	54·98 54 95 54 90 54 87	7 50 7 40 7 20 7 00	*9 2* Mean temp G = 17 60	W 244 902		30 575" 61 2° F.	(5)	Gas 105	$t = 548\ 7°$ $pt = 508\ 4°$ $d = 40\ 3°$
5 5 5.15 5 30 5 40	A 18 2 . R 17 0 A. 17 8	.. R_{24} 2 074 R_{34} 630 ..	48 75 48 80 48 84 48 90	5 90 6 20 6 58 7·02	*8 35* Mean temp G = 17 30	W 156 830		30 586" 62 0° F.	(6)	Gas 50	$t = 360·9°$ $pt = 345·8°$ $d = 15·1°$
6 25 6 32 . 6 40 . 6 52 7 0 . . 7 10	A 18 5 R. 17 5 A 18 4 R 17 7	. .. R_{24} 1 858 R_{34} 573 ..	38 70 38 78 38 82 38 70 38 66 38 60	6 3 7 1 7 4 6 4 6 0 5 4	*8 45* Mean temp G = 17 45	W. 98 585		30 586" 60 0° F. 30 578" 59 8° F.	(7)	Gas 35	$t = 278\ 4°$ $pt = 270\ 9°$ $d = 7\ 5°$
March 1 10 35 10 47 .	R. 15 7 M 18 6 A 16 3	R_{24} 1 119 R_{34} 407	21 28 21 29	4 55 2 13	8 8 8 7	52 99 52 53	22 56 22 46		(8)	At air temp	$pt = 15·44°$ $t = 15\ 22°$ $mk = 10·577$
11 18 . . 11 28 11 30 11.31 . .	R. 15 7 A 18 1 R 15 9 A 18 6	R_{24} 1·785 .. R_{34} 548 ..	37 05 37 13 37·42 37 44	5 75 6·6 10 0 10 2	*8 2* 16 7° 16·9° 17 2°	W 75 550		30 630" 60·7° F.	(9)	Gas 32	$t = 255\ 5°$ $pt = 249·5°$ $d = 6·0°$
11 43 . . 11.45 11 51 . . 11.57	R 16 1 A 19 0 R. 16·3 A 18·8	R_{24} 1 795 R_{34} 550 ..	37 52 37 58 37 54 37·50	4·91 5 01 5 15 4·82	17·5° 17·7° 17·9° 18·1°	W. 83 345		30·630" 60·5° F	(10)	Gas 32	$t = 259\ 5°$ $pt = 253·3°$ $d = 6·2°$

TABLE A.–v.—Comparison of Platinum and Air Thermometers—(*continued*).

Date and time	Auxiliary thermometers	Observed resistances: Connection.	Observed resistances: Spiral	H_2SO_4 U-Gauge: v	H_2SO_4 U-Gauge: x'	Mercury manometer: M_L	Mercury manometer: M_R	Barometer.	Number	Remarks.	Results
12 30 12 32 12 34 12 36 12 42 12 47 12 51	° R 16 6 A 17 8 A 17 6 R 16 7 .	 R_{24} 2 106 R_{34} 626	44 85 44 90 44 95 45 00 45 18 45 28 45 34	5 95 6 35 6 72 7 12 8 52 9 30 9 75	18 50° 18 55° 18 50° 18 45°	W 166 760		30 622″ 62 2° F	(11)	Gas 50	$t = 380\ 7°$ $pt = 364\ 1°$ $d = 16\ 6°$
5 8 . 5 10 . 5 15 5 20 . 5 25	R 17 7 A 18 6 . A 19 0 R 17 8	 R_{24} 2 131 R_{34} 628 .	45 86 45 84 45 72 45 56 45 50	5 10 4 95 4 30 3 32 2 80	17 8° 17 9° 18 0°	W 177 530		30 596″ 60 3° F.	(12)	Gas 50	$t = 391{\cdot}8°$ $pt = 378\ 5°$ $d = 18\ 3°$
6 1 . 6 2 . 6 4 . 6 6 .	A 19·3 R. 18 1 A 19·0	. . R_{24} 2 669 R_{34} ·779 .	58 30 58·20 58 10 58 05	8 3 7 9 7 5 7 3	. . 18 6°	W. 263 218		30 610″ 59 5° F.	(13)	Gas 115	$t = 602\ 3°$ $pt = 556\ 2°$ $d = 46\ 1°$
6.40 . 6 45 . 6 50 6 54 . . 6 57 . 7 4 .	R. 18 1 A. 19 4 . . A. 19 4 R. 18 1	. . . R_{24} 2 683 R_{34} 765 . .	59 35 59 17 59 25 59 30 59 32 59 31	2 32 1 55 1 60 1 80 1 90 1 85	18 8° 18 9° 19·0°	W. 277 295		30 617″ 59 2° F	(14)	Gas 100	$t = 624\ 2°$ $pt = 572\ 9°$ $d = 51\ 3°$
March 2 . 11 30 . 11 35	M 18 2 A. 16 5 R. 15 7	. . R_{24} 1 122 R_{34} 409	21 31 21 32 21 33	4 40 4 50 4 75	8 9 8 8 8 7	62 47 62 47 62 44	31 94 31 97 31 94	30 658″ 61 7° F	(15)	At air tempera-ture	$pt = 15\ 94°$ $t = 15\ 72°$ $mk = 10\ 570$
12 55 1 5 1.11 . 1.15 . . 1 25	A. 17 7 R 15 8 M 17 5 . A. 17 1	 R_{24} 1 073 R_{34} ·390 .	20 225 20 225 20 225 20 225 20 225	4 25 2 65 7 17 2 52 2 65	8 7 8 8 8 9	62 38 61 88 61 05 59 69 59·69	29 37 29 13 27 60 26 95 26 95	30 662″ 63·7° F.	(16)	In ice at 0°	$R_0 = 19\ 674$ $mk = 10{\cdot}571$
6 52 . . 6.56 . . 7.2 . . 7.17 . .	A 18 4 R. 17 4 M 19 2 A. 18·6	R_{24} 1 408 . . R_{34} ·502 . .	27 100 27 100 27 100 27 100	6 20 1·80 1 80 5·77	8 9 . 8·8	58 36 57 15 57 13 58 19	41 40 40 92 40·90 41 27	30 626″ 61 7° F	(17)	In steam	$mk = 10\ 571$ $R_1 = 26\ 348$ $t = 100\ 58°$ $R_1/R_0 = 1\ 3395$
March 4 . 12.11 . 12.22 . .	A. 11 5 R. 9·9 M 11·5	. . R_{24} 1·104 . .	20 95 20·97 20·99	3 50 3 75 4 00	 9·3 . .	57 26 57·24 57 24	26 33 26 32 26 33	30 516″ 58 0° F.	18)	At air tempera-ture.	$pt = 10\ 52°$ $t = 10\ 37°$ $mk = 10{\cdot}563$
1.5 . 1.15 . . 1.30 . .	A 12 7 R. 11 3 M 13·1	R_{24} 1·395 . . R_{34} 490	27 124 27 124 27·123	4 80 1 68 6·30	9 1 8 9 8 8	61 54 60·70 61 88	45 02 44 70 45 11	30 500″ 57·0° F	(19)	In steam	$R_1/R_0 = 1\ 3395$ $t = 100\ 48°$ $mk = 10\ 563$
March 4. [illegible] [illegible] [illegible]	Fresh-filling with air, while in steam. R 14·5 [illegible] [illegible]	 R_{24} 1·396 . .	 27 103 27 102 27·101	 5 85 2·10 7 87	 8·7 8.6 8 5	 32 12 31 12 32·54	 57·27 56·99 57·33	30 464″ 58 0° F.	(20) *a*	In steam	$V_1 = 63\ 41$ c c. $mk = 17{\cdot}755$ $t = 100\ 44°$

TABLE A.–v.—Comparison of Platinum and Air Thermometers—(*continued*).

Date and time	Auxiliary thermo meters	Observed resistances		H_2SO_4 U-Gauge		Mercury manometer		Barometer	Number	Remarks.	Results.
		Connection	Spiral	x	x'	M_L	M_R				
7.20 .	G 14°25	.	27 095	2 80	14 4°	W 254 437			b		
7 30 7 35 7.42	A 16 1 M 16 8 R 15 4	R_{24} 1 395	27 094 27 094 27 094	4 60 2 90 2 80	.. 14 7° .	54 07 53 69 53 69	28 73 28 59 28 61	30·448″ 58 0° F	c	Mercury bulb A emptied	$V_1 =$ 63 41 c c $mk =$ 17 755 $t =$ 100 43°
March 6 10 0 A.M.	A 12 5 R. 12 7	R_{24} 1 111	21·096	2 95	8 9	34 38	36 56	30 276″ 55 3° F	(21)	At air tempera-ture	$t =$ 12 32° $pt =$ 12 49° $mk =$ 17 743
March 7 11 40	A 12 6 R. 12 7	R_{24} 1 108	21 056 21 058	2 12 2 25	8 4 .	36 75 36 76	38·65 38 66	30 366″ 59·0° F.	(22)	At air tempera-ture	$t =$ 11 74° $pt =$ 11 90° $mk =$ 17 736
12 58 1 0 1 8 1 10 1 15 .	A. 13 4 R. 13 0 R 13 0 A. 13 4	R_{24} 1 851 R_{34} 551	39 10 39 20 39 45 39·57 39 65	5 47 6 15 7 90 8 75 9 35	9 5 12 0° 12 1° 12·2°	G 12 0° W 463 627		30 358″ 59 0° F.	(23)	Gas 40	$t =$ 288 7° $pt =$ 280 6° $d =$ 8 1°
1 42 . . 2 1	A 14 0 R 13 2 A. 14 1	. R_{24} 2 470 R_{34} 700	54 05 54 10 54 17	9 30 9 55 9 80	13 0° 13 1° 13 2°	G 12 8° W 592 907		30 350″ 58 5° F.	(24)	Gas 100	$t =$ 532 2° $pt =$ 496 0° $d =$ 36 2°
March 8 . 1 5	A 13 8 R 13 4	R_{24} 1 115	21·180	2 85	8 3	39 83	42 53	30 214″ 61 0° F.	(25)	At air tempera-ture	$pt =$ 13 76° $t =$ 13 57° $mk =$ 17 736
1 51 2 6 . .	R. 13 8 A 14 8	R_{24} 1 407 R_{34} 501	27 090 27 090	3 00 4 50	7 8 7 7	27 00 27 35	53·28 53 34	30 198″ 60 7° F.	(26) a		$t =$ 100 19°
2.17 . 2 22 2 28	M 15 0 A. 14 7 R. 14 0	. R_{24} 1 408	27 088 27 088 27·088	4 47 4 52 4 58	13 6° 13 7° 13·8°	G 13 5° W 257 608		30 190″ 60·8° F	b		$R_1 =$ 26 343 $R_1/R_0 =$ 1 3395 $V_1 =$ 63 41 $mk =$ 17 744 $t =$ 100 18°
2 33 2 44 2 50 2 57	A 15 4 R 14 2 M. 15 1 R 14 4	. . ..	27 087 27 086 27 085 27 084	4 65 3 10 2 85 5 03	13 9° 14 0° 14 1° 14 1°	51 85 52 55	27 42 27·72	30 177″ 60 6° F.	c		$t =$ 100 17°
March 9 12 20	A 13 8 R 14 0	. R_{24} 1·114	21·170 21 172	4 00 6 65	9 5 .	50 23 50·82	53 05 53 22	30 068″ 61 0° F	(27)	At air tempera-ture	$pt =$ 14 02° $t =$ 13 83° $mk =$ 17·721
1 5 . 1 15 1.20	R. 14 0 A 14 0 M. 13 9	R_{24} 1·072 R_{34} 394	20 231 20 231 20 231	5·25 2 81 3·60	9·1	50 04 49 50 49 68	49·00 48 85 48·90	30·060″ 60 6° F.	(28)	In ice	$R_0 =$ 19 668 $mk =$ 17 720
March 11. 10 15	A 16 2 R 18·0	R_{24} 1 123	21 380	3·20	6 4	47 80	51·40	30 070″ 61 0° F.	(29)	At air tempera-ture	$pt =$ 16 05° $t =$ 15 84° $mk =$ 17·730

TABLE A.–v.—Comparison of Platinum and Air Thermometers—(*continued*).

Date and time	Auxiliary thermometers	Observed resistances		H_2SO_4 U-Gauge.		Mercury manometer		Barometer	Number	Remarks	Results.
		Connection	Spiral.	x	x'	M_L	M_R				
March 11 11 50 11 54 11 58 12 0 12 5 .	° A 17 4 R. 14 2 .. A 17·6 R 14 6	.. R_{24} 2 173 R_{34} 587 .. .	48 57 48 60 48 66 48 67 48 75	9 42 9 60 9 90 9 97 10 5	.. 16 5° 16 6°	G 16·0° W 568 235		30 050" 64 2° F.	(30)	Gas 70	$t = 440 9°$ $pt = 417 1°$ $d = 23 8°$
12 40 12 41 12 46 12 52	A 17 8 R. 14 9 R. 15 1	. R_{24} 2 622 R_{34} ·746 .	57 77 57 80 57 99 58 10	2 87 2 97 3 37 3 75	17 1°			30 032" 65 7° F.	(31)	Gas 125	$t = 599 8°$ $pt = 552 6°$ $d = 47 2°$
5 30 .	A 17 0 R. 16·4	. R_{24} 1·128	21 45 21 44	3 75 5 91	8 8 ..	45 71 46·28	50 07 50 23	29·936" 63 0° F	(32)	At air temperature	$pt = 17 91°$ $t = 17 66°$ $mk = 17 731$
March 12 12 15 . .	R 14 2 A 16 2	R_{24} 1 117	21 20	2 73	8·7	46 84	50 23	29 969" 62 8° F.	(33)	At air temperature	$pt = 14 18°$ $t = 13 99°$ $mk = 17·723$
1 13 1 19 . 1 20 . . 1.23 1.24 .	A. 17 0 R. 14 6 M. 18 5 .. .	.. . R_{24} 1 730 R_{34} ·530 ..	36 55 36 48 36 30 36 17 36 10	7 85 6 23 4 43 2·70 5 30	.. .. 8·7 . ..	6 37 6 38 6 38 6 37 6 29	70 42 70 30 70 30 70·22 69·16	29 984" 63·70° F	(34)	Gas 35	$t = 241 76°$ $pt = 236 00°$ $d = 5 76°$
1 35 1.40 .	A. 17·0 R. 14·7	R_{24} 1 708 .	35·78 35·78	8·15 8·15	15 55° 15 65°	W 430 000		Bar same	(35)	Gas same	$t = 232 90°$ $pt = 228 08°$ $d = 4 82°$
2 55 3 2 .	A. 16·2 R. 15 5	R_{24} 1·700 R_{34} 536	35·29 35 32	4 07 4 20	15 75°	W. same		30 010" 61 8° F.	(36)	Gas same	$t = 225 71°$ $pt = 221 14°$ $d = 4 57°$
3 20 . 3 22 . 3 28	A. 16·2 R 15 6 M. 16 7	R_{24} 1·702 ..	35 39 35·40 35 40	6 54 4 15 6 07	8 9 ..	6 95 7 10 7 03	66·60 67 31 66 78	30 025" 61·4° F.	(37)	Gas same	$t = 227 59°$ $pt = 222 50°$ $d = 5 09°$
March 17(?) 6.35 . . 6.41 . . 6.47 .	.. A. 15·7 R. 15·9 M. 15·6	.. R_{12} 1 131	21 50 21·49 21 47 21·45	6·30 6 02 4 11 3 61	9 3	43 68 43·68 43 34 43 35	47 28 47·28 47 18 47·18	30·100" 59·7° F.	(38)	Cooling	$pt = 18 32°$ $t = 18 08°$ $mk = 17 713$

Explanation of the Tables.

The dates and times are given to show the sequence of the observations, the rate of variation of the temperature, to indicate how far the results may be relied on, and to trace the historical element.

The auxiliary thermometer readings are corrected for zero error. A. gives the temperature of the air near the gauge; R., of the air inside the resistance-box; M., of the mercury of the manometer. The box used in this series of experiments was a

legal ohm-box by ELLIOTT, nominally correct at 18° C., to which temperature the resistances are reduced. The temperature coefficient of the box was determined to be 00023 per 1° C., the coils being of platinum-silver wire The relative values of the resistances were correct to more than 1 in 10,000

The resistances observed are given in legal ohms, but have to be corrected for the temperature of the box R The correction for the resistance of the connecting wires is $\frac{1}{2} R_{24} + 008$.

By observing also the resistance R_{34} of the capillary tube electrode, the mean temperature of the corresponding 15 cm. of the capillary tube could be determined, and an allowance for its varying temperature approximately made by reckoning part with the bulb and part at air temperature. In most cases the temperature was varying; and an exact balance was observed with the rocking commutator as previously described, and the simultaneous reading x of the U-gauge was recorded. When the temperature was too steady to admit of this a fifth figure was obtained by observing galvanometer throws and interpolating. The column headed x' gives the reading of the level of the mercury by the scale S In taking observations at constant pressure, mercury was allowed to run out from the bulb A (see fig. 6, Plate 12) and weighed in a beaker. The observed weight in grammes W. of the mercury and beaker is given in the manometer column; the manometer was of course unnecessary in taking observations at the atmospheric pressure The temperature indicated by a delicate thermometer immersed in the water surrounding the bulb A is then given in the x' column. The readings given of the thermometers and barometer are mean or typical readings . all are not given, to save space. The observations are numbered to correspond with Plate 13.

No permanent change of resistance of the platinum wire was with certainty detected in this series of experiments, in spite of the prolonged heating to which the spiral was subjected, and although it was in contact with the glass of the bulb for a considerable portion of its length. The apparent change of temperature coefficient from $R_1/R_0 = 1{\cdot}3391$, Series IV., to 1 3395, Series V , was probably due to the fact that the position of the loose comparison-electrode was shifted so as accurately to correspond with the single electrode inside the capillary tube ; it had been accidentally misplaced through an error in measurement in Series IV. The apparent change in the value of R_0 in Series A.–V. from 19·674 to 19 668 (the latter value exactly was observed again one month latter, on April 18) was probably partly due to a change in the comparison-electrode, the copper portion of which was accidentally fused in re-coating with hard glass on March 4, so that it had to be re-made, and its resistance was slightly increased ; and partly, perhaps, to a change in the plug contacts in the 10-ohm arm of the bridge, which were thoughtlessly altered on that day to measure another resistance. The value $R_0 = 19{\cdot}668$ should, of course, be used in reducing the observations from March 4 onwards.

On March 4, while the bulb of the air thermometer was immersed in steam, it was

filled afresh with dry air at 103 cm. pressure, corresponding to a zero pressure of almost exactly one megadyne per sq cm By allowing the mercury to run out till the bulb A was emptied, and again measuring the pressure, and assuming the formula $pv = mk\,\theta$, the volume of the bulb up to the Cu-Pt junctions was calculated to be $V_1 = 63\ 41$. A repetition of the same operation on March 8 with the same air gave the same value.

The observations were all reduced, assuming $V_0 = 63{\cdot}29$ as the volume of the bulb at 0° C., and its expansion to be that given by the formula on page 168 (linear expansion of glass). The correction for glass expansion amounts to nearly 12° at 600° C., and must be considered subject to greater uncertainty than the other elements of the calculations, the apparent discrepancies between observations at constant volume and pressure may be in part due to errors of this nature.

The method of approximation used in previous portions of the paper in working out the results fails in the case of observations at constant pressure. Latterly I have used a Fuller's spiral slide-rule, which gives results correct to 1 in 10,000 at least. This is so exceedingly expeditious and convenient that the formula may be worked out directly with great rapidity. If p be the pressure of the air enclosed, V the volume of the bulb, θ its temperature, and v any other portion of the volume which is at temperature θ', the formula of the air thermometer may be stated thus—

$$p\,\{ V/\theta + \Sigma(v/\theta') \} = mk, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

where Σ implies that the summation is extended to all portions of the apparatus occupied by the mass of air under observation, and mk depends on the mass of air enclosed, and is theoretically constant for each filling. To make the statement more definite, we will define θ to be such a function that the above equation is true, p being constant and equal to one atmosphere: θ is, then, the temperature by normal air thermometer at constant pressure, and is nearly equal to the absolute temperature on the thermo-dynamic scale

From Formula I. we obtain, as the value of θ,

$$\theta = V / \{ mk/p - \Sigma\,(v/\theta') \}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

To illustrate the corrections involved, we will proceed to calculate Observation (5) :—

Mean value of resistance observed, 54·909 legal ohms. Correction for connecting wires, $2{\cdot}516/2 + {\cdot}008 = 1{\cdot}266$. R corrected, 53·643. Temperature of box, 16 9. Temperature correction $= -(54)\,(1{\cdot}1)\,({\cdot}00023) = -{\cdot}014$. R corrected, 53 629. $R/R_0 = 2{\cdot}7258$. Assuming $R_0 = 19{\cdot}674$ from Observation (16),

$$pt = (R/R_0 - 1)/{\cdot}0033947 = 508{\cdot}4°\ \text{C.}$$

Resistance of capillary tube electrode, $R_{34} = {\cdot}750$. Resistance of the platinum portion, [illegible]. Mean temperature, 340°. 5·2 cm. of this portion of the capillary tube is [illegible] at air temperature, the remaining 10 cm. at the temperature of the

bulb. Thus :—5 2 cm. = 15 (508 − 340)/(508 − 18). The volume of this portion (5·2 cm.) (by curve constructed from calibration data) is ·030 cub. cm. This has to be deducted from V_0, the volume of the bulb, and added to v, the volume at air temperature.

The volume of the bulb up to the platinum-copper junction of the electrode is calculated, assuming that air satisfies Boyle's Law, from the observations (20) and (26), both of which give the same result, $V_1 = 63{\cdot}41$ c.c., $V_0 = 63\ 29$ c.c.

In our case $V_0 = 63\ 29 - {\cdot}03 = 63\ 26$ c c.

The glass expansion factor corresponding to $pt = 508{\cdot}4$ is taken from the curve whose equation is given in the section on the expansion of hard glass, p. 168, $V = V_0(1 + \gamma\theta) = (63{\cdot}26)\ (1\ 0124)$. Small errors are less likely to be introduced if this correction is applied last of all to the value of θ.

To calculate the volume at air temperature $\Sigma\ (v/\theta')$:—

The volume from division $x' = 9$ of the scale S, and $x = 0$ of the scale FG, to the platinum-copper junction of the capillary tube electrode was determined by previous calibration to be 0·708 c.c., and is assumed to be, at the temperature of the air, A = 18 2 ; $\theta' = 291{\cdot}1$.

The mean value of x, the reading of the gauge in Observation (5), is 7 21.

The corresponding volume is (7 21) (0232) = 167 c.c.

Thus $v' = 708 + {\cdot}167 + {\cdot}030 = 905$, and $v'/\theta' = {\cdot}003110$.

Mercury was allowed to run out by way of P_3 into a beaker till the difference between the pressures of the internal and external air came within the range of the H_2SO_4 gauge The weight of the mercury and beaker was W = 244 902, that of the beaker 31·220. The same beaker was used in all the weighings here given.

Since the value of the temperature of the bulb A observed with the mercury thermometer G cannot be correct to more than 1 in 5000 (1/20 degree), it is needless to reduce the weight to *vacuo* or apply any of the other small corrections. The density of the mercury at 17° 55 C. being 13 553 grms. per c.c., the volume of 213·682 grms is 15 668 c.c. Since the mercury stood at $x' = 9\ 2$ just before it was allowed to run out, we must add (·2) (0127) c.c. to reduce it to $x' = 9$.

Thus the auxiliary volume was 15·770 c.c at a temperature

$$17^\circ{\cdot}55 + 272{\cdot}90 = 290^\circ{\cdot}45 \text{ C. by air thermometer.}$$

Thus $\quad v/\theta' = {\cdot}054290\,; \quad v'/\theta' = {\cdot}003110\,; \quad \Sigma\,(v/\theta') = {\cdot}057400.$

The acid stood at 2·30 in the gauge when the pressure was the same in either limb, and by experiment it was found that the reducing factor 0·136 was required to give the pressure correction in terms of mercury. Thus the required correction in Observation (5) is (7·21 − 2·30) ·136 = ·670.

The barometer reading reduced to centimetres is 77 659 cm., and the temperature 16° 2 C. ; correction for temperature,— ·228 cm. ; correction for level and capillarity, + ·045 cm.

Correction for pressure of acid in gauge, + ·670.

Corrected value of pressure of air in thermometer, 78 146 cm.

The value of the constant mk was calculated each day by Formula I. from the observations at air temperature and at the fixed points, assuming the values $\theta_0 = 272°{\cdot}90$ C., $V_0 = 63\ 29$ c c By observing the resistance of the platinum spiral the temperature of the air in the bulb could be very accurately deduced. On the morning of February 28 the value of mk observed was 10·578, Obs (2), and next morning 10 577, Obs (8). The agreement is very satisfactory, and perhaps closer than we have any right to expect, owing to the uncertainty of the pressure measurements. Using this value in Formula II, we obtain

$$mk/p = {\cdot}13535\,; \qquad \theta = V/\{mk/p - \Sigma\,(v/\theta')\} = 821°{\cdot}6 \text{ C},$$
$$t = 821{\cdot}6 - 272{\cdot}9 = 548°\,7 \text{ C.}, \qquad d = t - pt = 40°\,3 \text{ C}$$

This value is represented by the cross marked (5) on the curve of difference of temperature, Plate 13. All the observations in this series were calculated in this kind of way; but, of course, the work was done on a regular systematic form, which ensured accuracy and rapidity, so that it should be free from serious errors.

Graphic Representation of Results.

The best method of representing the results graphically is that which REGNAULT adopted in his comparison of mercury thermometers, namely, that of drawing the curve of difference of temperature This has the advantage of indicating all errors and deviations on a very large scale. In the curve diagram, Plate 13, fig 9, the abscissa represents temperature Centigrade by air thermometer on a scale of 10° to $\frac{1}{4}$-inch, and the ordinate the difference of temperature between the platinum and air thermometers at the rate of 1° to $\frac{1}{4}$-inch.* On this scale errors of 1 in 10,000 are distinctly appreciable in the upper part of the scale, as they correspond to a distance of $\frac{1}{50}$-inch or $\frac{1}{2}$-millimetre. Observations taken near the standard pressure of 76·0 cm in Series V. are represented by large crosses (+); observations at other pressures by small crosses (×). The Arabic numbers belong to Series V.; the Roman numbers to other series

If the variation of R follows the parabolic formula

$$R/R_0 = 1 + \alpha t + \beta t^2,$$

we shall have $d = t - pt = \delta[(t/100)^2 - (t/100)]$, where $\delta = -\ 10{,}000\beta/(\alpha + 100\beta)$ The parabola in Plate 4 corresponds to the value $\delta = +\ 1{\cdot}57$.

Since $$\alpha + 100\beta = {\cdot}0033947,$$

this gives $\alpha = +\ {\cdot}0034480$, $\beta = -\ {\cdot}000000533$, but the change of resistance does not seem to follow this law with absolute accuracy.

Observation (1) was taken within an hour of drying the air thermometer by repeated exhaustion and re-admission of clean dry air at 550° C.

* The curves are reduced to half-size in the plate.

The value $mk = 10{\cdot}578$ is assumed in its reduction. This is a little uncertain, because in Observation (2) the sun had been allowed to shine on half of the barometer in the room upstairs, causing the high reading 74°·7 F. of the attached thermometer, and rendering the temperature correction uncertain; the same uncertainty, of course, applies to the reduction of Observations (3), (4), (5), into which the same value of mk enters. A change of less than 1 in 1000 in the value of mk or V would suffice to account for the deviation of this group from the curve. Observations (4) and (5) indicate a difference of 1° C. between the values of the temperature by air thermometer at constant pressure and at constant density respectively at 540° C. This may be similarly due in part to errors in the assumed volume or expansion of the glass, or merely to changes in the temperature distribution between Observations (4) and (5).

Between 200° and 300° C. the observations seem to deviate in a regular way, as indicated by the dotted curve, from the parabola. That this deviation is not merely due to errors of observation is, I think, indicated by the curves of comparison of different platinum wires, which seem to show that the increase of resistance does not strictly follow the parabolic law, but is affected by some minor and comparatively unimportant variations which, from their very smallness (being of the order of 1° C.), are difficult to take account of otherwise than in a graphic record. Moreover, the observations in question were taken on different days and under different circumstances, so that it would be difficult to account for their agreement on any other hypothesis than that of some constant error affecting either the platinum or the air thermometer.

Observations (11) and (13) differ excessively from those in their immediate neighbourhood. On reference to the Table A.–v., it will be noticed that they were both taken with the temperature varying much too fast for purposes of accuracy; on the day in question the gas pressure was very variable, and it was difficult to get good observations at steady temperatures Observation (11) was further vitiated by the fact that some of the water had leaked from the glass case surrounding the bulb A, and its level had been allowed to get too low. Observations (37) and (34) were taken at constant density, and differ from (35), (36), in the same direction and for the same reasons that (5) differs from (4). This apparent difference is greater than that indicated by theory, and may be due to some error in the value of V, the volume of the bulb; further and more careful experiments are required to clear up this point.

Observations at steady temperatures from the previous series of experiments (Tables A.–II., III., IV.) are entered in the diagram by means of small circles; they are seen to agree with the curve as well as can be expected, considering the various errors pointed out in each case to which the earlier experiments were liable. Had the observations been more complete, the agreement would probably have been more perfect. In order to render them comparable with the results of Series A.–V., the same value of the coefficient of dilatation, namely, $\theta_0 = 272{\cdot}90$, has to be employed in their reduction in place of the coefficient $\theta_0 = 270{\cdot}1$, which is used in the Tables A.–II. and III. in the earlier portions of the paper. This anomalous value of θ_0

was probably due to an imperfection in the drying apparatus which I subsequently discovered. The exit tube from the last drying bottle was plugged with cotton and glass wool, to stop dust, so tightly that the air could not pass freely, and was connected to the apparatus by an old and rather rotten piece of black seamed rubber tubing through which damp air probably leaked when it was suddenly connected with the vacuum of the air thermometer bulb. The whole drying apparatus was, therefore, set up afresh before Series A.–IV. with new sulphuric acid U-tubes (prepared by STAS' method) and seamless solid rubber tubing, and the joints made perfect with hot paraffin wax. The whole was tested, and found to be perfectly air-tight.

Assuming that the anomalous value of $\theta_0 = 270{\cdot}1$ was due to the presence of water vapour, the error may be eliminated beyond 100° C. by applying from this point onwards the usual value of the coefficient of dilatation for dry air, namely $\theta_0 = 272{\cdot}90$. A correction $(t - 100)\frac{2{\cdot}8}{273}$ has, therefore, to be added to the values of t given in the Tables A.–II., A.–III.

The values of pt require no correction, and are calculated, assuming the coefficient ·003460.

Thus reduced, the latter observations in Table V., p. 192, become the following:—

TABLE A.–VI.

Number of observation.	Temperature		Difference observed	Difference by curve
	By air thermometer	By platinum thermometer		
	°	°	°	°
III., 9	268·5	264·8	3·7	7·1
I., 6	282·1	277·5	4·6	8·1
II., 1	356·1	343·1	13·0	14·2
III., 7	401·0	382·1	18·9	18·9
III., 1	472·6	447·1	25·5	27·5
III., 8	584·5	542·6	41·9	44·6

Mean error, 2·1° C.

The broken line on Plate 13, fig. 9, is used for deducing the temperature t by air thermometer from the formula

$$t - pt = 1{\cdot}57\{(t/100)^2 - (t/100)\},$$

the temperature pt being known by observation.

It gives at once the difference $(t - pt)$ in terms of pt as abscissa.

It is readily constructed by measuring off from each point of the first curve, backwards along the abscissa, a distance equal to one-tenth of the ordinate.

By using a curve like this an accuracy of $\frac{1}{10}$th degree Cent. at 600° may be obtained for the relative values of t deduced from observations with a known platinum wire, or with platinum wires which have been compared with a known wire. The *absolute*

value of t may be in error to the extent of about 1°, as it depends on the air thermometer observations. These, however, may be rendered still more accurate, and the uncertainty further reduced; they do not affect the claim of the platinum thermometer to rank as a *practical* thermometric standard.

The platinum air thermometer gives us the means of comparing theory with experiment, and of testing the accuracy of the formula

$$pv = mL\theta$$

at high temperatures by taking observations at various pressures in rapid succession, while the temperature is maintained nearly constant. The platinum spiral may at least be relied on to give any small changes of temperature which may occur. Nevertheless, for experiments of such delicacy it was felt to be necessary to maintain the conditions as uniform as possible, because any slight variation might introduce errors comparable with the differences that might be expected to exist between the indications of the air thermometer at constant pressure and at constant density.

Consequently, when on my return from India in February, 1887, I set to work again with the same glass air thermometer, a small gasometer was ordered in order to keep the gas pressure perfectly constant during any observation. This want had not been so much felt in August, when there were few students at work and little gas was being used, but the unsteadiness of the gas pressure was a great source of annoyance during the series of observations given in Table A.–v, which were taken during February and March in full term time, pending the arrival of the gasometer. On one occasion the temperature rose 100° in five or ten minutes, owing to change of gas pressure. Such changes of temperature would have been less material, had the platinum spiral been uniform, so that its mean temperature would have been more nearly that of the air enclosed. The gasometer arrived about the end of April; it leaked badly, and had to be mended. After that it performed satisfactorily, and a few observations were taken at steady temperatures one morning with its aid. Unfortunately, the observations at constant density were pushed too far this time, and the capillary tube gave way at about 550° C. under two atmospheres' pressure.

Porcelain Air Thermometer.

For higher temperatures it is necessary to use an air thermometer of porcelain. The difficulties to be encountered are that the wire must not touch the glaze and cannot safely be fused through the porcelain air-tight at any point exposed to the high temperature; and that the expansion of the material must be determined with some accuracy. To meet these difficulties, a somewhat peculiar type of instrument has been made.

A cylindrical bulb of porcelain, AB, Plate 12, fig. 8, 3 cm. diam., 200 c.c. capacity, opens at either end into thick side-tubes AC, BD, of small rectangular bore, 3 mm.

by 1 mm. (one of which is shown in section, full size, at E), which extend 6 inches on either side beyond the tube gas-furnace in which the bulb is heated. The side-tubes AC, BD, are fitted to contain double electrodes of thickish platinum wire, to the ends of which, at A and B, a single length of fine wire is fused, which extends through the bulb from A to B By means of these double electrodes we can determine the resistance of the fine wire independently of its connections, and we can also determine the mean temperature of the side-tubes AC, BD. One end D is fixed and connected with wax to the tube of the pressure-gauge. The free end C is closed with wax, and observed by means of a reading microscope. The expansion of the bulb itself is thus simultaneously observed, and the electrodes pass air-tight at the points C and D, which are never heated. Moreover, the form of the bulb renders it suitable for heating in a tube gas-furnace, which is very convenient.

Comparison of Platinum Wires.

Owing to the importance, from a thermometric point of view, of the comparison of different platinum wires, another comparison coil was wound with greater precautions, to ensure perfect symmetry The wires compared were pieces from the reels 1 and 3 used in making the platinum spirals for the air thermometers of Series III. and V. respectively. They were fused on to platinum electrodes and wound on terra-cotta slips, as previously described (pp. 199, 200)

The following observations were taken .—

TABLE P -II.

Time.		Resistance observations			Approximate temperature		Difference $pt_3 - pt_1$	Number of observation
Date	Hour	R + z	(R + z)/(R′ + z)	z	*pt*	*t*		
1886.					°	°	°	
Aug. 21	1 to 3 P.M	5 0410	·61024	0363	0	0	0	(1)
Aug. 23	10 to 11 30 A M.	6 7874	·61353	0460	100 06	100 06	0	(2)
„	12.30 P M	5·905	61217	0405	47 7	47 4	−0 06	(3)
„	1 20 to 2.46 P M	14·55	·61771	0807	547	593	+1 14	(4)
„	2 50 to 3 40 P.M.	12 83	61706	·0733	448	476	+1 08	(5)
„	3 40 to 7 19 P.M.	10 70	·61611	0632	326	339	+0 84	(6)
Aug. 24	10 A M	5·357	61103	0378	18·5	18 3	−0 03	(7)
„	10.30 A M to 12 NOON	15·46	·61812	0870	601	659	+1 05	(8)

An attempt was made to take an observation at a temperature of about 800°, but the insulation was not quite good enough ; the temperature was allowed to fall again and the following observations were taken :—

TABLE P.–III.

Time		Resistance observations			Approximate temperature.		Difference $pt_3 - pt_1$	Number of observation
Date	Hour	R + z	(R + z)/(R′ + z)	z	pt	t		
1886					°	°	°	
Aug. 24	12 NOON to 1 P M.	15 55	61801	0875	603	661	+0·39	(1)
”	1 to 1 30 P M.	14 83	61775	0839	562	611	+0 45	(2)
”	1 40 to 2 20 P M	12 96	·61702	0740	455	484	+0 51	(3)
”	2 20 to 3 P M.	10 88	61600	0635	335	34 9	+0 50	(4)
”	6 30 P M.	5 51	61074	0384	26 1	26	+0 07	(5)
Aug. 25	11 15 A M	5 398	61067	0379	20 0	20	0	(6)
In steam	12 NOON to 1 30 P M.	6 800	61306	0458	100 17	100 170	0	(7)
Aug 27 (In ice)	11 30 A.M to 1 30 P M	5 0537	60986	0357	0	0	0	(8)

The value of t is taken from the curve $t - pt = 1\ 57\ \{(t/100)^2 - (t/100)\}$. The observations are reduced as follows —

Let R be the resistance of the wire pt_1, R′ that of pt_3, r the ratio R/R′, and z the measured resistance of the connecting wires.

$$\text{Then } r = (\mathrm{R} + z) / (\mathrm{R}' + z)\{1 + (r - 1)\, z/\mathrm{R}\} \quad (z \text{ being small}).$$

Let τ stand for $100/(\mathrm{R}_1/\mathrm{R}_0 - 1)$, so that τ is a constant exactly analogous to θ_0 in the case of the air thermometer.

$$\text{Then } pt_1 = \tau_1 (\mathrm{R}/\mathrm{R}_0 - 1)\,; \qquad pt_3 = \tau_3 (\mathrm{R}'/\mathrm{R}'_0 - 1).$$

And the difference $pt_3 - pt_1$ may be put into a form more convenient for calculation, namely,

$$pt_3 - pt_1 = (\tau_3 - \tau_1)\,(\mathrm{R}/\mathrm{R}_0 - 1) - \tau_3\,(\mathrm{R}/\mathrm{R}_0 - \mathrm{R}'/\mathrm{R}'_0). \quad . \quad . \quad . \quad (\mathrm{T})$$

The small difference $(\tau_3 - \tau_1)$ may be accurately determined from the observations in ice and steam, for we have

$$1/\tau_1 - 1/\tau_3 = (\mathrm{R}_1/\mathrm{R}_0 - \mathrm{R}'_1/\mathrm{R}'_0)\,/100\,;$$

whence

$$\tau_3 - \tau_1 = \tau_1\tau_2\,(\mathrm{R}_1/\mathrm{R}_0 - \mathrm{R}'_1/\mathrm{R}'_0)\,/100,$$

and the small difference $(\mathrm{R}/\mathrm{R}_0 - \mathrm{R}'/\mathrm{R}'_0)$ is accurately given by the formula $(\mathrm{R}/\mathrm{R}_0 - \mathrm{R}'/\mathrm{R}'_0) = \mathrm{R}/\mathrm{R}_0\,(r - r_0)\,/r$, where r is the ratio R/R′. The only quantity, therefore, requiring the fullest accuracy of observation is $(r - r_0)$, the change of r. Since this is small, approximate values of the others will suffice.

From the observations (1) and (2) of Table P.–II. the deduced value of $\tau_3 - \tau_1$ is $6^\circ{\cdot}36$; from Observations (7), (8), Table P.–III., $\tau_3 - \tau_1 = 6^\circ{\cdot}23$. These values are used in the reduction of Tables P.–II. and P.–III. respectively. A rise of zero

amounting to nearly 1° C. was caused in each wire by the half-hour's exposure to the temperature of 800° in contact with clay and in the presence of copper, and probably tin and zinc The comparison at higher temperatures has to be effected by a different method to secure good insulation and guard the wire from contamination, but the experiments have not yet been completed. The results of the comparison are shown graphically in Plate 13, fig 10. The dotted portions of the curves are somewhat hypothetical, but there is naturally a family resemblance between them. They all show a comparatively sudden change (due to pt_1) somewhere between 200° and 300°, which, though small in amount, is distinctly marked; it is in each case followed by a flattening due to similar changes in the other wires, differing chiefly in phase A similar deviation is also indicated by the most accurate of the air thermometer observations (see page 221). The differences between different thermometers of pure platinum wire are seen to be very small, and to be capable of accurate measurement. I do not think they can possibly be due to any constant error affecting the method of comparison, as I have been unable to think of any that would at all account for the observed phenomena; but I have procured a stock of various specimens of platinum for more extended investigations.

Comparison of Platinum and Iron.

The comparisons of platinum and iron wires given in the earlier part of the paper were felt to require confirmation. The agreement with the exponential empirical formula was so close that I was inclined at the time to think that the deviations from it were errors of observation without meaning Commercial tinned iron wire had been used (the tin having been removed with glass paper as far as possible), and sufficient precautions had not been taken to avoid oxidation. The wire had ultimately reached a very steady state, but not until it had become covered with a film of black oxide A new comparison coil was therefore wound with *pure* iron wire and platinum wire No. 2 By an improved method of winding very good insulation was secured, and the wire was carefully heated in dry nitrogen and very little oxidised. (The steam which always escapes from the clay when heated probably produced the slight oxidation observed) The results of this comparison are given in Table F.–II.; they are simultaneous mean values of the resistances deduced, as before, from several observations at steady temperatures.

To facilitate the comparison of the two series of observations, the results of the first series are reproduced in Table F.–I., and reduced on the same plan as those in Table F.–II.

TABLE F.–I —Comparison of Platinum (No. 1) and Commercial Iron Wire. (The observations in this Table are the same as those in Table XI., p. 206)

Resistances observed		Temperatures deduced.			Difference observed, $fe_1 - pt_1$	Difference calculated by formula $(fe - pt)$	Deviation from parabola, obs. — calc.	Number of observation
Platinum.	Iron	pt_1	t	fe_1				
		°	°	°	°	°	°	
1	1	0	0	0	0	0	0	(1)
1 1225	1·1732	35 41	35 05	32 8	− 2 6	− 3·5	+ 0 9	(2)
1 3453	1 5263	99 79	99 79	99 79	0	0	0	(3)
1·5990	2 0036	173 2	175 2	190 3	+ 17 1	+ 19 8	− 2 7	(4)
1 9220	2·7692	266 5	273 9	335·4	+ 68 9	+ 71 5	− 2 6	(5)
2·2283	3 6715	355 0	370 7	506 4	+151 4	+150 5	+ 0 9	(6)
2 5548	4 8396	449 4	477 9	726·0	+276 6	+270 9	+ 5 7	(7)
2 8840	6 2667	544 5	590 1	998 4	+454 0	+433 9	+20 1	(8)
3 0314	7 0566	587 1	641 8	1148 2	+561 1	+521 6	+39 5	(9)

The value of t is given by the formula

$$t - pt = 1\ 57\{(t/100)^2 - (t/100)\}. \qquad (d)$$

The value of the difference is calculated from the formula

$$fe - pt = 15{\cdot}0\{(t/100)^2 - (t/100)\}. \qquad (fe)$$

The temperature by iron wire is given by the formula

$$fe = 100(\mathrm{R}/\mathrm{R}_0 - 1)/(\mathrm{R}_1/\mathrm{R}_0 - 1),$$

where $\mathrm{R}_1/\mathrm{R}_0 = 1{\cdot}5274$.

TABLE F –II.—Comparison of Platinum (No. 2) and *Pure* Iron Wire.

Resistances observed.		Temperatures deduced.			Difference observed, $fe_2 - pt_2$	Difference calculated by formula $(fe - pt)$	Deviation from parabola, obs — calc	Number of observation
Platinum	Iron	pt_2	t	fe_2				
		°	°	°	°	°	°	
1 0610	1 0863	18 15	17 92	16 48	− 1 67	− 2 2	+0 53	(1)
1 3360	1 5238	100	100	100	0	0	0	(2)
1 9578	2 9777	290 4	299 7	377 6	+ 87 2	+ 89 7	−2 5	(3)
2 2088	3 6914	359 8	376·1	513 8	+154 0	+155 8	−1 8	(4)
2 5102	4 7820	449 4	477 9	722 1	+272 7	+270 9	+1 8	(5)
2 8578	6 2825	552 9	600 1	1008 5	+455 6	+450 1	+5 5	(6)

The formulæ are as before, but the values of $\mathrm{R}_1/\mathrm{R}_0$ are different. (See Obs. (2).) The deviations are taken from the last column and set down on Plate 13, fig. 11; the abscissa is the approximate temperature by air thermometer; the similarity of the curves shows that the resistance-variations of the wires are similar in character.

Herein we see the treachery of the common empirical formula. Had we only the

observations in Table II., we might naturally suppose that the parabolic formula accurately represented the phenomenon, and that the numbers in the "deviation" column were errors of observation; and regret that after all our trouble the results were so inaccurate On the other hand, had we only the observations in Table F.–I., we might be carried away (as I was) with the idea that the exponential formula was the true solution of the problem, and that the small deviations were accidental, as in this case they very well might have been. But, having both series, F–I. and F.–II., in every way independent, and yet exhibiting such a marked similarity, we are led to inquire what it means. Each curve is marked by a small subsidiary wave at about 400° before the final upward rush This is not altogether fanciful, but is just such in position and magnitude as would be produced by an irregularity in the resistance-variation of the platinum of the order of 1° taking place between 200° and 300°; for in calculating the value of t and $(fe - pt)$ we have assumed smooth parabolic formulæ, which take no account of the small irregularities which are shown to exist by the curves of comparison of platinum wires, and one of which is indicated also by the air thermometer observations. In the curve F.–II. the phase of this subsidiary wave is retarded. Now the curve P.–I., containing the comparison between platinum wires (1) and (2), which were used in the comparisons F.–I., F.–II., respectively, shows that the curve of pt_2 is affected by a deviation similar in magnitude and sign to that of pt_1, but taking place about 100° later. If this curious correspondence of the curves had been foreseen, the comparisons between the platinum wires would have been made more extensive and complete, and, in particular, the form of the curves between 100° and 300° would have been carefully explored.

Professor TAIT has observed that at about a dull-red heat the resistance of iron increases very rapidly; the thermo-electric and mechanical qualities of the metal also change suddenly about this point. It is well known that very slight impurities exert a very marked effect on the softening point of iron; we need not be surprised, therefore, that the upward rush in the case of F–II does not come within the limit of observation, like that of F.–I. I propose to make a direct comparison between the wires F.–I. and F.–II. to confirm this. Unfortunately, my first impression on reducing the observations was that they disagreed completely; the nature of the agreement was not noticed till the apparatus had been dismounted and the wires used for other purposes, so that no more observations could be taken with the same coils.

The curves F.–I., F.–II., sufficiently illustrate the futility of applying empirical formulæ calculated from observations between narrow limits for purposes of extrapolation. I have known cases where an empirical formula, calculated by the method of least squares from observations at *varying* temperatures with a *mercurial* thermometer between 20° and 100° C., has been applied to estimate temperatures above 1000°. This is an extreme instance of a method to be avoided.

Between the limits 0° and 500°, however, the parabolas—

$$R/R_0 = 1 + 0{\cdot}0045346t + 0\ 000007034t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

$$R/R_0 = 1 + 0\ 0045658t + 0\ 000007082t^2, \quad . \quad . \quad . \quad \quad . \quad . \quad \text{I.}$$

both of which correspond exactly to the formula—

$$fe - t = 13{\cdot}43\ \{(t/100)^2 - (t/100)\},$$

represent the resistance-variations of the wires I. and II. with an accuracy of about 1 per cent.

Benoit gives for Iron—

$$R/R_0 = 1 + 0\ 004516t + 0\ 000005828t^2; \qquad R_1/R_0 = 1{\cdot}5099,$$

for Steel—

$$R/R_0 = 1 + 0{\cdot}004978t + 0{\cdot}000007351t^2, \qquad R_1/R_0 = 1{\cdot}5713.$$

These correspond to

$$fe - t = 11{\cdot}43\{(t/100)^2 - (t/100)\} \quad . \quad \text{(Iron)}$$

and

$$fe - t = 12\ 87\{(t/100)^2 - (t/100)\} \quad . \quad \text{(Steel)}$$

respectively.

Matthiessen, by observations between 0° and 100° with a mercury thermometer, found for very pure iron the formula

$$R_0/R = 1 - \ 0051182t + {\cdot}000012915t^2,$$

whence $R_1/R_0 = 1{\cdot}6197$. But this kind of formula is quite inadmissible for extrapolation, as it makes R a maximum when $t = 198°$. Between the limits 0° and 100° C. the formula of the parabolic type which agrees most nearly with it is

$$fe - t = 12{\cdot}74\{(t/100)^2 - (t/100)\}.$$

He similarly found for two specimens of commercial platinum, between 0° and 100°,

$$R_0/R = 1 - {\cdot}0027225t + {\cdot}000005776t^2 \quad . \quad . \quad (1); \text{ whence } R_1/R_0 = 1{\cdot}2730.$$
$$R_0/R = 1 - {\cdot}0025777t + {\cdot}000005054t^2 \quad . \quad . \quad (2), \text{ whence } R_1/R_0 = 1{\cdot}2613.$$

Wire (1) at 50° gives $pt = 50\ 70$. No. (2) gives $pt = 50{\cdot}20$. These differ by half a degree, whereas the pure wires of Tables P give $pt = 50{\cdot}40$ approximately, and differ by only one or two hundredths of a degree. Matthiessen's formula (1) makes R a maximum when $t = 236°$, and is of course inadmissible for extrapolation.

Comparison of Platinum and Platinum-Silver Alloy.

This is of little interest, except that the platinum-silver wire is used for standard resistance-coils. It is quite unsuited for thermometry, because of its small coefficient and variable composition.

The resistance of the platinum-silver wire was found to be increased about 5 per cent. by annealing for three hours at a red heat till it reached a steady state, while its mean coefficient between 0° and 100° fell from ·000279 before annealing to ·000236 after. For this reason it seems desirable that the wire used in the construction of standard coils should be well annealed.

The resistance-variation of this specimen of platinum-silver between 0° and 650° C. was well enough represented by the formula

$$R/R_0 = 1 + \cdot0002327t + 0\cdot000000033t^2.$$

Comparisons at High Temperatures.

The coil used in the experiments recorded in Table P.–I., p. 202, was tried in the porcelain tube alone, without the iron tube, at the highest temperature the gas-furnace would produce with the air blast. The insulation was too bad and the thermo-electric effects too large to allow of accurate measurements and so, when the apparatus cooled, a copper tube was inserted inside the porcelain. On again heating the tube the thermo-electric effects had disappeared, but the resistance was found to change enormously. When cold, the resistance of the wire was doubled, and it had become quite brittle. The copper tube came out as bright as if it had been heated in hydrogen; metallic copper was deposited all over the cooler parts of the electrodes and terra-cotta; the porcelain tube was bent with the heat. It is, therefore, necessary to try some other method at high temperatures. The apparatus is not yet complete, but promises well. I hope soon to be able to extend the comparisons accurately to 1000° at least.

NOTE.

[The experiments on surface condensation, mentioned on page 181, were not sufficiently precise to be applied to the case of the air thermometer; they only served to confirm the results of recent experiments, to show that the phenomenon is dependent in great measure on the past history, and to emphasise the importance of eliminating aqueous vapour and CO_2. It has not been thought worth while to use Equation (E), p. 179, in the reduction of the observations, partly because they are not yet sufficiently accurate, and partly because it is doubtful whether Equation (E) itself is strictly applicable beyond the limits of the experiments on which it is founded. The corrections it [illegible] are of the same order as the errors of observation, and anyone who thinks it [illegible] easily apply them.—July 11, 1887.]

VII. *On the Distribution of Strain in the Earth's Crust resulting from Secular Cooling; with special reference to the Growth of Continents and the Formation of Mountain Chains.*

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(1) THE reasoning of this paper is based upon the results of Sir W. THOMSON's and Professor G. H. DARWIN's well-known and independent researches on the rigidity of the Earth, upon Sir W. THOMSON's investigation on the secular cooling of the Earth, and, lastly, upon the beautiful contraction theory of mountain evolution which these researches lead up to and support. Its objects are to determine the distribution of strain in a solid globe resulting from secular cooling, and to examine the effects which this distribution must have upon the form of the great features of the Earth's surface.

In the first part of the paper I shall suppose the Earth to be bounded by a smooth spherical surface, and to be made up of a very great number of very thin concentric spherical shells, each shell being so thin that the loss of heat throughout it may be considered uniform. In the latter part the effects of inequalities on the Earth's surface upon the results so obtained will be alluded to. The argument urged against the contraction theory by the Rev. OSMOND FISHER will also be incidentally considered.

I. *The Distribution of Strain in the Earth's Crust resulting from Secular Cooling.*

(2) In his memoir on the secular cooling of the Earth, Sir W. THOMSON works out the case of the conduction of heat in "a solid extending to infinity in all directions, on the supposition that at an initial epoch the temperature has had two different constant values on the two sides of a certain infinite plane," and he shows that, for "a globe 8000 miles in diameter of solid rock, the solution will apply with scarcely sensible error for more than 1,000,000,000 years." The solution he gives is

$$v = v_0 + \frac{2V}{\sqrt{(\pi)}}\int_0^{x/2\sqrt{(\kappa t)}} dz \,.\, \epsilon^{-z^2};$$

where "κ denotes the conductivity of the solid, measured in terms of the thermal capacity of the unit of bulk; V, half the difference of the two initial temperatures; v_0, their arithmetical mean; t, the time; x, the distance of any point from the middle plane; v, the temperature of the point x at time t."*

(3) Differentiating v with respect to t, we obtain

$$\frac{dv}{dt} = -\frac{V}{2\sqrt{(\pi\kappa)}}\frac{x}{t^{\frac{3}{2}}}\cdot\epsilon^{-x^2/4\kappa t},$$

the rate of cooling at the point x at time t. Differentiating this expression with respect to x, we find

$$\frac{d^2v}{dx\,dt} = \frac{V}{2\sqrt{(\pi\kappa)}}\frac{1}{t^{\frac{3}{2}}}\left(\frac{x^2}{2\kappa t} - 1\right)\epsilon^{-x^2/4\kappa t}.$$

Hence $d^2v/dx\,dt$ is equal to zero when $x = \sqrt{(2\kappa t)}$. In other words, at any given epoch, the rate of cooling is a maximum at the depth for which x is equal to $\sqrt{(2\kappa t)}$.†

Hence the rate at which any shell parts with its heat increases to a certain depth below the Earth's surface, where it is a maximum, after which it decreases towards the centre, and the depth of the surface of greatest rate of cooling is continually increasing, and *varies as the square root of the time that has elapsed since the consolidation of the globe.*

(4) Consider any two consecutive spherical shells below the surface of greatest rate of cooling. Since the upper shell cools the more rapidly, its inner surface would, if free, contract more than the outer surface of the shell below; but, being forced to remain of the same radius as the latter after its contraction, it follows that the upper shell must be stretched or rent in order to rest upon the lower. Owing to the great pressure at that depth, and also to the slow rate of cooling, there can be little doubt but that the upper shell will be stretched and not rent.

Consider, again, two consecutive shells above the surface of greatest rate of cooling. In this case the lower shell cools the more rapidly; the inner surface of the upper shell, if free, would not therefore contract so much as the outer surface of the shell below. The upper shell must then either be stretched less than the lower, or must be crushed and folded in order to rest upon it. It will be shown afterwards that, above the surface of greatest rate of cooling, the amount by which each shell is being stretched gradually diminishes towards the surface of the Earth, until at a certain depth it is zero (the shell at this depth being now unstrained through cooling), and that, outside this depth, every shell is being crushed or folded.

* 'Edinb. Roy. Soc. Trans.,' vol. 23, 1864, pp. 161–162; THOMSON and TAIT's 'Natural Philosophy,' App. D, §§ (l), (o).

† This paragraph is the substance of a letter by Professor G. H. DARWIN on "The Formation of Mountains and the Secular Cooling of the Earth," published in 'Nature,' Feb. 6, 1879 (vol. 19, p. 313). If the time be 98,000,000 years, the depth at which the rate of cooling is greatest is about 53 miles

Hence *folding by lateral pressure takes place only to a certain depth below the Earth's surface; at this depth it vanishes, and, passing through it downwards, folding gives place to stretching by lateral tension.**

(5) In order to find the depth at which folding by lateral pressure vanishes, I propose to solve the following problem —

A globe, of radius r, is surrounded by a number of concentric spherical shells, called A_1, A_2, A_3 . . , of thicknesses a_1, a_2, a_3 . . ., respectively The globe remaining at its initial temperature, the shell A_1 is cooled by $t_1°$, the shell A_2 by $t_2°$ in the same time, and so on. The linear coefficient of expansion being e, and the same for all the shells, it is required to find the distribution of strain resulting from this method of cooling.

[(6) Consider the shell A_1. If the globe were absent, the radius of its inner surface would become $r(1 - et_1)$. It is, however, obliged to remain of radius r, and it must, therefore, be stretched. Assuming the stretching to be uniform throughout the shell, and expressing the fact that the volume of the shell after cooling $t_1°$ is equal to its original volume multiplied by $1 - 3et_1$, it will be found that the radius of the outer surface of the shell is

$$r_1 - \frac{r_1^3 - r^3}{r_1^2} et_1,$$

where

$$r_1 = r + a_1$$

Proceeding in a similar manner with the other shells, and remembering that the radius of the inner surface of any shell after cooling must be the same as the radius of the outer surface of the shell below after its cooling, then the radius of the inner surface of the shell A_{n+1} becomes

$$r_n - \frac{(r_n^3 - r_{n-1}^3)\,et_n + (r_{n-1}^3 - r_{n-2}^3)\,et_{n-1} + \quad + (r_1^3 - r^3)\,et_1}{r_n^2}.$$

Now, let $a_1 = a_2 = \ldots = a$; also let $t_1 = \delta t_0$, $t_2 - t_1 = \delta t_1$, . . ., $t_{n+1} - t_n = \delta t_n$. If the shell A_{n+1} had been allowed to contract, as if the globe and n interior shells were absent, the radius of its inner surface would have been $(r + na)(1 - et_{n+1})$. Hence the amount by which a great circle of the inner surface of this shell is stretched is proportional to

$$\frac{e}{(r+na)^2}\left[(r + na)^3\,\delta t_n + (r + \overline{n-1}\,.\,a)^3\,.\,\delta t_{n-1} + \ldots + r^3\,.\,\delta t_0\right],$$

that is, if the shells be supposed infinitely thin and infinitely great in number, to

$$\frac{e}{(r+na)^2}\int_r^{r+na} z^3 \frac{dt}{dz}\,dz.$$

* Whilst folding and crushing by lateral pressure must be accompanied by a development of heat, as pointed out by Mr. R MALLET and others, stretching by lateral tension, on the other hand, must be accompanied by a cooling of the masses stretched.

If this expression for any shell be negative, then that shell will be folded and not stretched]*

(7) Comparing the above with the notation used by Sir W. THOMSON, it will be seen that t is proportional to his dv/dt; his x corresponds to $c - z$, where c is the radius of the Earth, and z is the distance of any point from the centre of the Earth. Hence the dt/dz of the preceding paragraph is proportional to $-(d/dx)(dv/dt)$ in Sir W. THOMSON's notation.

Now

$$\frac{d}{dx}\left(\frac{dv}{dt}\right) = \frac{V}{2\sqrt{(\pi\kappa)}} \cdot \frac{1}{t^{\frac{3}{2}}}\left(\frac{x^2}{2\kappa t} - 1\right)\epsilon^{-x^2/4\kappa t}$$

Substituting in the integral in § (6), we have

$$\delta_{n+1} = \frac{\mu}{t^{\frac{3}{2}}(r+na)^2} \cdot \int_{c-r}^{c-r-na} (c-x)^3 \left(\frac{x^2}{2\kappa t} - 1\right)\epsilon^{-x^2/4\kappa t}\, dx,$$

where μ is a constant.

No practical use can be made of this expression, and we are, therefore, obliged to return to the last expression in § (6), in order to find the radius of the shell which experiences no strain through cooling.

(8) If, in the expression for dv/dt in § (3), we put $x = 2\sqrt{(\kappa t)}\, z$, we find that, at any given time, the rate of cooling is proportional to $z\epsilon^{-z^2}$. At the depth where the rate of cooling is greatest (for which $z = 1/\sqrt{2}$) the value of $z\epsilon^{-z^2}$ is ·42888. Again, at the depth for which $z = 4·00$, the value of this expression is ·00000045014. Hence, at any time, the rate of cooling at the latter depth is about one-millionth of its value where it is greatest. For our present purpose I shall, therefore, assume that below the depth for which $z = 4·00$ the rate of cooling is practically insensible, and it will be seen afterwards that the quantities neglected do not appreciably affect the results.

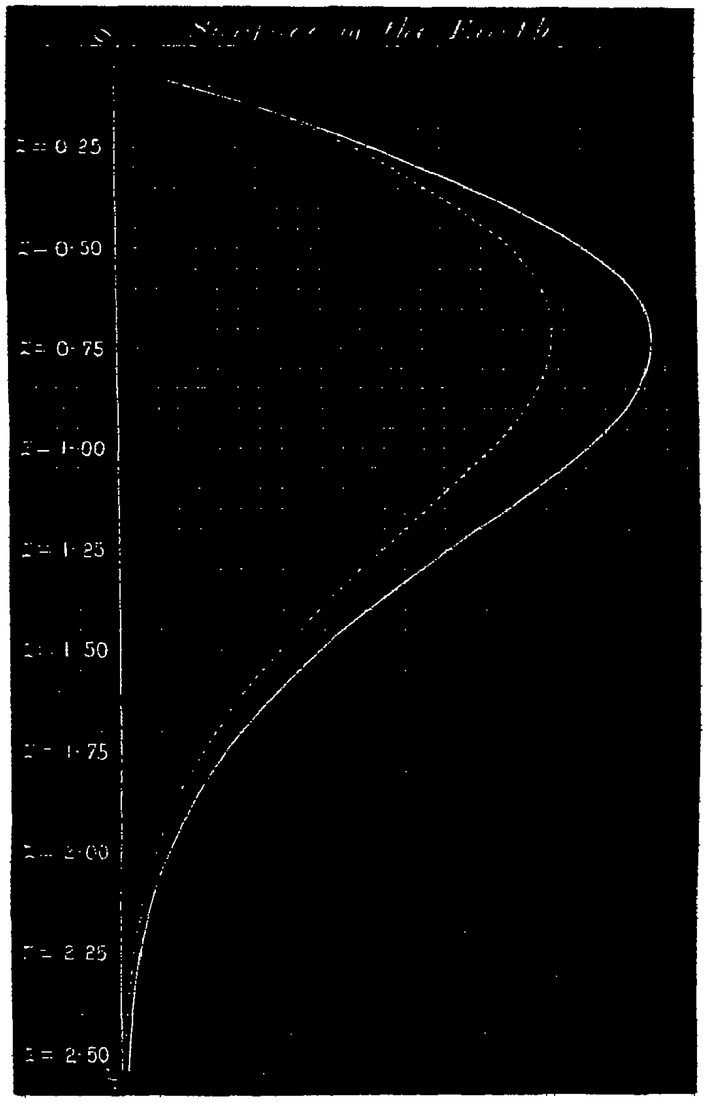
If the temperature of melting rock be 7000° F., the consolidation of the globe may have taken place about 98,000,000 years ago; but, as Sir W. THOMSON remarks, "we are very ignorant as to the effects of high temperatures in altering the conductivities and specific heats of rocks, and as to their latent heat of fusion. We must, therefore, allow very wide limits in such an estimate as I have attempted to make; but I think we may with much probability say that the consolidation cannot have taken place less than 20,000,000 years ago, or we should have more underground heat than we actually have, nor more than 400,000,000 years ago, or we should not have so much as the least observed underground increment of temperature."†

* The paragraphs within square brackets (added June 21, 1887) are abridged from the proof [illegible] the original paper communicated to the Society, partly because the proof there given was [illegible] and partly because another and more elegant proof is given, in the note following [illegible] DARWIN.

† [illegible] vol. 23, pp. 160–161; 'Natural Philosophy,' App. D., §§ (j), (l).

If the time that has elapsed since consolidation be taken as 174,240,000 years, the depth for which $z = 4{\cdot}00$ is 400 miles. For simplicity, let us adopt this value for the present.

(9) The radius of a sphere equal to the Earth in volume being about 3959 miles, I therefore suppose the Earth at the present time to be made up as follows —A central globe, 3559 miles in radius, at a temperature of 7000° F., which as yet has not sensibly cooled; surrounded by 400 concentric spherical shells, each one mile in thickness, the rate of cooling in each shell being uniform throughout, and equal to its value at the outer surface of that shell. On these assumptions I have calculated, from



the last expression in § (6), the proportional amounts by which the four hundred shells are stretched or folded.

The results of the calculation are shown in the accompanying figure, in which the dotted curve represents the rate of cooling, at the time considered, from the surface down to the depth for which $z = 2{\cdot}55$, and the continuous curve represents, at the same time, the amount by which a great circle of any shell is being stretched or folded, the folding by the part of the curve to the left of the axis Ox, and the stretching by the part to the right.

(10) The continuous curve also represents approximately the volume that, in a given time, is stretched or folded of any shell. For, let r be the radius of the internal surface of any shell, a its thickness; let r' and a' be what these values would naturally become by cooling if the shell were isolated; and let $r' + \delta$ be the radius of the globe on which this shell is obliged to fit, δ being small compared with r' If the shell were isolated, its volume after cooling would naturally be $4\pi r'^2 a'$. But the volume of the shell of radius $r' + \delta$ and thickness a' is $4\pi a' (r'^2 + 2r'\delta)$. Hence the amount of the shell that is stretched or folded is equal to $- 8\pi a' r' \delta$, the shell being stretched or folded according as δ is positive or negative. This expression, at a given time, varies nearly as δ, and therefore the continuous curve of the figure may be considered to represent very fairly the amount of rock stretched or folded at any depth.

(11) Without attributing much weight to the numerical results of these calculations—for, on account of our ignorance on many points, they are given rather for their qualitative than their quantitative value—the following conclusions may be deduced from them, taking t provisionally at 174,240,000 years —

1. Folding by lateral pressure changes to stretching by lateral tension at a depth of about 5 miles.

2. Stretching by lateral tension, inappreciable below a depth of about 400 miles, increases from that depth towards the surface; it is greatest at a depth of 72 miles, that is, just below the surface of greatest rate of cooling;* after this, it decreases, and vanishes at a depth of about 5 miles.

3. Folding by lateral pressure commences at a depth of about 5 miles, and gradually increases, being greatest near the surface of the Earth.†

(12) Since, at the depth at which folding by lateral pressure vanishes, the thin spherical shell cools and naturally contracts without straining, it follows that the folding of the outer crust is exactly the same as it would be if the whole globe beneath the unstrained shell were to cool uniformly throughout, and at the same rate as at the unstrained surface.

It will be seen, in the next paragraph, that the depth of this surface increases with the time since consolidation. Hence a part of the crust at one time stretched by lateral tension may at some later period be folded by lateral pressure.

* If $t = 174,240,000$ years, the depth of the surface of greatest rate of cooling is about 71 miles As the surface of greatest stretching should be just below the surface of greatest rate of cooling, this close approximation indicates the degree of accuracy of the assumptions made in § (9).

† The limited depth to which crust-folding extends may, perhaps, be considered as an argument against the contraction theory on the hypothesis of solidity, inasmuch as room is apparently not afforded for the accumulation of sediment to the estimated thicknesses of 40,000 feet in the Alleghany Mountains [illegible] in the Rocky Mountains. But, assuming such estimates to be correct, it should be [illegible] depth of the unstrained shell has been calculated on the supposition that the [illegible] is smooth and spherical; and it is probable that the existence of the surface [illegible] account for folds amply large enough to bury the thickest known masses of sediment.

(13) Let us suppose, for a moment, that the rate of cooling is always inappreciable at the depth for which $z = 4{\cdot}00$, *i.e.*, for which $x = 2\sqrt{(\kappa t)} \,.\, 4{\cdot}00$. This depth is continually increasing, and varies as the square root of the time that has elapsed since the consolidation of the globe. Dividing the crust above this depth always into the same number of spherical shells, the thicknesses of these shells therefore increase in proportion to the square root of the time. Now, the rates of cooling of any two particular shells have at any time the same proportion to one another, and therefore the proportional values of δt_0, δt_1, δt_2, . . . , in § (6) are always the same. But the cubes of the radii of the shells above the surface of greatest rate of cooling diminish in a less ratio, as the time increases, than the cubes of those below that surface; so that the depth of the unstrained surface would increase in a proportion rather greater than the square root of the time. On the other hand, the rate of cooling of any particular shell varies inversely as the time;* and, although at any time, the rate of cooling at the depth for which $z = 4{\cdot}00$ is always about one-millionth of its rate at the depth where it is greatest at that time, yet in early periods the rate of cooling might be sensible at a depth greater than that for which $z = 4{\cdot}00$. This would have the effect of slightly diminishing the proportion alluded to above. Hence, within certain limits, it is true that *the depth of the unstrained surface increases as the square root of the time that has elapsed since the consolidation of the globe.*

From the value given in § (11) we can therefore determine approximately the depth of the unstrained surface at any other time.

If we assume, as is generally done in the theory of the Conduction of Heat, that κ, the rate of conductivity, is independent of the temperature, then the rate of cooling of any particular shell varies only when the time changes, and therefore the depth of the unstrained surface is, at any time, independent of the initial temperature of the globe.

(14) Making use of the conclusion of the last paragraph, we can also determine approximately the law according to which the amount of rock stretched or folded in a given time changes. Considering any shell above the unstrained surface, the amount of it folded in a given time has been shown, in § (10), to be $8\pi a' r' \delta$. Now, approximately, r' may be considered constant, δ to vary inversely as the time, and a' directly as the square root of the time; so that $8\pi a' r' \delta$ varies nearly inversely as the square root of the time. This is the case with every shell above the unstrained surface, and therefore with the total folding of all such shells. Hence *folding by lateral pressure was effected most rapidly in the early epochs of the Earth's history, and, since then, the total amount of rock folded in any given time decreases nearly in proportion as the square root of the time increases*

(15) The same law being approximately true of the total amount of rock stretched by lateral tension, it follows that the ratio of the amount of rock stretched to the

* For $dv/dt = -V/\sqrt{\pi} \,.\, z\, \epsilon^{-z^2}\; 1/t$, which varies as $1/t$, since z is a constant for any particular shell.

amount folded in a given time is very nearly constant, but in reality slightly diminishing as the time increases.

Hence the ratio of the areas contained by the parts of the continuous curve in the figure with the axis Ox, to the right and left of that axis, may be considered to represent roughly the ratio of the total amount of rock stretched to the total amount folded since the earth solidified.*

II. *The Rev.* O. FISHER's *Argument on the Insufficiency of the Contraction Theory.*

(16) It may be well here to refer to the objection urged against the contraction theory by the Rev. O. FISHER, an objection which has by some been considered conclusive against either the contraction theory, as usually held, or the assumptions on which that theory is founded. In its latest form this argument will be found in a paper in the 'Philosophical Magazine' for February, 1887.† Briefly it may be summed up as follows:—

Mr. FISHER's argument is based on the assumptions that, initially, the Earth was practically solid, its surface spherical, and its whole mass at the high temperature of 7000° F. throughout; and that it cooled down from this condition to that implied by Sir W. THOMSON's solution. Further, he takes the case which he considers most favourable to the advocates of the contraction theory, and supposes that each spherical layer, in sinking down by reason of the cooling of the mass beneath, maintained its horizontal extension, so that the whole of the compression, introduced by its having to fit on the shrunken nucleus, was rendered available for producing corrugations. He thus finds that "if all the elevations which would have been produced by compression, through the contraction of the Earth cooling as a solid, were levelled down, they would form a coating of about 900 feet in thickness above the datum level, which would be the surface, had the matter of the crust been perfectly compressible, so that compression would not have corrugated it." He concludes, therefore, that, "if we take into consideration the land and the ocean-basins, the existing inequalities of the surface are greater than can be accounted for by the theory of compression through contraction by cooling of a solid globe, even upon the too highly favourable suppositions made in the present paper."‡

This argument seems to me inconclusive on several grounds.

* Roughly, this ratio is about 340 to 1. But this relates only to folding resulting directly from secular cooling, the surface of the Earth being smooth and spherical. The total amount of rock-folding must, of course, be far in excess of that indicated by this ratio; see footnote to § (23)

† "On the Amount of the Elevations attributable to Compression through the Contraction during Cooling of a Solid Earth," 'Phil. Mag.,' vol. 23, 1887, pp. 145–149. The original form of the argument is contained in a paper "On the Inequalities of the Earth's Surface viewed in connection with the Secular Cooling," 'Cambridge Phil. Soc. Trans.,' vol. 12, 1879, pp. 414–433, and in 'The Physics of the Earth's Crust' (1881), chaps. v. and vi. See also a letter to 'Nature,' Nov. 23, 1882 (vol. 27, pp 76–77)

(17) First, the problem investigated is not one according to nature. For it takes no account of the time during which the cooling has taken place, and, therefore, being independent of the time, the effect must be the same as if the cooling were suddenly accomplished. But sudden cooling involves the impossible supposition that κ, the rate of conductivity, should be infinite

(18) But, again, let us for a moment imagine that the temperature of the Earth's crust (supposed solid) did suddenly change from a uniform one throughout to its present condition as given by Sir W. THOMSON's solution. Let us further assume the Earth's surface to have been initially spherical, and the Earth to be divided into a very great number of very thin shells by spheres concentric with the Earth; the shells being so thin that the cooling throughout each may be considered uniform.

Then the loss of heat experienced by any one of these shells is $V - v$, where V was its initial and v is its present temperature. And, as v increases with the depth from the surface, it follows that $V - v$ diminishes as the depth increases.

Consider any two consecutive shells. Since the upper shell is cooled by the greater amount, its inner surface would, if free, contract more than the outer surface of the shell below; but, being forced to remain of the same radius as the latter after its contraction (and straining), it follows that the upper shell must be stretched in order to rest upon the lower

The same reasoning applies to every pair of consecutive shells from the greatest depth at which $V - v$ is a sensible quantity to the very surface of the Earth. Hence, on the assumption of sudden cooling throughout the whole of the Earth's crust that has so far undergone cooling, there ought only to be crust-stretching by lateral tension, and not any crust-folding by lateral pressure.

But, as shown above in § (11), lateral pressure, brought into action by secular cooling, does produce folding in the Earth's crust—to a limited depth, it is true, and in rock that may already once have been stretched by lateral tension.

I maintain, therefore, that an argument built up on the virtual hypothesis of a sudden cooling of the globe, however favourable to the contraction theory the other data employed may be, loses its force when we consider the natural process of a continuous and gradual cooling.

(19) Lastly, even if it were to be proved that the volume of rock folded by lateral pressure due to secular cooling is insufficient to produce the existing inequalities of land and ocean-basin, it would in no wise follow that the mountain-chains alone could not be so produced. And it should be remembered that this is all that is generally asserted by the advocates of the contraction theory. Mr. FISHER's argument assumes that the Earth's surface was initially spherical. But Professor B. PEIRCE* and Professor G. H. DARWIN† have shown that vast wrinkles would be formed on the

* "The Contraction of the Earth," 'Nature,' Feb. 16, 1871 (vol 3, p. 315), reprinted from the 'Proceedings of the American Academy of Arts and Sciences,' vol. 8.

† "Problems connected with the Tides of a Viscous Spheroid," 'Phil. Trans.,' 1879 (Part 2), pp. 539–593.

surface of a once viscous Earth by "the diminution of oblateness arising from the diminished velocity of rotation upon the axis" resulting from tidal friction; wrinkles that would be amply large enough to form the foundations of the continental masses. In order to prove the insufficiency of the contraction theory, it is necessary, therefore, to show that folding by lateral pressure, both directly and indirectly, by acting on vast masses of sediment, is incapable of producing, not the average height of the inequalities of the Earth's surface, but the total amount of rock-folding in all past and present mountain-chains

III *The Effects of Crust-Stretching and Folding on the Evolution of the Earth's Surface-Features*

(20) It has been already stated that much weight cannot be attached to the numerical results obtained in this paper. They are evidently dependent on the assumptions made at the commencement. It should also be remembered that the coefficient of expansion (e) has been supposed constant at whatever temperature the loss of heat takes place, and there is reason for believing that this is not strictly true. And again, it was assumed that, on solidifying, the surface of the Earth was a smooth sphere, although by the interference of tidal friction there must previously have been raised up masses forming the nuclei which, by continual growth of folded rock and mountain-range along their borders, have developed into our present continental areas. Clearly the existence of these great masses from the very beginning must have had an all-important influence on the subsequent evolution of the Earth's surface-features.

(21) "In the case of the Earth," says Professor G. H. DARWIN, "the wrinkles would run north and south at the equator, and would bear away to the eastward in northerly and southerly latitudes; so that at the north pole the trend would be north-east, and at the south pole north-west. Also the intensity of the wrinkling force varies as the square of the cosine of the latitude, and is thus greatest at the equator, and zero at the poles. Any wrinkle when once formed would have a tendency to turn slightly, so as to become more nearly east and west than it was when first made.

"The general configuration of the continents (the large wrinkles) on the Earth's surface appears to me remarkable when viewed in connection with these results."

And again, indicating a physical cause of continental permanence, he says, "But, if this cause was that which principally determined the direction of terrestrial inequalities, then the view must be held that the general position of the continents has always been somewhat as at present, and that, after the wrinkles were formed, the surface attained a considerable rigidity, so that the inequalities could not entirely subside during the continuous adjustment to the form of equilibrium of the Earth adapted at each period to the lengthening day. With respect to this point it is worthy of remark

that many geologists are of opinion that the great continents have always been more or less in their present positions."*

(22) Now, soon after the formation of these wrinkles, that is, in the initial period of the Earth's history as a solid, or nearly solid, globe, the unstrained shell must have been very close to the surface of the Earth, and the surface of greatest stretching also so near to it that stretching by lateral tension must have affected the form of the surface features. But, owing to the pressure of the continental wrinkles, the amount of stretching under them must have been very much less than under the great oceanic areas. Thenceforward, therefore, *crust-stretching by lateral tension must have taken place chiefly beneath the ocean-basins, deepening them and intensifying their character.* And, in leading to the continual subsidence of the ocean-bed, it is evidently a physical cause of the general permanence of oceanic areas: a cause, it is true, continually receding from the surface, and diminishing in intensity with the increase of time, but probably even now not quite ineffective.

(23) Again, the amount of crust-stretching by lateral tension being greatly in excess of the amount of crust-folding by lateral pressure due to secular cooling, it follows that folding beneath the ocean-bed will do little but diminish its rate of subsidence. The effects of folding in changing the form of the Earth's surface features will therefore be most apparent in the continental areas, especially in those regions where the change of vertical pressure above the folded layers diminishes most rapidly, *i e.*, near the coast-lines where the slope towards the ocean depths is greatest. It is perhaps worthy of remark that these are the districts where earthquake and volcanic action are now most prevalent †

In the coast regions, moreover, the products of continental denudation are chiefly deposited, and the rock-folding due simply to secular cooling produces in vast masses of sediment still more crushing and folding.‡ The direction of the folds will be perpendicular to the average slope of the surface above them, *i.e.*, they will generally be parallel to the coast-line. Hence *the continents will grow by the formation of mountain chains along their borders.*

(24) In a given time, the amount of rock-folding resulting from secular cooling was greatest in the first epochs of the Earth's history, and diminished as the time increased. It does not necessarily follow that the early mountain ranges were the loftiest and most massive, but probably they were , and very possibly also the displacement, by crushing and folding, of two neighbouring portions of rock was greatest in early times. But,

* 'Phil Trans ', 1879, pp 589, 590

† Professor J. MILNE, "Note on the Geographical Distribution of Volcanoes," 'Geol. Mag.,' vol. 7, 1880, pp 166–170.

‡ Professor J D DANA thus summarises the history of the Alleghany and other mountain-chains:— "First, a slow subsidence or geosynclinal . . . and, accompanying it, the deposition of sediments to a thickness equal to the depth of the subsidence; finally, as a result of the subsidence, and as the climax in the effects of the pressure producing it, an epoch of plication, crushing, &c., between the sides of the trough." 'Phil Mag ,' vol 46, 1873, p 49.

taking into consideration the whole surface of the globe, *the process of mountain-making diminishes with the increase of the time, and so also does the rate of continental evolution.*

(25) It cannot be said that the contraction theory on the hypothesis of solidity is entirely free from objections. Two very obvious ones have already been alluded to in the course of this paper, namely (1) The small calculated depth of the unstrained surface, especially in early geological periods; and (2) The small proportion of folded rock to stretched rock directly produced by secular cooling. But I do not think that these objections are by any means fatal to the theory. Assuming the Earth to be practically solid, and to have been originally at a high temperature throughout, I believe it may be concluded that the peculiar distribution of strain in the Earth's crust resulting from its secular cooling has contributed to the permanence of ocean-basins, and has been the main cause of the growth of continents and the formation of mountain chains.

VIII. *Note on Mr.* DAVISON'S *Paper on the Straining of the Earth's Crust in Cooling.*

By G. H. DARWIN, *LL D., F.R.S, Fellow of Trinity College, and Plumian Professor in the University of Cambridge.*

Received June 15,—Read June 16, 1887.

MR DAVISON'S interesting paper was, he says, suggested by a letter of mine published in 'Nature' on February 6, 1879. In that letter it is pointed out that the stratum of the Earth where the rate of cooling is most rapid lies some miles below the Earth's surface. Commenting on this, I wrote —

"The Rev. O. FISHER very justly remarks that the more rapid contraction of the internal than the external strata would cause a wrinkling of the surface, although he does not admit that this can be the sole cause of geological distortion. The fact that the region of maximum rate of cooling is so near to the surface recalls the interesting series of experiments recently made by M. FAVRE ('Nature,' vol. 19, p. 108), where all the phenomena of geological contortion were reproduced in a layer of clay placed on a stretched india-rubber membrane, which was afterwards allowed to contract. Does it not seem possible that Mr. FISHER may have under-estimated the contractibility of rock in cooling, and that this is the sole cause of geological contortion?"

Mr. DAVISON works out the suggestion, and gives precision to the general idea contained in the letter. He shows, however, that there is a layer of zero strain in the

Earth's surface, and that this layer, instead of that of greatest cooling, must be taken to represent FAVRE's elastic membrane

It appears that the mathematical discussion of the problem in his paper is unnecessarily laborious,* and he has not made various important deductions as to the integral results of distortion and as to the magnitude of the effects to be expected I, therefore, offer the following note with the intention of rendering more complete an important chapter in the mechanics of geology.

When a spherical shell expands with rise of temperature, it may be said to stretch, in one sense of the word. If the shell were one of the layers of the Earth, such a stretching would have no geological effect, for it would merely involve a change of density. The term "stretching" then requires an explanation in connection with Mr. DAVISON's paper. The stretching which we have to consider is, in fact, simply the excess of the actual stretching above that due to rise of temperature. The negative of such stretching is a contraction, and it would actually be shown by a crumpling of strata.

If ρ be the density of a body, and ϵ its modulus of linear expansion for temperature, then it is obvious that when the temperature is raised θ degrees the density becomes $\rho(1 - 3\epsilon\theta)$. Now suppose that a spherical shell of radius r expands so that its radius becomes $r(1 + \alpha)$, and suppose that at the same time its temperature is raised θ degrees. Then, if k be the modulus of stretching, so that unit length is stretched by a length k, we have, in consequence of the above explanation of stretching,

$$\alpha = k + \epsilon\theta. \qquad \qquad (1)$$

Now let us consider the geometry of changes in a sphere such that a shell of internal radius r, thickness δr, and density ρ, expands until its internal radius r becomes $r(1 + \alpha)$ and its density ρ becomes $\rho(1 - 3\epsilon\theta)$.

The external radius $r + \delta r$ clearly then becomes

$$r(1 + \alpha) + \delta r\left[1 + \frac{d}{dr}(r\alpha)\right].$$

Thus the mass of the shell $4\pi r^2\rho\,\delta r$ becomes

$$4\pi r^2\rho\,\delta r\left[1 + 2\alpha + \frac{d}{dr}(r\alpha) - 3\epsilon\theta\right].$$

Then, since the mass remains unchanged, we must have

$$2\alpha + \frac{d}{dr}(r\alpha) - 3\epsilon\theta = 0.$$

This "equation of continuity" may clearly be written

$$\frac{d}{dr}(r^3\alpha) = 3\epsilon\theta r^2.$$

* *Cf* foot-note, p 234

Substituting for α in terms of the modulus of stretching as given by (1), we have

$$\frac{d}{dr}(kr^3 + \epsilon\theta r^3) = 3\epsilon\theta r^2 .$$

Hence

$$\frac{d}{dr}(kr^3) = -\epsilon r^3 \frac{d\theta}{dr},$$

and therefore

$$k = -\frac{\epsilon}{r^3}\int r^3 \frac{d\theta}{dr}\,dr, \quad \ldots \ldots \quad (2)$$

where the integral is taken from r down to such a depth that there is no change of temperature.

If, now, θ represents the rise of temperature per unit time, and if we replace k by dK/dt, the rate of stretching per unit time, and if we make application of (2) to the case of the Earth, and write v for the temperature of the Earth at a depth x below the surface, and c for the Earth's radius, we have

$$\frac{dv}{dt} = \theta,$$

$$c - x = r;$$

and (2) becomes

$$\frac{dK}{dt} = -\frac{\epsilon}{(c-x)^3}\int_c^x (c-x)^3 \frac{d^2v}{dx\,dt}\,dx \quad \ldots \ldots \ldots \quad (3)$$

In inserting the limits to the integral, it is assumed that the temperature at the Earth's centre is sensibly constant.

The amount by which a great circle of radius $(c - x)$ is being stretched per unit time is

$$2\pi(c-x)\frac{dK}{dt}.$$

This expression, with the above value (3) for dK/dt, is Mr. DAVISON's result.

We know from THOMSON's solution for the cooling of the Earth that, when x exceeds a small fraction of the Earth's radius the temperature gradient and its rate of variation in time are very small. Hence, when x is not very small, $d^2v/dx\,dt$ is very small; therefore we may with sufficient approximation replace $(c-x)^3$ under the integral sign by $c^3 - 3c^2x$. Outside of the integral we may simply neglect x. Also the limit c of the integral may be replaced by infinity, and this is desirable because THOMSON's solution is really applicable to an infinite slab and not to a sphere. With these approximations we get

$$\frac{dK}{dt} = \epsilon\int_x^\infty \left(1 - \frac{3x}{c}\right)\frac{d^2v}{dx\,dt}\,dx. \quad \ldots \ldots \ldots \quad (4)$$

Integrating, with regard to time, from the time t to zero, we shall get the total amount of stretching in the layer x from the epoch of consolidation down to time t. Thus

$$K = \epsilon \int_x^\infty \left(1 - \frac{3x}{c}\right) \frac{dv}{dx} dx. \qquad (5)$$

Now with THOMSON'S notation

$$\int_x^\infty \frac{dv}{dx} dx = \mathrm{V} - v$$

Since

$$\frac{dv}{dx} = \frac{\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}} e^{-x^2/4\kappa t},$$

$$\int_x^\infty x \frac{dv}{dx} dx = \frac{\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}} \int_x^\infty x e^{-x^2/4\kappa t} dx = \frac{2\mathrm{V}\kappa t}{(\pi\kappa t)^{\frac{1}{2}}} e^{-x^2/4\kappa t} = 2\kappa t \frac{dv}{dx}.$$

Hence (5) becomes

$$K = \epsilon \left[\mathrm{V} - v - \frac{6\kappa t}{c^2} \cdot c \frac{dv}{dx}\right].* \qquad (6)$$

This expression gives us the total amount of contraction since consolidation in terms of the temperature and the temperature gradient. I shall return to this expression later.

Differentiating (6) with respect to the time, we have

$$\frac{dK}{dt} = \epsilon \left[-\frac{dv}{dt} - \frac{6\kappa}{c} \frac{dv}{dx} - \frac{6\kappa t}{c} \frac{d^2 v}{dx\, dt} \right].$$

But

$$\frac{dv}{dt} = -\frac{x}{2t} \frac{dv}{dx}, \qquad \text{and} \qquad \frac{d^2 v}{dx\, dt} = \frac{1}{2t} \left(\frac{x^2}{2\kappa t} - 1\right) \frac{dv}{dx}.$$

Hence

$$\frac{dK}{dt} = \frac{\epsilon}{2t} c \frac{dv}{dx} \left[\frac{x}{c} - \frac{12\kappa t}{c^2} - \frac{6\kappa t}{c^2} \left(\frac{x^2}{2\kappa t} - 1\right)\right],$$

or

$$\frac{dK}{dt} = \frac{\epsilon}{2t} c \frac{dv}{dx} \left[\frac{x}{c} - \frac{3x^2}{c^2} - \frac{6\kappa t}{c^2}\right]. \qquad (7)$$

When x exceeds more than a small fraction of the Earth's radius dv/dx is very small; hence in (7) we may regard x/c as small, and write the equation

$$\frac{dK}{dt} = \frac{\epsilon}{2t} c \frac{dv}{dx} \left[\frac{x}{c} - \frac{6\kappa t}{c^2}\right]. \qquad (8)$$

* If x had not been treated as small, we should have got

$$\epsilon (\mathrm{V} - v) \frac{c(c^2 + 6\kappa t)}{(c - x)^3} - \frac{2\kappa t}{(c - x)^3} \left[\frac{c^3 - (c - x)^3}{x} + 4\kappa t\right] \epsilon \frac{dv}{dx},$$

and it is easy to see that this leads to the same result as the above for all practical purposes.

When $x = 0$, dK/dt is negative, and hence at the surface there is contraction or crumpling The crumpling continues for some depth downwards, and vanishes when

$$\frac{x}{c} = \frac{6\kappa t}{c^2}.$$

Taking, with THOMSON, the foot and year as units, κ appears to be about 400; if, therefore, t is τ million years, $\kappa t = 4 \times 10^8 \tau$. Now, c being 21×10^6 feet, $\kappa\tau/c^2 = \tau/10^6$ approximately, and

$$x = \frac{24 \times 10^8}{21 \times 10^6} \tau \text{ feet,}$$
$$= 114\,\tau \text{ feet.}$$

If τ be 100, $x = 2$ miles.

Thus, if the time since consolidation be 100 million years, the present depth of the stratum of no strain is 2 miles, and the depth is proportional to the time since consolidation. With a greater value of κ the depth is greater

With regard to the value of dK/dt at greater depths, we observe that at a few miles below the stratum of zero strain $6\kappa t/c^2$ becomes negligible compared with x/c. Hence dK/dt is approximately proportional to $x\,dv/dx$. Now, if we take the figure drawn in THOMSON and TAIT's 'Natural Philosophy,' Appendix D, and augment or diminish each ordinate NP′ proportionately to the corresponding abscissa, it is clear that we shall get just such a curve as that drawn by Mr. DAVISON. His curve might thus have been drawn without any computations at all. The function $x\,dv/dx$ is proportional to dv/dt, and thus his dotted curve is of just the same kind as the other, excepting close to the surface.

Now let us return to the expression for the integral stretching, viz.—

$$K = \epsilon\left[(V - v) - \frac{6\kappa t}{c^2}\, c\, \frac{dv}{dx}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We have

$$\frac{dv}{dx} = \frac{V}{(\pi\kappa t)^{\frac{1}{2}}} e^{-x^2/4\kappa t}.$$

Hence, if x be small and t large (both of which conditions apply to the present time near the Earth's surface),

$$\frac{dv}{dx} = \frac{V}{(\pi\kappa t)^{\frac{1}{2}}}.$$

Hence

$$K = \epsilon\left[V - v - \frac{6(\kappa t)^{\frac{1}{2}}}{c\pi^{\frac{1}{2}}} V\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now, as we have seen above, with such values as those with which we have to deal, $6\kappa t/c^2$ is a small fraction, notwithstanding that it increases with the time.

Hence, for the upper layers, we have approximately

$$K = \epsilon(\text{V} - v). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Thus it appears that the integral effect is always a stretching, and that it is the same in amount at whatever speed the globe cools. The fact that, if the globe cools suddenly, the integral effect must be stretching has been pointed out by Mr DAVISON.

If we differentiate (10), we have

$$\frac{dK}{dt} = \epsilon\left[-\frac{dv}{dt} - \frac{3\kappa}{c}\cdot\frac{\text{V}}{(\pi\kappa t)^{\frac{1}{2}}}\right]$$

But

$$-\frac{dv}{dt} = \frac{x}{2t}\frac{dv}{dx},$$

and, near the surface,

$$\frac{dv}{dx} = \frac{\text{V}}{(\pi\kappa t)^{\frac{1}{2}}}$$

Hence

$$\frac{dK}{dt} = \epsilon\frac{dv}{dx}\left(\frac{x}{2t} - \frac{3\kappa}{c}\right)$$

$$= \frac{\epsilon}{2t}c\frac{dv}{dx}\left(\frac{x}{c} - \frac{6\kappa t}{c^2}\right),$$

and thus we find equation (8) again, as ought to be the case This may also be written—

$$\frac{dK}{dt} = \frac{\epsilon}{2t}\frac{\text{V}}{(\pi\kappa t)^{\frac{1}{2}}}\left(x - \frac{6\kappa t}{c}\right).$$

We must now see whether the amount of crumpling of the surface strata can be such as to explain the contortions of geological strata.

It must be borne in mind that, from a geological point of view, contraction is not the negative of stretching. When a stratum is stretched it may perhaps be ruptured, and rock may be squeezed up into the crack, at least for the strata which are very near the surface, and therefore not under great pressure; but when compressed the stratum is no doubt crumpled. Hence it is insufficient to know that the integral effect from the time of consolidation is a stretching; for that stretching may be merely the excess of a stretching over a crumpling. Now we have found above that the depth of the stratum of no strain is given by

$$x = \frac{6\kappa t}{c},$$

Hence, at the time t', given by $t' = cx/6\kappa$, the surface of no strain was at depth x; and at all later times than t' the surface of no strain lies deeper. Therefore, to find

the total amount of crumpling at any depth, we require to find the integral effect taken from t' to t, which is greater than t'.

The integral stretching from consolidation to time t is

$$K = \epsilon\left[\mathrm{V} - v - \frac{6\kappa t}{c}\frac{dv}{dx}\right].$$

Now near the surface v is nearly equal to $v_0 + x\dfrac{dv}{dx}$;

hence

$$K = \epsilon(\mathrm{V} - v_0) - \epsilon\frac{dv}{dx}\left[x + \frac{6\kappa t}{c}\right].$$

But

$$\frac{dv}{dx} = \frac{\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}}e^{-x^2/4\kappa t} = \frac{\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}} \text{ near the surface.}$$

Hence

$$K = \epsilon(\mathrm{V} - v_0) - \frac{\epsilon\mathrm{V}}{(\pi\kappa)^{\frac{1}{2}}}\left[\frac{x}{t^{\frac{1}{2}}} + \frac{6\kappa t^{\frac{1}{2}}}{c}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

At the time t', given by $t' = cx/6\kappa$,

$$K = \epsilon(\mathrm{V} - v_0) - \frac{\epsilon\mathrm{V}}{(\pi\kappa)^{\frac{1}{2}}}\left[x^{\frac{1}{2}}\left(\frac{6\kappa}{c}\right)^{\frac{1}{2}} + x^{\frac{1}{2}}\left(\frac{6\kappa}{c}\right)^{\frac{1}{2}}\right] = \epsilon(\mathrm{V} - v_0) - \frac{\epsilon\mathrm{V}}{(\pi\kappa)^{\frac{1}{2}}}\cdot 2\left(\frac{6\kappa}{c}\right)^{\frac{1}{2}}x^{\frac{1}{2}}. \quad . \qquad (13)$$

This gives the total stretching from the time of consolidation until the surface of no strain has got down to x.

If, therefore, we subtract (13) from (12), we get the total stretching between the time t' and the time t, and the result is obviously—

$$K = -\frac{\epsilon\mathrm{V}}{(\pi\kappa t)^{\frac{1}{2}}}\left[x - 2\left(\frac{6\kappa t}{c}\right)^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{6\kappa t}{c}\right].$$

This is clearly

$$K = -\epsilon\frac{dv}{dx}\left[x^{\frac{1}{2}} - \left(\frac{6\kappa t}{c}\right)^{\frac{1}{2}}\right]^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

This expression is essentially negative, and therefore the total effect from t' to t is a crumpling, as was foreseen.

This integral crumpling vanishes at the same place as does dK/dt, that is to say, when

$$x = \frac{6\kappa t}{c}.$$

This, we have shown above, will be at a depth of 2 or 3 miles. The amount of crumpling at the surface is given by putting $x = 0$, and

$$K = -\epsilon \frac{6\kappa t}{c} \frac{dv}{dx}.$$

Now we have seen that, if τ be the number of millions of years since consolidation,

$$\frac{6\kappa t}{c} = 114\,\tau \text{ feet},$$

and

$$\frac{dv}{dx} = 0^{\circ}{\cdot}02 \text{ Fahr. per foot}.$$

Hence

$$K = -\tau\epsilon \times 2{\cdot}28.$$

The total amount by which a great circle is contracted is 25,000 K miles; and, judging by the coefficient of expansion of metals, ϵ may be about 5×10^{-5}.

Thus, with these rough data, the amount of crumpling of a great circle of radius c is

$$2\pi c K = \tau \times \left(\frac{5}{10^5}\right) \times (2{\cdot}5 \times 10^4) \times 2{\cdot}28 = 2{\cdot}85\,\tau \text{ miles}.$$

Thus, in 10,000,000 years, $28\frac{1}{2}$ miles of rock would be crumpled up.

The area of rock crumpled is $4\pi c^2 . 2K$, and with these numerical data

$$4\pi c^2 . 2K = 4c\,(2\pi c\,K).$$

Now $c = 4000$ miles, and therefore

$$4\pi c^2 . 2K = 22{,}800\,\tau \text{ square miles}.$$

Thus, in 10,000,000 years, 228,000 square miles of rock will be crumpled up and piled on the top of the subjacent rocks.

The numerical data with which we have to deal are all of them subject to wide limits of uncertainty, but the result just found, although rather small in amount, is such as to appear of the same order of magnitude as the crumpling observed geologically.

The stretching and probable fracture of the strata at some miles below the surface will have allowed the injection of the lower rocks amongst the upper ones, and the phenomena which we should expect to find according to Mr. DAVISON's theory are eminently in accordance with observation. It therefore appears to me that his view has a strong claim to acceptance.

IX. *Transmission of Sunlight through the Earth's Atmosphere.*

By CAPT. W. DE W. ABNEY, *R.E.*, *F.R.S.*

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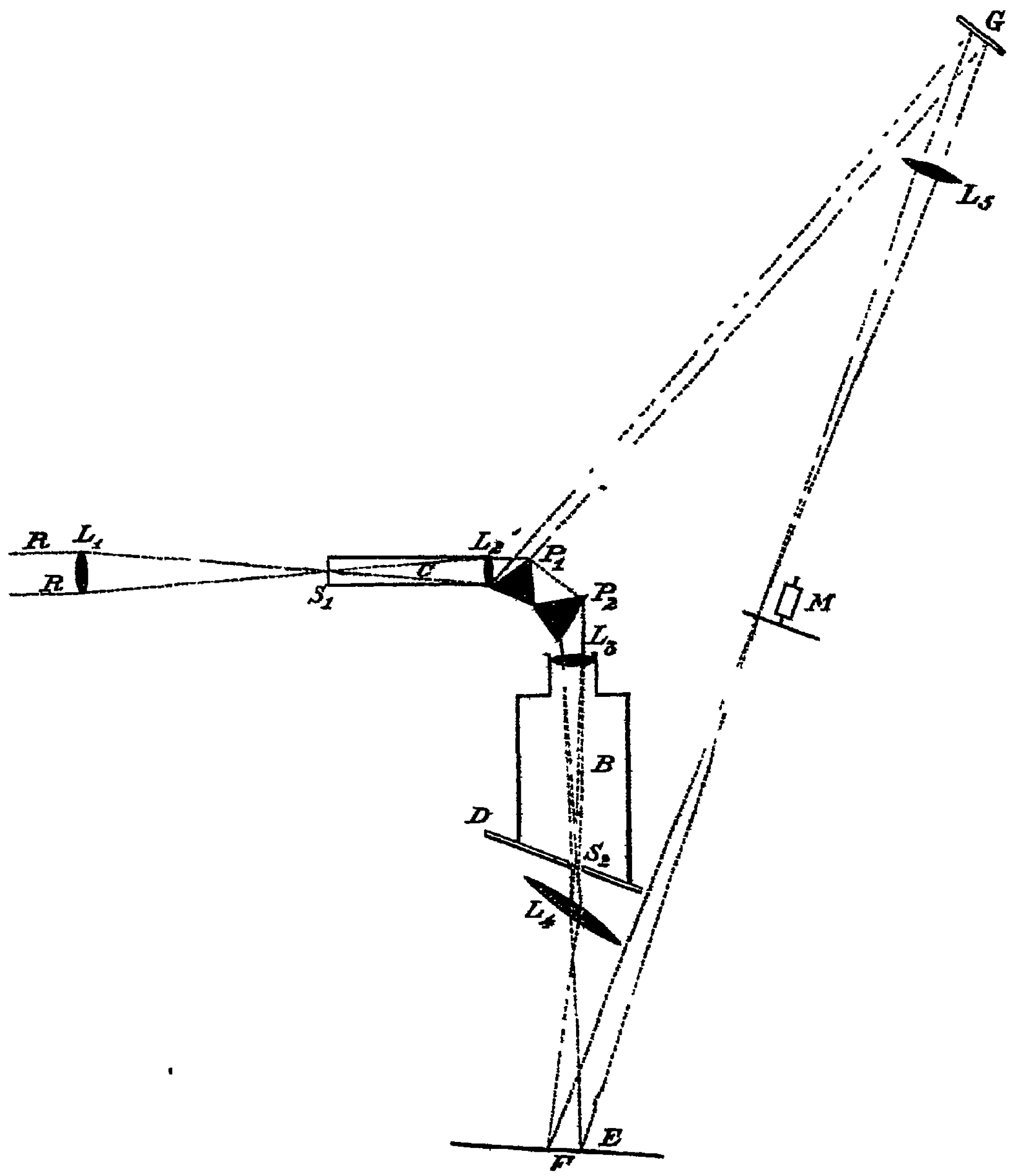
IT will be in the recollection of the Royal Society that early last year General FESTING and myself brought forward a method of colour photometry, which is described in a paper in the 'Phil. Trans.,' 1886.

In a postscript, dated June, we added a description of a somewhat modified apparatus in which any variation in the intensity of the light producing the spectrum under measurement was equally reproduced in the comparison-light

This was effected by causing a part of the light which passed through the slit of the spectroscope to act as the comparison-light. For the sake of easy reference the diagram of the modified apparatus is reproduced in this paper.



R, R, are rays coming from a heliostat, and a solar image is formed by a lens L_1 on the slit S_1 of the collimator C. The parallel rays produced by the lens L_2 are partially refracted and partially reflected. The former pass through the prisms P_1, P_2, and are focussed to form a spectrum by a lens L_3 on D, a moveable screen, in which is a slit S_2. The rays coming through S_2 are collected by a lens L_4 to form a monochromatic image at F of the near surface of the second prism. When D is removed the image of the surface of the prism is white, obtained by tilting the lens L_4 at an angle as shown.

The reflected rays from P_1 fall on G, a silver-on-glass mirror. They are collected by L_5, and form a white image of the prism, also at F.

At M is a small electromotor carrying a disc with moveable sectors, so that any amount of incident light may be cut off.

By this arrangement both the comparison-light and the spectrum itself are formed by the light coming through the slits.

§ I. *Graduation of the Sectors.*

As the intensity of the comparison-light can only be altered by opening or closing the sectors of the rotating discs, it is essential that their graduation shall be good. As a matter of fact, the circles were carefully divided, but the graduation was only regarded as sufficiently accurate when the aperture used was more than 5°, since for very small angles the ratio of any small error in graduation to the aperture used might then not be negligible, which it would be where the aperture was fairly large. As it almost always happened that coloured light had to be measured, which was very much dimmer than the comparison-light diminished by an aperture of 5° in the discs, it became necessary to adopt some other means To accomplish this, one of the two following methods was employed 1st, either the light was absorbed by means of coloured glasses; or 2nd, a mirror of unsilvered glass was used. Both of these plans enabled readings of coloured light of feeble intensity to be measured with apertures of the rotating discs, commencing at 90°. With such modified reflected light it was rarely necessary to close the sectors to less than 5°. The glasses used were red, green, blue, or violet, repeated experiment having shown to General FESTING and myself that it makes no difference in the result what colour the comparison-light may be. In every case the ratio of the diminished intensity to the original intensity of the comparison-light was determined, and the measures of the former reduced to the scale of the latter. It has been almost a surprise to find what great accuracy can be attained in reading by the new method. It is a matter of the greatest ease to read within 1 per cent. of the mean of a series of observations when the source of illumination is the sun; the coloured shadows are so steady that the very slightest movement of the slit across the spectrum is at once perceived.

§ II. *Method of observing with Sunlight.*

During the whole of these observations but one instrument was used, so that they compare absolutely with each other. A silvered surface was used to reflect the sunlight on to the condenser, and the image of the central portion of the sun's disc was focussed on the slit and kept upon it. It became a matter of importance that the mirror should remain untarnished, as General FESTING and myself have found that tarnish materially alters the proportions of the reflected rays. These results we intend to bring forward shortly.

§ III. *Times and Places of Observation.*

From November, 1885, to January, 1887, at my laboratory at South Kensington, on every possible occasion when my official duties would allow, I have taken observations of the colour composition of sunlight. But as those before June 1886 were taken by our first plan, I have not included them here. Where my laboratory is situated, in South Kensington, the E.N E., E, and S.E. are quarters in which the densest parts of commercial London are situated, and consequently, if a wind from any of these quarters prevails, the observations are liable to be marred by smoke; but to the N.N.E, N., N.W, W., and S there is a comparative freedom from any such source of error; and with a wind blowing from those quarters the sky can be as blue as it is in the most uninhabited part of the country. This is important to remark, since it may be thought that the neighbourhood of a big city like London is an unsuitable one for making sunlight observations. In fact, a comparison of results obtained at South Kensington with those obtained in what is termed pure country air, shows that the latter has no perceptible advantage over the former when the wind is in a suitable quarter.

In September, 1886, I took my apparatus to Switzerland, eventually bringing it to my old observing station on the Riffel, Zermatt, at the height of about 8,000 feet. On three different days I made observations under the most favourable circumstances, which will be described later on. In England my observations were usually made at about either 10, 12, 2, or 5 o'clock; sometimes more than one set was taken on each day. I thought it inadvisable that anyone except myself should take readings, in order to avoid any readings which might be difficult of collation.

§ IV. *Length of Time required for a Set of Observations.*

It may be interesting to note the time it takes to complete a whole set of observations. The way I proceeded was as follows:—The adjustment of the standard light was so made that in most cases the full aperture, 90°, of the rotating discs allowed light to pass which nearly coincided with the illuminating value of the rays of

maximum intensity The aperture was then closed to read 80°, then 70°, and so on till 20°, after which every 5° were used till the final opening was itself 5°. The coloured glass was then placed in front of the comparison-light, or the plain glass substituted for the mirror, and the readings recommenced from 90° until 5° was again reached. The light from the spectrum at this point would ordinarily be so feeble that it would be impossible to read rays of feebler luminosity. Three readings of each setting of the sectors were made and the mean taken. This involved thirty-three readings, or sixty-six in all, and the reverse order of the observations was again carried out, *i e.*, commencing with 5° aperture and the absorbed (or partially reflected) light up to 90° aperture, and then from 5° back to 90° with an unabsorbed comparison-light. The time of the commencement of each cycle of readings was noted, and the mean taken as the correct time applicable to the mean of the whole set. Twenty-five minutes generally sufficed to make the double set of measures. For two hours on each side of noon the alteration in the sun's zenith distance during twenty-five minutes is not sufficient to make any very material alteration in the relative proportions of the rays, and the mean of the two sets may be taken to be the reading of all the rays at the mean time of observation.

Later in the day, when the sun's zenith distance rapidly increases, the mean values derived from the cycles of observations may not exactly correspond to the mean time, though, perhaps, not very far from it.

§ V. *Comparison of Results.*

I am not at all disposed to think that observations taken during a whole day are likely, as a rule, to give a true value for the coefficients of transmission of the different rays, more particularly in a climate such as that of England. The atmospheric conditions often vary greatly between the evening and morning, and I have come to the conclusion that by combining observations of the sun at different altitudes, but at approximately the same hour, the minimum values of absorption for each ray are more likely to be correctly determined. As a rule, just the contrary mode of proceeding has been taken. In the determination of atmospheric absorption of stellar or lunar light by BOUGUER, SEIDEL, PRITCHARD, MULLER, PICKERING, and LANGLEY the days' observations have been compared together. It is probable that the atmospheric conditions obtaining at night are more equable than those in the day.

§ VI. *Atmospheric Conditions most suitable at the Time of Observation.*

A hazy day is essentially an unfavourable day for taking such observations as I had in view, and I have only used, in my final result for minimum absorption, observations made on such days as appeared suitable from a meteorological aspect. A still day is usually a slightly hazy day, and I preferred, where possible, to utilise those

days on which there was a breeze blowing from a favourable direction, such as I have already indicated; and, if the sky were partially cloudy, only those days on which the clouds were collected in cumuli, leaving fairly large spaces in the sky of as dark a blue as is obtainable in England.

I would here interpolate that, from the nature of the loss of light by transmission through the atmosphere, it would be impossible to find a sky of that blue-black which is found at high altitudes, the most that can be expected is a deep blue.

Besides the ordinary meteorological observations which were taken at the Museum at South Kensington, I have had the advantage, through the kindness of Mr. Whipple, and by permission of the Meteorological Council, of obtaining complete records from the Kew Observatory for the days I wished to utilise. As this observatory is but a few miles distant from my place of observation; and, as these records are compiled with every accuracy, I have utilised them for my work.

§ VII. *Atmospheric Conditions at the Riffel.*

The morning atmospheric conditions of the Riffel were perfection on each day of my observations, but after 2.30 P.M. they were unsatisfactory. In the mornings the wind was north, a quarter which is well known to Alpine men as a "fine" quarter, and the sun rose with a whiteness of surface which I suppose we can never see at low elevations. The sky was then intensely blue, and improved to a blue-black as the sun gained in altitude. The distant Oberland ranges of mountains were well defined, no visible haze intervening. The shadows on the distant Bietschhorn were black, and the snow-capped summits stood out with almost undimmed whiteness. At Thebes, in Egypt, where I was for three months in 1874–75, I often remarked upon the depth of shadows of the rocks of the Lybian range, distant some three miles from my station; but the shadows in the Oberland Alps, lying some twenty miles or more away, were on this occasion even darker, showing that the haze caused by dust, which is always more or less prevalent in Egypt, was almost absent. The presumable absence of water particles as well as of dust on my days of observation at the Riffel made the atmosphere clearer than it was in Egypt even under the most favourable conditions. I am not stating this only from recollection, but I have records of the fact in photographs which I took at the time at both places. This state of the atmosphere lasted, as I have said, till the afternoon, when a battle for supremacy was waged between the north wind and the wind coming over from the Italian side of the range. The sky then became more or less hazy, and the observations of sunlight were discontinued. The sky at mid-day, as I have said, was of a blue-black, and in fact, with a pocket spectroscope, the spectrum could barely be seen. On a previous occasion, when photographing the sky spectrum at the same place, and, as far as can be judged, under precisely the same conditions, the exposure necessary to give to the plate was at least some seven or eight times that required in England to obtain similar results,

indicating an absence of scattered light, and a consequent increase in direct sunlight. I think, for observations on visible radiation, such a locality as I selected in the Alps is preferable to one of equal altitude in a warmer climate. In Professor LANGLEY'S report* on his observation at Mount Whitney, he describes the atmosphere below him as being filled with dust, his lower observing station lying in it. Now in the Alps, on days such as I have described, this dust haze is absent. I have looked down some 10,000 feet and failed to discover it. For these observations perfect dryness of air is not a necessity, though undoubtedly for observations of the dark rays and extreme red rays it must be; aqueous vapour, except it be present in very large quantities, does not affect the visible radiation from the sun at these high altitudes. When pointing a pocket spectroscope at a distant horizon, I have failed to see any of the rain-bands even when the atmosphere was notoriously damp. The only time at an elevation of 8,000 feet when I have seen the rain-band has been when rain has been freely falling. I am not asserting that the rain-bands are never present except under such circumstances, but only that I have not seen them. It has been necessary to be somewhat prolix in describing the atmospheric conditions at the Riffel, as my standard solar spectrum is derived from my observations taken on September 15th, 1886, at noon.

§ VIII. *Law of Diminution of Light.*

The observations carried on throughout the year were undertaken more to obtain a meteorological record than for any other purpose, and it was not till November last that any comparison of the curves I had plotted was undertaken, nor did I opine that any set law governed the absorption of the different rays. LANGLEY'S results, contained in the volume I have already quoted, rather forbade the idea that any exact or even approximate law prevailed, since certainly his results gave no clue to any. The plotting of the observations was made on squared paper, and through the points thus obtained a smooth curve was drawn by hand; and it will be seen that all the observations lie very close indeed to the curve so drawn. From the curves the value of illumination at each unit of my scale was noted and tabulated. From such tabulation it became easy to try whether any particular law governed the loss of light by different rays after transmission through a thickness of air. Thus the intensities of visible radiation, as measured at the Riffel on September 15th, could be compared with those at South Kensington on July 1st.

A first trial of Lord RAYLEIGH'S theoretical law for loss of light caused by the scattering of small particles, $I' = I\epsilon^{-kx\lambda^{-4}}$ (where I' and I are the transmitted and original intensities, k a constant, x thickness, and λ the wave-length of any ray), [illegible]

* 'Professional Papers of the Signal Service (U.S.A.),' No. XV., "Researches on Solar Heat and its Absorption by the Earth's Atmosphere," LANGLEY.

with the observations at these two dates, showed that the observed values could be accounted for by the theory that the loss at the station of lower altitude was due solely to the suspension of fine particles in the atmosphere. Other days' observations in England were compared in the same way, and in very nearly every case the above law held good, kx, of course, varying as the zenith distance of the sun on the different days varied. A comparison was then made *inter se* of the different days' observations, and the above law, as it should, still held good. Now, on the different days in England on which the observations were made, the mean time of observation was known, and consequently the air-thickness The air-thicknesses were reduced to a uniform barometric pressure of thirty inches of mercury for one atmosphere.

§ IX. *Conditions of Comparison between the Riffel and South Kensington.*

Evidently the air-thickness at mid-day on the 15th September at the Riffel could not be compared with the air-thicknesses at sea-level, as it by no means followed that the atmospheric conditions were the same, *i e.*, that the scattering particles assumed by the above theory were as abundant at an altitude of 8,000 feet or more as in London at sea-level in a unit volume It should be remarked that this does not affect the value of the comparison of the observations made between the two places, since the law would equally apply, were the particles per unit volume more or less The reduction of the observations showed that the particles were fewer in number, which is equivalent to observing sunlight through an atmosphere less than unity at sea-level, though the absolute air-thickness was slightly greater than unity. It is manifest that there is very great utility in making the illuminating value of the spectrum at the Riffel the standard with which to compare the values obtained with greater air-thicknesses at sea-level; since the differences in the proportions of the different transmitted rays are very much accentuated, and any slight errors in observation, which would mask the results when the differences between the observed air-thicknesses are small, are eliminated.

§ X. *Minimum Loss of Light.*

For the purpose of obtaining the minimum loss of light, I have selected observations made on seven days, all of which are compared with the Riffel observations. The atmospheric conditions were favourable, as will be seen by the meteorological Tables annexed. These days are divided into two groups of three, leaving one odd day. The first three days are June 4th, July 5th, and July 21st, when the observations were made with a thickness of about 1·3 atmosphere. The second group comprises observations 29th October, 4th November, and 18th November, when equally satisfactory atmospheric conditions existed, with thickness of about 3·3 atmospheres. The seventh day is 14th October, when the thickness of atmosphere at the time of

observation was about 2 atmospheres. From the first group a mean of the air-thicknesses is taken, and also the mean coefficient, and the second group is treated in the same way. Thus

	Air-thickness	k
June 4.	1 265	00190
July 5.	1·373	00211
July 21	1 295	00220
Means	1·311	00207

	Air-thickness	k
October 29	3 404	·00472
November 4	3·428	·00500
November 18	3·160	·00412
Means	3·331	·00461

To ascertain the coefficient for one atmosphere, we use the formula

$$\frac{x' - x}{z' - z} = \mu,$$

where x' and x are the coefficients for the two groups, and z' and z the air-thicknesses. In the case before us we thus have

$$\mu = \frac{x' - x}{z' - z} = \frac{\cdot 00461 - \cdot 00207}{3\cdot 331 - 1\cdot 311} = \cdot 00126\,;$$

that is, the formula

$$\text{I}' = \text{I}\epsilon^{-kx\lambda^{-4}}$$

becomes

$$\text{I}' = \text{I}\epsilon^{-\cdot 00126\lambda^{-4}x},$$

x being air-thickness in terms of one atmosphere at sea-level.

For October 14th the air-thickness is 1·973, and the coefficient is ·00284; according to the above formula, the last should be ·00249.

Combining October 14th with the second group, we have $\mu = \cdot 00130$, which is not very different from that derived from Groups 1 and 2.

Now, the accuracy of reading may vary about 2 per cent. in the yellow of the spectrum, and perhaps more in the blue, and it is believed that greater accuracy at

present is not obtainable than the figure in the 4th place of decimals. The adopted minimum loss of light from scattering is therefore represented by $I' = I\epsilon^{-0013x\lambda^{-4}}$.

Taking the mean of a large number of days, I find that the average value of κ is 0017.

§ XI *Langley's Coefficients of Absorption.*

Turning to the work which LANGLEY has published in the volume already alluded to, I find no single instance of coincidence with the above law. For instance, at page 151 he gives a Table of coefficients of transmission, which are as follows in the visible spectrum:—

λ	3580	3830	4160	4400	4680	5500	6150
Coefficient of transmission for one atmosphere	449	531	600	636	·677	·734	·781

Taking λ 6150, which agrees nearly with my scale number 47, and λ 4400, which agrees nearly with my scale number 57, it will be seen that the coefficient of absorption increases nearly by λ^{-2} instead of by λ^{-4}. How this wide discrepancy arises, I am at a loss to understand There is one point, however, to be remembered, which is in favour of the observations I have recorded, that within certain limits I was quite independent of any small haze which might have passed between the slit of the collimator and the sun, as the spectrum was compared with one portion of the light coming through the slit, whilst another portion formed the spectrum itself. Now any small amount of white haze would probably give a general absorption without in any large degree altering the *relative proportions* of the component rays, whilst the bolometer readings would be affected considerably, but this would not be sufficient to account for the systematic and great differences between our two results. A glance at the curve of illumination of the solar spectrum will show that, as far as λ 6500 at least, there can be no difficulty in making visual measures. With the width of slits used, the light coming through scale number 47, a candle had to be placed but 5 inches off the screen on a bright summer day at noon to balance the illuminated shadows; and, as the colour of the candle-light and that of the D-light is very nearly of the same tint, no difficulty could be found in judging of the value of the latter according to the old method,* and this gave identical results with the new method adopted, and described at the beginning of this paper. For that reason I have selected LANGLEY'S coefficients of transmission of λ 6150 to compare with mine, as no physiological objections can be brought against such comparison.

The following in Column II. are the coefficients he tabulates for a stratum of air equivalent to a column of 1 decim. of mercury for this wave-length —

* "Colour Photometry," 'Phil Trans.,' 1886

I	II	III	I	II	III
February 15	963	·753	May 2	·974	819
March 23 . .	971	799	May 29 . .	·973	·813
,, .	960	733	,, . .	·976	·831
,,	967	775	June 22 . . .	979	·850
,,	961	·740	September 12	·958	·722
,,	968	·781	,, .	·959	·728
March 29 .	·945	·651	September 15 .	·978	·844
,,	·9*9	. .	,, .	·976	831
March 31	930	·578	,,	·975	·826
April 24	·975	826	November 25 .	967	·775
,, . .	976	831			
May 1 . .	961	740			

Now the mean of these numbers in Column II., which are LANGLEY'S measures, gives his adopted coefficient of absorption as ·968 for an air-thickness, equivalent to 1 dcm. of mercury, and the coefficient of transmission for one atmosphere (760 mm. of mercury) as 781. The numbers in Column II. are apparently very concordant, but the coefficient of absorption, it must be recollected, is only for a thickness of atmosphere of 1 dcm. of mercury, and the differences are much more apparent if we reduce them all to coefficients for one atmosphere, as we have done in Column III. For instance, if we take the highest and lowest values in Column II., which are ·930 on the 31st March and ·979 on the 22nd June, and reduce them to the coefficients for one atmosphere, we find that they are ·850 and ·578 respectively, and the mean of the series is ·774. Similarly at page 25, Table VI., of LANGLEY'S work the highest and lowest coefficients of absorption for an atmosphere equivalent to 1 dcm. of mercury are ·980 and ·872, which at one thickness of atmosphere are ·858 and ·351.

Again, in the Table p. 151 for λ 440, the highest and lowest coefficients for an atmosphere equivalent to 1 dcm. of mercury are ·974 and ·907, which at one atmosphere give coefficients of ·819 and ·476, whilst the lowest but one, which is ·935, gives ·601 for one atmosphere.

These results have been brought forward to show that in the limits of visibility of the spectrum there are wide discrepancies in the coefficients of transmission, as might be expected, in what may be called uncompensated readings, and it appears that by adopting some method analogous to that which General FESTING and myself adopted in our apparatus—of balancing one side of the bolometer with the spectrum and the other with a fraction of total sunlight—perhaps such great differences in the coefficients might be eliminated. It is at first sight somewhat misleading to tabulate coefficients of such small thickness of atmosphere, since in every case the apparent deviation from the mean will appear slight.

* This figure is obliterated in my copy.

§ XII. *Loss of Light other than by Scattering.*

The question arises if there is a loss of light from any other source than the scattering by small particles, and an answer is at once furnished by every-day observation. We know that it is; that a hazy day diminishes sunlight, though often without materially altering the ratio of the components of the light Now light, *quâ* light, I did not measure on all these occasions, but here the value of the measures taken by means of the candle comes in, assuming that the candle is invariable or only varies slightly. Mr. VERNON HARCOURT tested a candle such as we employ, and got from it a result more satisfactory than with a standard candle; and, as the same brand of candle is always employed, it is believed that the same measurements hold good.

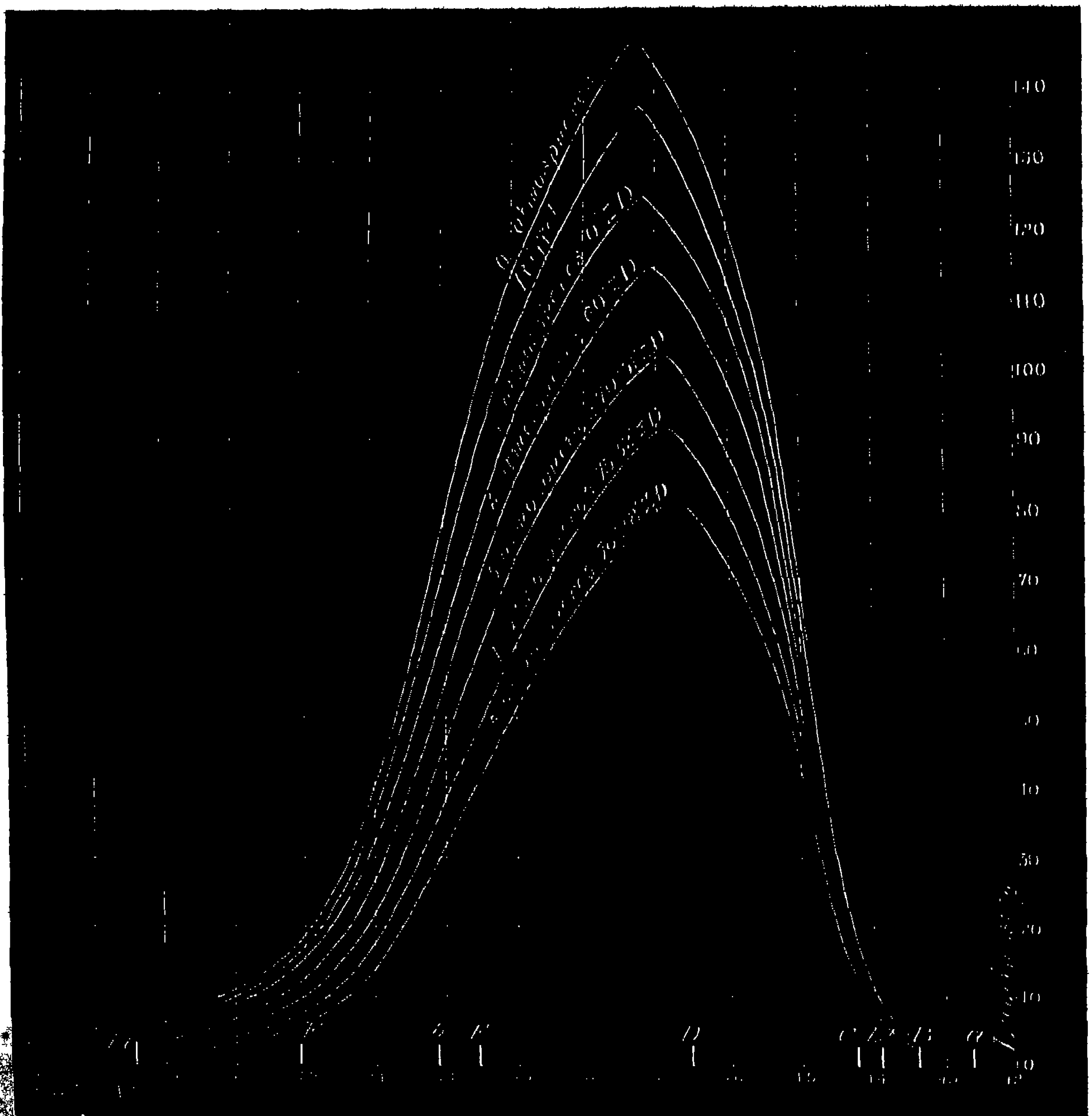
With the instrument unchanged in any respect, the slits having the same width in both cases, an estimation of the total illuminating value of sunlight can be at once ascertained. It happens that on two days which I have selected, viz., 4th June and 29th October, what I may call the candle-value of the spectrum was ascertained. Now the illuminating value of the spectrum, taken by means of the candle, as already has been said, is not such a satisfactory method as that I have latterly employed, and on which the foregoing measures have been based; but an average value for k can be readily ascertained. On the 4th June, about a quarter of an hour later than the time when the printed observations were made, the value of the ray at 47 on my scale was 80, and for the 29th October, also about a quarter of an hour later, it was 62, or 1 to ·775. The value for this ray at the adopted times of observation on the above days is 114·3 and 89·0 respectively, or 1 to ·778. This shows that on these two days any loss of light, except that due to scattering, was very small. On some other days, however, I adopted a different plan, which I had carried out, not for the purpose of applying them to these results, but for estimating the photographic values of skylight and sunlight. Fortunately, also, these measures were carried on at the Riffel: observations of no small value, as it proves.

§ XIII. *Loss of Total Light by Transmission through the Atmosphere.*

Before entering into this more fully, something must be said as to the value of total sunlight as derived from my observations, assuming for the present that there is no diminution of light except from the scattering by the small particles. Having obtained the coefficient of transmission per atmosphere for each ray, it is easy to construct curves representing the luminosity for every air-thickness, and, having so constructed them, to find the value of the areas of each. These areas then represent the values of total illumination which would be observed, were the total light under measurement, as General FESTING and myself have shown in the paper before quoted. Having found the areas, it is easy to see if any law holds good connecting total light

and air-thickness. This latter problem is one which astronomers have long studied, and their researches point out that, if I′ and I be the values of the light before and after traversing an air-thickness, θ be the zenith distance, and α the constant of transmission, then $I' = I\alpha^{-\sec\theta}$.

Diagram 1.



Owing to refraction, the formula $\alpha^{-\sec\theta}$ cannot apply accurately where θ is very large. This was pointed out by BOUGUER, who calculated the variation of the thickness of the atmosphere traversed at different zenith distances, allowing for refraction. This calculation was subsequently verified by Professor FORBES, and the results are tabulated in his BAKERIAN Lecture "On the Transparency of the Atmosphere and the Law of Extinction of the Solar Rays in passing through it," which appeared in 'Phil. Trans.,' 1842, Part II. Since this volume may be difficult to refer to, I quote part of his Table to show these variations.

Z.D	Sec Z D	Forbes' Value	Bouguer's Value
0	1 0000	1 0000	1 0000
10	1 0154	1 0164	1 0153
20	1 0642	1 0651	1 0642
30	1 1547	1·1556	1 1547
40	1 3054	1 3062	1.3050
50	1 5557	1 5550	1 5561
60	2 0000	1 9954	1 9903
70	2 9238	2 9023	2 8998
75	3 8637	3 8087	3 8046
80	5 7588	5 5711	5 5600
82 30	7 6613	7·2343	—
85	11 4737	10 2165	10 2002
86	14 3356	12 1512	12 1401
88	28 6537	18 8825	19 0307
90	Infinite	35 5034	35 4955

Astronomers have made various estimations of the value of what they term absorption, and perhaps no astronomical problem has received more attention than this one. The determination of the coefficient of absorption is a necessary preliminary for ascertaining star magnitudes, and thus has an importance peculiarly its own Professor LANGLEY, to whose work I shall presently have to refer, made two estimations, one on Etna at 4,000 feet elevation, and another at Mount Whitney, at a still higher altitude. At both of these localities he found a value for the coefficient of transmission to be ·88, though at the latter station he disclaims any great accuracy as likely, which is indeed the case, considering the method he caused to be employed. The other values obtained were as follows :—

PRITCHARD at Cairo and at Oxford, ·843 and ·791 respectively; BOUGUER ·812; SEIDEL ·794, and MULLER ·825 : or a mean of ·804 at low-level stations.

Professor LANGLEY, assuming from his observations with the bolometer, regards these results as being liable to error. He says, "For be it observed in general terms that, since the rays with large coefficients are represented by diminishing geometric progressions, whose common ratio is near unity, these rays will persist, whilst others with small coefficients are early extinguished. . . . . But what we desire now further to point out is that, according as the difference of these coefficients of transmission for the different portions of the light of the same star is greater, so will the error of the result in treating them as equal be larger : a consequence so obvious that it is only necessary to make the statement in order to have its truth recognised.

"Since it has now been demonstrated that the formula ordinarily employed leads to too small results, it might properly be left to those who still employ it to show that their error is negligible, but this has never been done. There is possibly an impression that if there were any considerable error its results would become apparent in such numerous observations as have been made all over the world in stellar photo-

metry during this century. But it is, in my opinion, a fallacy to think so; and I believe, as I have elsewhere tried to show, that the error *might* be enormous—that the actual absorption *might* be twice what it is customarily taken, or 40 per cent. instead of 20 per cent., without the errors being detected by such observations as are now made."

It will be found from my observations, and also from those of Professor LANGLEY himself, that the error made by astronomers in not taking into account the different coefficients of absorption of the different rays is negligible.

I will first of all take a value which was derived for atmospheric absorption in which k was ·001183. (Be it remembered, this is not one I adopt, but it was a value obtained by certain combinations.) The areas obtained for the curves of illumination of 0, 1, 2, 3, 4 atmospheres, and which would have been the values observed by any integration method, were as follows:—

740, 657, 572, 505, 441.

Now, as the least atmosphere through which an observer can observe at sea-level is 1, we may take 1 and 4 atmospheres as lying on the true curve and calculate the others from them, using the formula $I' = I\epsilon^{-\mu x}$, where μ is the coefficient of absorption and x the thickness in atmospheres. We find

$$\log 657 = -\mu + \log I$$
$$\text{and } \log 441 = -4\mu + \log I,$$
$$\text{from which } \mu = \cdot 1324.$$

The calculated values for 0, 2, and 3 atmospheres are then 749, 573, and 503 respectively, values which might be said to be identical with the above.

It may be thought that it is owing to a happy accident that these numbers are so close. If we take the value of the coefficient $k = \cdot 0015$, we find that the values of the areas for 0, 1, 2, 3, and 4 atmospheres are—

730, 625, 534, 459, and 396

respectively. Using the logarithmic formula, we find that the value of μ is ·1529 and the values of light passing through the above thicknesses are—

730, 627, 538, 461, and 396

respectively. In this case, if a be the coefficient of transmission,

$$a^x = \epsilon^{-\mu x},$$

and

$$a = \cdot 858.$$

Once more we may take the coefficient of scattering as $k = \cdot 0019$, and we find that areas of the curves for 0, 1, 2, 3, and 4 atmospheres are—

930, 755, 623, 513, and 418.

The values derived from the formula are as follows—

917, 755, 620, 508, and 418.

This gives a, the coefficient of transmission, $= \cdot 822$. It may be imagined that the same result would not obtain if the coefficients of absorption for the different rays were other than those obtained from Lord RAYLEIGH'S law. We have at hand values of the atmospheric absorption for the different rays which Professor LANGLEY has adopted. They are to be found in the volume to which we have referred.

For the purpose in view, I have assumed that my estimation of the illuminating value for one atmosphere with a coefficient of ·001183 holds good, and constructed curves with the coefficients of absorption which he gives, and then taken the areas as giving the value of total illumination at different thicknesses. The values are, for 0, 1, 2, 3, and 4 atmospheres—

1,000, 657, 393, 236, 142,

and the values obtained by using the logarithmic formula are—

1009, 657, 394, 237, and 142,

a sufficiently close agreement to need no comment; but it should be remarked that, as $\mu = \cdot 5109$, the coefficient of transmission $a = 6$ nearly, a value far lower than has been found by astronomers.

Finally, I give the results by the above method, which are obtained from the areas of the curves derived from the minimum coefficient of absorption ($k = \cdot 0013$), which are as follows for 0, 1, 2, 3, and 4 atmospheres, viz. :—

762, 662, 578, 504, 439.

Taking 762 and 439 as points on the curve $I\epsilon^{-\mu x} = I'$, we get the following numbers ($\mu = \cdot 1385$)—

762, 664, 578, 504, 439;

which gives the coefficient of transmission $= \cdot 869$, which, it will be seen, is higher than any value assigned to the coefficient of atmospheric transmission that has been obtained by astronomers; but to this I shall refer presently.

§ XIV. *Observations of Total Light by Colour-blind Persons.*

It was interesting to ascertain what would be the effect of the observations of total light by a colour-blind person as regards accuracy of result in comparison with a normal-eyed person.

To ascertain this, it was supposed that the observer R* had taken the luminosity observations, and the curves for the sunlight were reduced proportionally by the amount by which the normal curve of the electric light was shown to be reduced by this observer—an inadmissible assumption, as General FESTING and myself have shown. The curves so reduced were plotted out, and the areas taken.

It was found that, when $k = \cdot 0013$, for 0, 1, 2, 3, and 4 atmospheres the areas were

532, 455, 389, 334, 287.

* See BAKERIAN LECTURE, 1886

Taking 532 and 287 as lying on the curve derived from $I' = I\epsilon^{-\mu x}$, the values obtained were as follows—

532, 456, 391, 335, 287;

and the value for the coefficient of transmission was $= \cdot 862$, a value somewhat lower than that obtained for the normal-eyed person, though not very far from it.

I think in the foregoing enough has been adduced to show that, whether the loss of intensity by different rays in passing through the atmosphere obeys Lord RAYLEIGH's law or follows the observations made by LANGLEY, astronomers have been quite warranted in using the logarithmic law, at all events in observations made to 75° zenith distance.

§ XV. *Photographic Values of the Integral Value of Light through different Air-Thicknesses.*

We thus see that as far as light, *quâ* light, is concerned the method of observing the integration of the different wave-lengths gives results concordant with those obtained by treating the light analytically. The point that next presented itself was as to whether the photographic values of radiation when treated in the same manner would give a similarly satisfactory result. If so, then the photographic values of sunlight which I had obtained could be applied to the question of total value of sunlight on any particular day.

In the 'Proceedings of the Royal Society' I have shown the photographic sensibility of a particular salt of silver for the different rays of the spectrum; and, knowing the day and hour on which that value was determined, and assuming the minimum value of the coefficient for scattering as applicable, after constructing the necessary curves and taking the areas, the following results were obtained at 1, 2, 3, and 4 atmospheres. The values of areas of the curves were—

621, 457, 340, 253,

and those derived from using the formula $I' = I\epsilon^{-\mu x}$, μ being $\cdot 2993$ and a $\cdot 74$, were—

621, 461, 342, and 253.

The maximum value of the spectrum on this salt of silver lies well in blue, and it was thought that a theoretical consideration of the value when the photograph was taken on a salt of silver which had a maximum in the extreme violet would be useful, since such a salt—the chloride—is used in ROSCOE's actinometer. The spectrum value of this salt had been previously carefully taken by myself, and the values were applied to each unit of my scale. The values obtained for 0, 1, 2, 3, and 4 atmospheres from the curves were as follow—

690, 413, 246, 147, 87;

the values from the formula $I' = I\epsilon^{-\mu x}$, μ being ·5192 and a 595, were—

689, 413, 248, 148, 87;

again a sufficiently close value to show that it may be safely used.

§ XVI. *Experimental Photographic and Optical Tests of the Logarithmic Formula.*

A great many tests with turbid water were made to ascertain if the same held good experimentally The same photographic preparation as that already referred to was used. A cell 6 inches long and 4 inches wide was employed, and different portions of the same photographic plate were used for obtaining impressions of the light acting. The light was allowed to fall through the cell, containing clear water, on an isolated portion of the plate $\frac{3}{4}$-inch square. Eight different exposures were given to various portions of the plate to form a scale of density of deposit. Exposures were given to the other parts of the same plate to light from a constant source passing through the 4-inch and the 6-inch thicknesses of clear and turbid water. The plate was then developed with ferrous oxalate,* fixed, and dried. The density of deposit of each square was next measured by a plan which I have described in the 'Photographic Journal.'

§ XVII. *Mode of measuring the Density of Deposit on a Photographic Plate.*

From the paper in question I have made the following extract.—

"The light, whatever it may be, is placed at A; a lens L_I at distance of its equivalent focus, in this case 9 inches The negative, N, is placed in front of this, and another lens, L_{II}, throws the image on the screen S, in front of which is a rod, R, whose shadow is cast by the light coming through L_{II}.

"This is the ordinary optical lantern form of apparatus. At one side, at a convenient distance, I place a mirror, M, with the angle so adjusted in azimuth that it reflects the light from A over the patch illuminated by the lens L_{II}. This naturally throws another shadow of the rod alongside the first shadow If desirable, I can place the other lenses, L_{III} and L_{IV}, the latter forming the image of L_{III}, upon the first patch of light. (As a rule, these last two lenses are unnecessary) The screen, S, may be transparent

* This developer was used for convenience, as it gave a black deposit, which was useful for subsequent measures

or opaque, whichever is deemed best Where the shadows of the rod fall, I cut out a square mask to enable me to view the two shades without distraction to my eyes by glare from adjacent parts. It will be seen that, as the light and the reflected beam are stationary, the method of varying distance cannot be adopted to vary the intensity of the light. To obtain the necessary variation, I employ revolving sectors. These sectors, being connected with a small electro-motor wheel, work with four Grove's cells The aperture of the sectors can be increased or diminished during motion by a simple arrangement. This is an admirable plan of graduating light, and answers for all purposes of the sort. It will now be apparent that, should the light vary, the results will not be vitiated in the least, since the original light is made to act as the comparison-light as well."

By this arrangement a "density curve" for the exposures given to the plate through the clear water was constructed, and the values of the other exposures through the turbid medium, in terms of the exposure through the clear water, were determined. A screen with squares of different translucencies, which had been carefully measured, was placed in front of the plate.

The following is an example .—

6-inch turbid water	
Opacity of screen.	Equivalent exposure through clear cell
39 5 29 0 22 0	= 9 = 6 5 = 5 0
90 5	20 5

Or it required 4·415 times the exposure through the 6-in. turbid cell to be equivalent to a unit exposure through the clear cell.

4-inch turbid water	
Opacity of screen.	Equivalent exposure through clear cell
39·5 29 0 22·0	= 14 = 11·5 = 8 0
90 5	33 5

Or it required 2·70 times the exposure through the 4-in. turbid cell to be equivalent to a unit exposure through the clear cell.

In the above example, for convenience sake, the clear cell was exposed for two minutes through rotating sectors having an aperture of 30°.

The 4-inch turbid cell was exposed for five minutes through rotating sectors having an aperture of 45°.

The 6-inch turbid cell was exposed for five minutes through rotating sectors having an aperture of 90°

Hence the above values have to be multiplied by 7·50 and 3·75 respectively, and we find the relative values of a unit exposure to be—

Clear cell	4-inch turbid	6-inch turbid
1	$\frac{1}{10\cdot125}$	$\frac{1}{38\cdot1}$

Using the logarithmic formula, we get, taking the clear cell and the 6-in. turbid cell as giving points in the curve,

$$\mu = \cdot5832;$$

and the value for the 4-in. cell becomes 1/10·3, which is sufficiently close to 1/10·125 to show that it holds good.

The optical values were also taken by means of the rotating sectors, and found to be—

Clear cell	4-inch turbid water.	6-inch turbid water
75·5	26·5	15·5

Using the first and last as giving points lying in the curve derived from the logarithmic formula, we get $\mu' = \cdot2639$, and find the values to be 75·5, 26·3, and 15·5; a coincidence which is nearly exact.

Now we find that

$$\frac{\mu}{\mu'} = \frac{5832}{2639} = 2\cdot21;$$

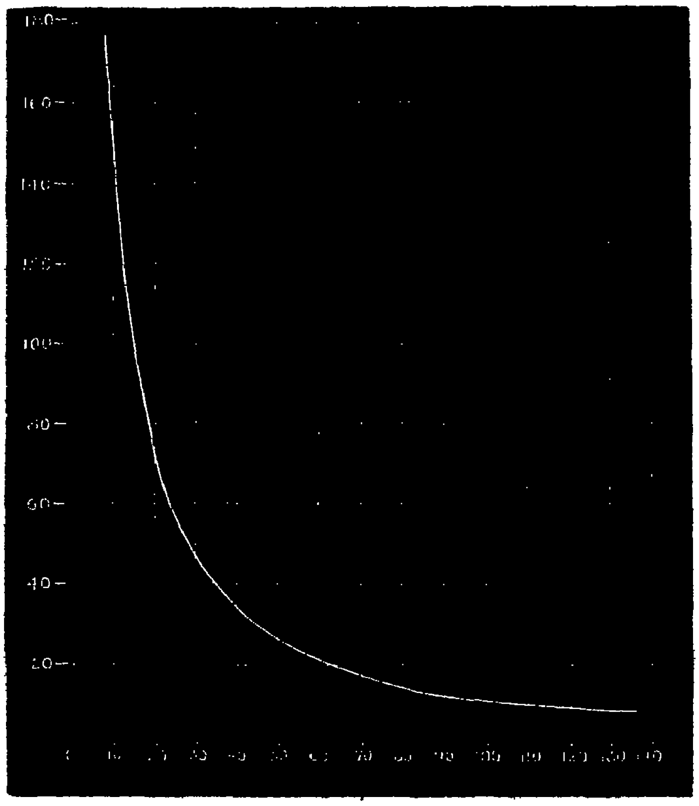
and if we compare the value of the optical measure of the areas of the sun curves with the measure of the photographically derived curves in the same way in the results we have given, it will be found that the factor μ'/μ is very nearly the same as the above. This close approximation leaves no doubt that the logarithmic formula is sufficiently exact to be employed.

Another example may be quoted, which will, with a diagram, still further explain the mode adopted. An exposure to lamp-light was given to portions of plates in succession by means of the rotating disc, for 15 seconds each exposure. The apertures were as follows, and the densities, measured as described, are placed in juxtaposition :—

Aperture	Relative transparency of negative
$8\frac{1}{2}$°	177
$18\frac{1}{2}$	83
$28\frac{1}{2}$	$51\frac{1}{2}$
$38\frac{1}{2}$	$34\frac{1}{2}$
$48\frac{1}{2}$	27
$58\frac{1}{2}$	22
$71\frac{1}{2}$	$16\frac{1}{2}$
$86\frac{1}{2}$	$12\frac{1}{2}$
$98\frac{1}{2}$	$11\frac{3}{4}$
138	$7\frac{1}{2}$
180	$5\frac{1}{2}$

Diagram 2 shows this plotted.

Diagram 2



Photographs were then taken through a clear-water cell, through 3·069-in. turbid water, and through 1·759-in. the same water, the light being admitted through varying apertures of the rotating disc. The results are as follows .—

CLEAR Cell; 20 sec. exposure.

Aperture.	Density	Equivalent on above scale	
48½ 38½ 28½ 18½	30½ 42 59 96	42 33 25 17	Therefore 1 = ·865. Reduced to standard of 1 min exposure, 1 = 2 595
134	.	117	

1¾-in. turbid; 1 min. exposure.

Aperture.	Density.	Equivalent on above scale.	
17½ 28½ 58½ 86½	104 60½ 29 18½	14 23 45 70	Therefore 1 = 795.
191	..	152	

3-in. turbid ; 3 min. exposure.

Aperture	Density	Equivalent on above scale	
$17\frac{1}{2}$ $28\frac{1}{2}$ $58\frac{1}{2}$ $86\frac{1}{2}$	84 51 $22\frac{1}{2}$ $13\frac{1}{2}$	18 28 59 84	Therefore 1 = ·984 Reduced to 1 min exposure, 1 = 328
191	.	189	

Using the first and last for points in the logarithmic curves, we get the observed values

2·595, ·795, and ·328 ;

calculated 2·595, ·786, and ·328 ;

$\mu =$ ·677.

The optical values were next observed, from which $\mu' = 306$ when $\mu/\mu' = 2{\cdot}21$, which agrees with the foregoing example.

§ XVIII. *The Measurement of the Photographic and Optical Values of Total Intensity equivalent to the Measurement of a Single Ray.*

A remarkable deduction now presents itself from the fact that, if we divide μ and μ' by k in the results given by plotting the areas, we find that the results are numbers which are about 105 and 235, and these represent wave-lengths 5570 and 4540 respectively, so that, if we observe the total value of light optically, it is equivalent to observing monochromatic light of λ 5570, and if we use bromoiodide of silver for registering the intensity it is equivalent to measuring a ray of λ 4540.

We may apply any of the results obtained by astronomers to find the value of k.

	Coefficient of atmospheric transmission	μ	k
LANGLEY (on Etna) .	880	1·274	00122
*PRITCHARD (at Cairo) . . .	·843	1·704	00164
,, (at Oxford) . .	791	2 333	·00224
BOUGUER	812	2 070	·00199
SEIDEL and PICKERING	794	2·311	00222
MÜLLER . . .	·825	1 924	00185

With these factors we may construct the curves of illumination for any air-thickness

* Professor PRITCHARD'S maximum value, as far as I have calculated it, is very close to mine, viz , 860.

It should be remarked that the methods adopted in determining the absorption by astronomers practically eliminated any variation due to haze, since the stars were compared *inter se*, which is what the method employed in the foregoing observations also does, though in a more complete manner

The above plan of obtaining the coefficient of k leads to another very important result, which is that on any day, by taking an optical measure of the *total* light with any standard, such as a candle, and also a photographic measure of the same, we can fix the coefficient k to be applied to each wave-length, and also the loss by general absorption, with very great exactness, as two equations are formed from which each can be deduced.

I would point out that even a simpler method is to expose a sensitive surface to rays coming through an orange medium and also through a blue medium, and measure the relative densities or blackness of the resulting photographs. The exposure through the orange medium will be equivalent to an optical measure.

It must be recollected, however, that each different sensitive photographic compound will give results for different parts of the spectrum.

§ XIX. *Application of the above Law for the Comparison of the Actual Variations in Sunlight.*

Now on various days exposures were made on sensitive paper during the time of observation, and the depth of density measured, and from these measures were derived the intensity of the light acting If the loss of light is alone caused by the scattering of small particles, and from no other cause, then the values of the ordinates of the different curves belonging to the days on which simultaneous observations were taken ought, at scale number 56, to be proportional to the values obtained from the photographs, the times being those given in the Tables. At the Riffel the photographic value on September 15th was 256; at South Kensington on July 21st, 172; on October 29th, 72; on November 4th, 72. The values of scale number 56, taken from the curves, are as follows:—Riffel, 2·45; 21st July, 1·74; October 29th, ·73; November 4th, ·7; which are fairly in accord. On the other hand, there are values, as for the 8th November and 23rd December, which are not concordant, showing that there was a general absorption of light as well as a scattering in its passage through the atmosphere.

§ XX. *Considerations as to the Amount of Scattering.*

We are bound to ask ourselves what is the cause of the different coefficients for the scattering of light in its passage through the atmosphere. It must be recollected that small particles of any kind will suffice for the purpose, and that it is merely a question of quantity which determines the coefficient. Greater or less amount of dust will affect the question, and dry weather is not the weather in which, from this cause, the scattering should be least. Again, small particles of water must always be more or less present, and it is believed by the writer that these are the most active

causes in diminishing the amount of transmitted light. Lord RAYLEIGH has shown that the sizes of the particles have a considerable effect in the scattering, and when we have white mist the scattering of light must be general rather than selective But it may happen that the general aggregate of particles present may be of sizes which to a greater or less degree refuse to scatter light of wave-lengths not taken between certain limits. Or the sizes of the particles may be so varied that, whilst one gives a limit of scattering for one certain wave-length, another may give a limit for another, and so on till the final outcome may be to give a loss of light not exactly varying as λ^{-4} This, it appears to me, may be the meaning of this law being obeyed on days which are perceptibly misty, as in some November and December days, and in which the integration method by photography is not in accord with the optical method adopted. On the whole, I am inclined to think that on fine days, near mid-day, with pure blue sky, the water particles are present in numbers, and, if dust be fairly absent, that then we get the minimum loss. At night, when the temperature is diminished, the water vapour probably condenses to give a larger number of small water particles, and hence star observations give a greater value for the coefficient of transmission than I have obtained for the minimum, though the mean value they have deduced is not far from the value I obtain when the coefficient kx is equal to ·0019, which is a value that on several occasions I obtained. That these water particles have much to say to the coefficient of transmission is shown by PRITCHARD's determinations at Oxford and Cairo respectively; the former gave a coefficient of ·791 and the latter ·841

§ XXI *Deductions from the Riffel Observations.*

It would be premature to deduce too much from the observations taken at the Riffel. It will be seen that the air-thickness at the Riffel at noon on the day observed is equivalent to 1 atmosphere at sea-level, and that our equivalent value is really a good deal less than that thickness. This means that there are at higher altitudes proportionally fewer particles to scatter the light than at sea-level. I refrain from giving the values I obtained near sunset at the same place, but the value of absorption I found to be startlingly smaller, so much so that my results must be repeated before I can vouch for the deductions to be made. It seems to me that in the Alps we have the most favourable conditions for studying the atmospheric permeability for light, owing, in proper seasons of the year, to the absence of dust. It should be pointed out that the radiations which act on our eyes as light are less absorbed by aqueous vapour than are those radiations which lie in the infra-red, and that it by no means follows that, if the district of the Alps is a good locality for observing the one, it is therefore also good for observing the other. As to that I express no opinion. Probably a spot like Mount Whitney, where LANGLEY observed, might be preferable, more particularly as the long waves are much less scattered than the short waves.

§ XXII. *Relative Colour Brightness of different Parts of the Solar Spectrum.*

ROOD, in his 'Modern Chromatics,' gives the value of the brightness of different parts of the spectrum under the head of different colours. His Table he constructed from VIERORDT's curve, which General FESTING and myself criticised in our paper on "Colour Photometry." Taking a June day near mid-day, the following values were derived, and they are compared with ROOD's —

ROOD's scale	ROOD's nomenclature of colour	ROOD's value of brightness	ABNEY's value of brightness
0 to 149	Red	54	65
149 „ 194	Orange-red	140	138
194 „ 210	Orange	80	61
210 „ 230	Orange-yellow	114	77
230 „ 240	Yellow	54	39
240 „ 344	Yellow-green and green-yellow	327	365
344 „ 447	Green and blue-green	134	183
447 „ 495	Cyan blue	32	33
495 „ 806	Blue and blue-violet	60	36
806 „ 1000	Violet	5	3

In this scale A = 0, a = 40 05, B = 74 02, C = 112 71, D = 220 31, E = 363 11, F = 493·22, G = 753 58, H = 1000

TABLE of Wave-lengths.

Scale number	λ	$\frac{1}{\lambda^4}$
44	662	52
45	629	64
46	601	76 5
47	577	90
48	557	104
49	538	120
50	519	138
51	502	158
52	487	178
53	474	198
54	464	216
55	454	235
56	445	255
57	436	276
58	428	298

South Kensington—place of observation		44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date 1886, June 5. Time · 10^h	I Ordinates at the Riffel	11·6	64 4	114	136	127	105	72	36	17	9 52	5 73	3·65	2 45	1·76	1 49
Secant Z D 1·258.	II Observed ordinates	9 5	51	88	102	92·5	75	50	24	11	5·8	3 4	2 1	1 35		
Barometer 30·159 inches.	I ÷ II Observed	12 2	1 28	1 29	1 33	1 37	1 4	1·44	1 5	1 55	1 64	1 69	1 74	1 82		
Air-thickness 1 265 atmospheres.	I ÷ II Calculated	12 4	1 26	1 3	1 33	1 37	1 41	1 45	1 51	1 57	1 63	1 69	1 75	1 82		
kx 00190.	III Ordinates reduced to compare with I	10 5	57	98 3	114·3	104 1	84	55·4	26·6	12·2	6·5	3 8	2 3	1 5		

South Kensington—place of observation.		44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date · 1886, July 1. Time · 2^h 30^m.	I Ordinates at the Riffel	11 6	64 4	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 76	1 49
Secant Z.D 1 30	II Observed ordinates	10	52 5	86	98	86	62	39	18	7 5	3 8	2 05	1 18	·71	46	33
Barometer 30 246 inches.	I ÷ II. Observed	11 6	1 24	1 32	1 39	1 51	1 67	1 87	2	2·27	2 5	2 79	3 08	3 45	3 83	4 38
Air-thickness . 1·31 atmospheres.	I ÷ II. Calculated	11 6	1 24	1 32	1 42	1 53	1 66	1 84	2 04	2·27	2 53	2 78	3 08	3 43	3 85	4 31
kx . ·00532.	III Ordinates reduced to compare with I.		45 7	76	85	73	55·8	34·6	15 5	6·5	3·28	1 82	1 04	61	41	3

South Kensington—place of observation.		44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date 1886, July 5. Time 2^h 45^m.	I Ordinates at the Riffel	11·6	64 4	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 76	1 49
Secant Z D. 1 364.	II. Observed ordinates	8 5	47	80	95	85	69	45	21 5	10	5 25	3	1 8	1 61	8	5
Barometer 30 243 inches.	I. ÷ II. Observed	1 36	1 37	1·42	1·43	1 49	1 52	1·6	1 67	1·7	1 83	1 91	2 03	2 04	2 2	2·29
Air-thickness . 1·373 atmospheres	I. ÷ II. Calculated	1 33	1 36	1·4	1 44	1 48	1 53	1·59	1 66	1·73	1 81	1 88	1 95	2 04	2 13	2 23
kx . ·00211	III Ordinates reduced to compare with I	10 3	55	95	114	102·4	81 4	53 7	25 9	11 6	6 3	3 6	2 2	1 4	9 5	7

South Kensington—place of observation.

Date: 1886, July 21.
Time: $2^h\ 10^m$.
Secant Z.D.. 1·806
Barometer: 29·815 inches.
Air-thickness: 1·295 atmospheres.
kx: 0022.

	Scale numbers.														
	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
I. Ordinates at the Riffel	11 6	64	114	136	127	105	72	36	17	9 52	5·73	3 65	2 45	1 76	1 49
II. Observed ordinates	9 5	50	87	100	93	71	48	23	10 5	5 6	3·2	2	1 3	85	7
I ÷ II { Observed	1 22	1 28	1 31	1 36	1 37	1 45	1 5	1·56	1 62	1 7	1 79	1 82	1 9	2 01	2 13
I ÷ II { Calculated	1 24	1 27	1 31	1 35	1 39	1 44	1 5	1 56	1 64	1 71	1 77	1 85	1 93	2 03	2 13
III. Ordinates reduced to compare with I	10 4	55·6	96 6	111·5	101·6	80 8	53	25 3	11 5	6 2	3 5	1 9	1 4	9	8

South Kensington—place of observation

Date: 1886, Oct. 14.
Time. $11^h\ 35^m$
Secant Z.D.. 1 993
Barometer: 29·710 inches.
Air-thickness. 1·973 atmospheres.
kx. 00284

	Scale numbers.														
	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
I Ordinates at the Riffel	11 6	64	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 76	1 49
II. Observed ordinates	9	47	80	92	83	64	42	20	9	4 7	2 7	1 6	1 05		
I ÷ II. { Observed	1 29	1 36	1 43	1 48	1 53	1 64	1 71	1 8	1 88	2 03	2 12	2 27	2 33		
I ÷ II. { Calculated	1 33	1 38	1 43	1 47	1 54	1 61	1 7	1 8	1 91	2 02	2 12	2 24	2 37		
III Ordinates reduced to compare with I	10	53 3	92	105	94 7	74 5	48 6	22 9	10 8	5 4	3 1	1 8	1 2		

South Kensington—place of observation

Date 1886, Oct. 29.
Time. $2^h\ 30^m$
Secant Z D. 3 369
Barometer· 30·320 inches
Air-thickness. 3 404 atmospheres.
kx 00472

	Scale numbers														
	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
I Ordinates at the Riffel	11 6	64 4	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 76	1 49
II Observed ordinates.	10	53	87	97	83	65	41	18	8 25	4	2 5	1 3	8		
I − II { Observed	1 16	1 22	1 28	1 4	1 54	1 61	1·76	2	2 12	2 38	2 5	2 71	3 06		
I − II { Calculated	1 17	1 24	1 31	1 4	1 5	1 6	1 76	1 93	2 12	2 33	2 54	2 77	3 05		
III. Ordinates reduced to compare with I	9 9	47 4	77 7	88 8	77 9	60	37 5	17	7 4	3 8			73		

South Kensington—place of observation		Scale numbers.														
		44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date. 1886, Nov 4. Time $2^h\ 15^m$ Secant Z.D 3·443. Barometer 29 667 inches. Air thickness. 3 428 atmospheres. kx 00500	I. Ordinates at the Riffel	11 6	64	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 76	1 49
	II Observed ordinates	10	52	85	97	83	62	40	18	8	4	2 25	1 2	8		
	I − II { Observed	1 16	1 23	1 32	1 41	1 53	1·67	1 8	2	2 13	2 38	2 60	3 04	3 22		
	I − II { Calculated	1 17	1 24	1 32	1 41	1 51	1 63	1 79	1 98	2 19	2 42	2 64	2 91	3 22		
	III. Ordinates reduced to compare with I	9	46 4	77 5	87 4	75 2	58	36	16 4	6 8	3 5	1 9	1 1	7		

South Kensington—place of observation.		Scale numbers.														
		44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date 1886, Nov 8. Time: $10^h\ 25^m$ Secant Z D 3 35 Barometer 29·789 inches. Air-thickness. 3 326 atmospheres. kx ·00592.	I Ordinates at the Riffel	11 6	64	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 79	1 49
	II. Observed ordinates	10 75	57	88	98	86	63	37	18	6 5	3 5	1 8	1 1	66	41	
	I ÷ II. { Observed	1 1	1 13	1 28	1 39	1 48	1 66	1 98	2	2 43	2 63	2 86	3 4	3 69	4 29	
	I ÷ II. { Calculated	1 1	1 19	1 27	1 39	1 51	1 66	1 85	2 08	2 34	2 64	2 93	3 2	3 69	4 20	
	III Ordinates reduced to compare with I	8 5	43 8	72 6	80	68 7	51 9	31 8	14 1	5 8	2 94	1 59	91	55	34	

South Kensington—place of observation		Scale numbers.														
		44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date 1886, Nov. 16. Time $2^h\ 35^m$. Secant Z D 5 042 Barometer 29 664 inches. Air-thickness 4·985 atmospheres kx 00712.	I. Ordinates at the Riffel	11 6	64	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45		
	II Observed ordinates	11	58	90	97	82	60	35	16	6 5	3 2	1 7	9	45		
	I ÷ II { Observed	1 05	1 1	1 27	1 4	1 56	1 75	2 06	2 25	2 62	2 97	3 38	4 05	4 45		
	I ÷ II { Calculated	1 06	1 15	1 26	1 39	1 53	1 72	2 05	2 25	2 61	2 98	3 41	3 9	4 5		
	III Ordinates reduced to compare with I	8	40 5	66 3	72	61	44 7	27	11 7	4 9	2 4	1 23	68	40		

South Kensington—place of observation.		Scale numbers 44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date · 1886, Nov. 18.	I Ordinates at the Riffel	11 6	64	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 76	1 49
Time: $11^h\ 10^m$	II. Observed ordinates		54	90	100	89	68	43	20	8 5	4 5	2 5	1 45	94		
Secant Z.D. 3·173	I + II. Observed		1 18	1 27	1 36	1 43	1 53	1 67	1 8	2	2 11	2 30	2 51	2 64		
Barometer: 30 inches.	I + II. Calculated		1 22	1 29	1 36	1 44	1 53	1 66	1 8	1 95	2 12	2 28	2 48	2 68		
Air-thickness: 3 173 atmospheres. kx: ·00412	III Ordinates reduced to compare with I	·	51 1	83 2	93 8	84 1	64 4	40 9	18 8	8 4	4 1	2 3	1 4	9		

South Kensington—place of observation		Scale numbers. 44	45	46	47	48	49	50	51	52	53	54	55	56	57	58
Date. 1886, Dec. 23.	I. Ordinates at the Riffel	11 6	64	114	136	127	105	72	36	17	9 52	5 73	3 65	2 45	1 76	1 49
Time: $1^h\ 45^m$	II Observed ordinates	10 5	60	95	100	83	61	35	14	6	2 8	1 4				
Secant Z.D 4 973.	I + II Observed	1 01	1 07	1 2	1 36	1 53	1 72	2 06	2 56	2 83	3 4	4 1				
Barometer · 29 801 inches.	I + II Calculated	99	1 09	1 22	1 36	1 53	1 75	2 03	2 42	2 87	3 4	3 96				
Air-thickness: 4·812 atmospheres. kx: ·00848.	III. Ordinates reduced to compare with I.	6 9	37 2	59 4	63 5	52 4	37 8	22 3	9 4	3 7	1 77	9				

July 1, 2h 35m			July 5			July 21			October 14, 11h 35m			October 29		
Aperture	Scale Numbers.		Aperture.	Scale Numbers.		Aperture	Scale Numbers		Aperture	Scale Numbers.		Aperture	Scale Numbers	
98	47 03	46 94	95	47·00		97 8	47 24	46 77	92	47 40	46 60	94	47 30	46 34
89	47 70	46 15	90	47 60	46 60	87	48 18	45 97	87	47 75	46 52	82 3	48 04	45 82
80 3	48 32	45 65	82	48 20	46 10	81 5	48 52	45 80	76	48 30	45 81	70	48 75	45 43
71 4	48 60	45 42	73	48 80	45 60	76 1	48 61	45 63	65	48 96	45 51	59	49 20	45 14
62 5	49 02	45 15	64	49 20	45 32	70 3	48 84	45 52	54	49 46	45 32	47	49 70	44 90
53 5	49·40	45 05	55	49 61	45·20	68 8	49 14	45 50	49	49 60	45 11	35	50 25	44 64
44 6	49 71	44 82	46	49 92	44 95	65 2	49 18	45 40	43 5	49 96	44 90	29 5	50 40	44 51
35 6	50·10	44 55	37	50·24	44 81	58 0	49 64	45 17	32 5	50 40	44 75	23 5	50 66	44 38
26 7	50 50	44 43	27	50 64	44 66	50 8	49 87	45 06	22	50 90	44 57	17 5	50 92	44 26
17 9	51 00	44 26	20	51·06	44·40	43·5	50 09	44 83	16 5	51 15	44 39	11 7	51 50	44 04
8 9	51 69	43 90	18	51 16	44 50	36 3	50 53	44 60	11	51 75	44 12	7 16	52 15	43 70
4 5	52 68	43·57	10	51 95	43 95	29 0	50 76	44 40	5 5	52·78	43 85	3 6	53 13	48 31
3 5	53 18	43 37	9 1	52 14	44 08	25 3	50 87	44 40	2 7	54 07	43 58	1 8	54 36	42 61
2 75	53 67	43·27	6 6	52 60	43 84	21 7	51 10	44 38	1 35	55 76	43 18	9	55 60	42 30
1 5	54 46	43·17	4 9	53·14	43·84	14 5	51 56	44 26	67	60 46	42 91			
1	55 46	43 07	3 3	53 75	43 54	10·8	51 90	44 03						
5	56·37	42·97	1 6	55 15	43 28	7 25	52 62	43 80						
25	58 90	42 57				5 43	53 15	43 63						
						3 62	53 84	43 58						
						2 17	54 75	43 35						
						1 45	55 89	43·01						
						72	57 94	42 67						
						24	61 59	42 21						

November 4, 2h 15m.			November 8, 10h 25m			November 16			November 18, 1h 40m			December 23, 1h 55m		
Aperture.	Scale Numbers.		Aperture.	Scale Numbers.		Aperture	Scale Numbers		Aperture	Scale Numbers		Aperture	Scale Numbers	
98 7	46·78	..	97	47 22	46 64	100	46 60	..	100	47 25	46 50	100	47 00	..
92·3	47 51	46·34	90	47 77	46 14	95·2	47·25	46·21	95	47 64	46 21	90	47 82	45 75
79	48·16	45 82	74 6	48 55	45 50	85 7	47 80	45 69	90	48 16	46 00	80	48 20	45 40
66	48·81	45 43	60	49 07	45 04	76 2	48 29	45 17	80	48 42	45 56	70	48 65	45 25
52·7	49 46	45 04	45	49 59	44 78	66 6	48 60	45 08	70	48 81	45 30	60	49 05	45 00
39 5	50 04	44 78	30	50 24	44·52	57 1	49 12	44 91	60	49 30	45 04	50	49 40	44 90
26·3	50 50	44·52	15	50 88	44 13	47 6	49 49	44 86	50	49 59	49 91	40	49 80	44 70
19·7	50·70	44·39	7 5	51 54	43 74	38 1	50 90	44·65	40	49 98	44 78	30	50 15	44 50
13 2	51·28	44 13	5 0	52 45	43 48	28 5	50 27	44 39	30	50 37	44 65	20	50 52	44 30
7 9	52 06	43 85	2·5	53 62	43 35	19	50 76	44 20	20	50 76	44 39	18	50 70	44 22
5·3	52 58	43 48	1·6	54 40	43·14	9 5	51 45	43 80	10	51 54	44 13	9	51 53	43 90
2 6	53·74	43 22	·8	55 70	42 83	5 2	52 06	43 48	5	52 58	43 61	6	52 02	43 74
9	55 70	42 96	4	57 26	..	3 4	52 75	43 30	2·5	53 62	43 48	3	52·91	43 66
·5	57 78	42 60				1 7	54 00	43 09	1 25	55 70	43 22	1	55 13	43 38
						1 2	54 33	42 83						
						6	55 82	42 71						

ADOPTED Readings.

Scale number	June 5	July 1	July 5	July 21	Oct 14	Oct 29	Nov 4	Nov 8	Nov 16	Nov 18 $11^h 10^m$	Nov 18 $1^h 10^m$	Dec 28	Riffel
44	9 5	10	8 5	9 5	9	10	10	10 5	11	..	..	10 5	11 6
45	51	52 5	47	50	47	53	52	57	58	54	57	60	64
46	88	86	80	87	80	87	85	88	90	90	90	95	114
47	102	98	95	100	92	97	97	98	97	100	102 5	100	136
48	92 5	86	85	93	83	83	83	86	82	89	88	83	127
49	75	62	69	71	64	65	62	63	60	68	65	61	105
50	50	39	45	48	42	41	40	37	35	43	40	35	72
51	24	18	21·5	23	20	18	18	18	16	20	17	14	36
52	11	7 5	10	10·5	9	8 25	8	6 5	6 5	8 5	7	6	17
53	5·8	3 8	5 25	5·6	4 7	4	4	3 5	3 2	4 5	3 5	2 8	9 52
54	3·4	2 05	3·0	3 2	2 7	2 3	2 25	1 8	1 7	2 5	1 9	1 4	5 73
55	2·1	1 18	1 8	1 7	1 6	1 3	1 2	1 1	·9	1 45	..	..	3 65
56	1·35	·71	1 6	1 3	1 05	·8	8	.	45	·94	.		2·45
57	..	·46	·8	·85	..	.	..	41	..	..	..	..	1 76
58		33	5	70	..	..	..	..		..	..	.	1 49

METEOROLOGICAL Data extracted from the Records of the Kew Observatory.

Date.	Hour.	Barometer readings reduced to 32° and to M S.L.	Thermometer		Solar radiation	Grass minimum	Wind.		Cloud			Remarks
			Dry.	Wet.			Direction	Velocity	Amount.	Form	Direction	
		inches	°	°	°	°		miles p h.				
June 4, 1886 .	10 A M	30 190	52·9	46 6		..	N.N.E.	14	2	small cu	..	Very fine day, with visibility in afternoon
	2 P.M	30 159	58·5	51 4	120	38 8	N N.E	11	4	ci str. / cu	N N E	
June 5 . .	10 A.M.	30·127	52 1	46 5	..	.	N E	11	2	ci	Slowly N E	Very fine day, mock-sun seen about 6 P M, and solar halo during evening
	2 P M.	30·067	58·4	51 0	117	36 5	N	10	4	ci / low ci str	N E	
July 1 . . .	10 A.M	30 285	63 4	54·9	.	..	N E	13	1	small cu	E N E	Very fine, ground fog at night
	2 P.M.	30·246	68 2	59 0	118	40	E.	13	3	cu.	E N E	
July 5 . .	10 A.M.	30·269	70·1	58 8	.	.	N N W	9	0		..	Very fine
	2 P.M.	30·243	76·5	62·1	133	49	N W.	9	0			
July 21 . .	10 A.M	29 867	74·6	68 6			S.E	13	6	cir : cir str / cu	S S W	Day fine, but night stormy
	2 P M	29 815	81·5	70 1	139		S.	25	6	cir. str., and cir str / cu	S S W	
Oct. 14 . .	10 A M.	29·695	51·7	48 5	..	.	W	10	1	sm. cu	W	Day fine; evening overcast, with lunar halo.
	2 P.M.	29 728	56 2	49 6	110	30	W.	12	7	cir str / cu	W N W	

METEOROLOGICAL Data extracted from the Records of the Kew Observatory—*continued*

Date	Hour	Barometer readings reduced to 32° and to M. S. L.	Thermometer		Solar radiation	Grass minimum	Wind.		Cloud			Remarks.
			Dry	Wet.			Direction.	Velocity	Amount	Form	Direction	
		inches	°	°	°	°		miles p h				
Oct. 29, 1886.	10 A.M	30·274	57·1	55·2		..	S.	13	7	st det / scud	S W	Fine
	2 P.M.	30·320	60·5	54·4	104	39	S W	15	5	st / cu	S W	
Nov. 4 . .	10 A.M.	29·911	45·5	43·5	..		W S W	9	1	high st.	?	Very fine day.
	2 P.M.	29·867	52·7	46·4	94	30	S W	13	8	cir str / str / cu	W S W / W / W.	
Nov 8	10 A M	29·789	37·6	36·5	..		S.W	5	0	.		Fine, but large amount of sheet cirrus in evening, with lunar halo
	2 P.M	29·703	45·9	39·6	89	23	S W	13	7	cir str / cu	S W	
Nov 16 . .	10 A M	29·618	45·3	42·5	..	..	W S W	6	6	low ci / det str	S S W / W S W	Fine day; rain during night
	2 P M	29·656	47·6	41·9	89	41	W	10	3	ci str / cu	S S W / W.	
Nov 18 . .	10 A.M.	29·984	45·4	41·9	.	..	W.	15	5	ci / det str	N W.	A very fine day
	2 P M	30·052	48·6	43·1	88	30·9	N W	14	2	ci st. cu	N W.	
Dec. 23. . .	10 A.M	29·743	39·0	35·5	.		W	20	3	ci str	N W	Fine and bright
	2 P.M	29·803	41·4	37·5	72	29·3	W N.W	13	9	ci str / ragged cu	N W	

X. *On Hamilton's Numbers.*

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and JAMES HAMMOND, *M.A. Cant.*

Received June 11,—Read June 16, 1887.

INTRODUCTION.

IN the year 1786 ERLAND SAMUEL BRING, Professor at the University of Lund in Sweden, showed how by an extension of the method of TSCHIRNHAUSEN it was possible to deprive the general algebraical equation of the 5th degree of three of its terms without solving an equation higher than the 3rd degree. By a well-understood, however singular, academical fiction, this discovery was ascribed by him to one of his own pupils, a certain SVEN GUSTAF SOMMELIUS, and embodied in a thesis humbly submitted to himself for approval by that pupil, as a preliminary to his obtaining his degree of Doctor of Philosophy in the University.* The process for effecting this reduction seems to have been overlooked or forgotten, and was subsequently re-discovered many years later by Mr JERRARD. In a report contained in the 'Proceedings of the British Association' for 1836, Sir WILLIAM HAMILTON showed that Mr. JERRARD was mistaken in supposing that the method was adequate to taking away more than three terms of the equation of the 5th degree, but supplemented this somewhat unnecessary refutation of a result, known *à priori* to be impossible, by an extremely valuable discussion of a question raised by Mr JERRARD as to the number of variables required in order that any system of equations of given degrees in those variables shall admit of being satisfied without solving any equation of a degree higher than the highest of the given degrees.

In the year 1886 the senior author of this memoir showed in a paper in KRONECKER's (better known as CRELLE's) 'Journal' that the trinomial equation of

* BRING's "Reduction of the Quintic Equation" was republished by the Rev. ROBERT HARLEY, F.R.S., in the 'Quarterly Journal of Pure and Applied Mathematics,' vol. 6, 1864, p. 45. The full title of the LUND Thesis, as given by Mr HARLEY (see 'Quart. Journ. Math.,' pp. 44, 45) is as follows: "B. cum D. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum, quae consent. Ampliss. Facult. Philos. in Regia Academia Carolina Praeside D. ERLAND SAM. BRING, Hist. Profess. Reg. & Ord. publico Eruditorum Examini modeste subjicit SVEN GUSTAF SOMMELIUS, Stipendiarius Regius & Palmcrentzianus Lundensis. Die XIV Decemb., MDCCLXXXVI, L.H.Q.S.—Lundae, typis Berlingianis."

30.9.87

the 5th degree, upon which by BRING's method the general equation of that degree can be made to depend, has necessarily imaginary coefficients except in the case where four of the roots of the original equation are imaginary, and also pointed out a method of obtaining the absolute minimum degree M of an equation from which any given number of specified terms can be taken away subject to the condition of not having to solve any equation of a degree higher than M.* The numbers furnished by HAMILTON's method, it is to be observed, are not minima unless a more stringent condition than this is substituted, viz., that the system of equations which have to be resolved in order to take away the proposed terms shall be the simplest possible, *i.e.*, of the lowest possible weight and not merely of the lowest order; in the memoir in 'CRELLE,' above referred to, he has explained in what sense the words weight and order are here employed. He has given the name of HAMILTON's Numbers to these relative minima (minima, *i.e*, in regard to weight) for the case where the terms to be taken away from the equation occupy consecutive places in it, beginning with the second

Mr. JAMES HAMMOND has quite recently discovered by the method of generating functions a very simple formula of reduction, or scale of relation, whereby any one of these numbers may be expressed in terms of those that precede it his investigation, which constitutes its most valuable portion, will be found in the second section of this paper. The principal results obtained by its senior author, consequential in great measure to Mr. HAMMOND's remarkable and unexpected discovery, refer to the proof of a theorem left undemonstrated in the memoir in 'CRELLE' above referred to, and the establishment of certain other asymptotic laws to which HAMILTON's Numbers and their differences are subject, by a mixed kind of reasoning, in the main apodictic, but in part also founded on observation.† It thus became necessary to calculate out the 10th Hamiltonian Number, which contains 43 places of figures. The highest number calculated by HAMILTON (the 6th) was the number 923, which comes third in order after 5 (the BRING Number), 11 and 47 being the two intervening numbers. It is to be hoped that some one will be found willing to undertake the labour (considerable, but not overwhelming) of calculating some further numbers in the scale.

The theory has been "a plant of slow growth." The LUND Thesis of December,

* For instance, an equation of not lower than the 905th degree may be transformed into another of that degree, in which the 2nd, 3rd, 4th, 5th, 6th, 7th, terms are all wanting, by means of the successive solution of a ramificatory system of equations, of no one of which the degree exceeds 6, whereas by the Jerrard-Hamiltonian method this transformation could not be effected for the general equation of degree lower than the 6th Hamiltonian Number, viz, 923. So for the analogous removal of 5 consecutive terms the inferior limit of degree of the equation to be transformed would be 47 by the one method, but 44 (the lowest possible) by the other In the case of 4 consecutive terms HAMILTON could not avoid being aware that 11, the 4th number which I have named after him, might be replaced by 10, as the lowest possible inferior limit of the equation to be transformed.

† In the 3rd section, communicated to the Society after the 1st and 2nd had gone to press, the empirical element is entirely eliminated, and the results reduced to apodictic certainty.

1786 (a matter of a couple of pages), HAMILTON's Report of 1836, with the tract of Mr. JERRARD therein referred to, and the memoir in 'CRELLE' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date.

§ 1 *On the Asymptotic Laws of the Numbers of* HAMILTON *and their Differences.*

Consider the following Table —

1	0	0	0	0	0	0	0	....
	1	1	1	1	1	1	1	...
		2	3	4	5	6	7	..
			6	15	29	49	76	....
				36	210	804	2449	.
					876	24570	401134	...
						408696	246382080	...
							83762796636	.
								..
								.

Any line of figures, say $p, q, r, s, t, \ldots \theta$, in the Table being given, to form the subsequent line $q_1, r_1, s_1, t_1, \ldots \theta_1$, we write

$$q_1 = \frac{p(p+1)}{1.2} + q.$$

$$r_1 = \frac{p(p+1)(2p+1)}{1.2.3} + pq + r.$$

$$s_1 = \frac{p(p+1)(p+2)(3p+1)}{1.2.3.4} + \frac{p(p+1)}{1.2} q + pr + s.$$

$$t_1 = \frac{p(p+1)(p+2)(p+3)(4p+1)}{1.2.3.4.5} + \frac{p(p+1)(p+2)}{1.2.3} q + \frac{p(p+1)}{1.2} r + ps + t.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\theta_1 = \frac{p(p+1)\ldots(p+i-1)(ip+1)}{1.2.3\ldots(i+1)} + \frac{p(p+1)\ldots(p+i-2)}{1.2.3\ldots(i-1)} q + \ldots + \theta.$$

If we call the n^{th} term of the m^{th} line $[m, n]$, the general law of deduction may be expressed by the formula

$$[m+1, n] = -\mathrm{B}_{n+1}([m, 1] - 1) + \sum_{i=0}^{i=n} [m, n+1-i]\,\mathrm{B}_i[m, 1],$$

where $\mathrm{B}_i k$ means the coefficient of z^i in $(1-z)^{-k}$.

The negative term $-\mathrm{B}_{\kappa+1}([m, 1]-1)$, it may be noticed, arises from decomposing the first term of $[m+1, n]$, as given by the original formulæ, into two parts, of which it is one.

Thus, *ex. gr*,

$$\frac{p(p+1)(p+2)(p+3)(4p+1)}{1\ 2\ 3\ 4\ 5}$$

is changed into

$$-\frac{(p-1)p(p+1)(p+2)(p+3)}{1\ 2\ 3\ 4\ 5}+\frac{p(p+1)(p+2)(p+3)}{1\ 2.3\ 4}p.$$

The numbers in the hypothenuse of this infinite triangle, viz.,

1, 1, 2, 6, 36, 876, 408696, 83762796636, 3508125906207095591916,
6153473687096578758445014683368786661634996, ... ,

are what I call the Hamiltonian Differences, or Hypothenusal Numbers*; and their continued sums augmented by unity, viz.,

2, 3, 5, 11, 47, 923, 409619, 83763206255, 3508125906290858798171,
6153473687096578758448522809275077520433167, ... ,

are what I call the Hamiltonian Numbers. The two latter of these have been calculated by means of Mr. HAMMOND's formula, presently to be mentioned, and the corresponding Hypothenusal Numbers deduced from them by simple subtraction. Their connection with the theory of the TSCHIRNHAUSEN Transformation will be found fully explained in my memoir on the subject in vol. 100 of 'CRELLE.' My present object is to speak of the numbers as they stand, without reference to their origin or application.†

* The other numbers of the "triangle," whose properties it may be some day desirable to investigate, may be termed co-hypothenusal numbers of *order* measured by their horizontal distance from the hypothenuse—their vertical distance below the top line denoting their *rank* In the sequel the development is given of the half of a hypothenusal number (of the first order) in a descending series of powers (with fractional indices) of the half of its antecedent, the coefficients in the principal part of such series being (not, as might have been the case, functions of the rank, but) absolute constants These may be termed the hypothenusal constants The values of the first four of them are shown to be 1, $\frac{4}{3}$, $\frac{11}{18}$, $\frac{10}{81}$

† The reader will be disappointed who seeks in HAMILTON's Report any systematic deduction of the numbers which I have called after his name He treats therein the more general question of finding the number of letters sufficient for satisfying any system of equations of given degrees by means of a certain prescribed uniform process whereby the necessity is obviated of solving any equation of a higher degree than the highest one of the given equations, and among, and mixed up with, other examples considers systems of equations of degrees 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5; 1, 2, 3, 4, 5, 6, for which the minimum numbers of letters required to make such process possible (when the equations are homogeneous) are 5, 11, 47, 923, respectively. Accordingly he has no occasion to employ the infinitely developable triangle which gives unity and cohesion to the problem which deals with an indefinite number of equations of all consecutive degrees from 1 upwards. This triangle, which plays an impor-

The question arises as to whether it is possible to deduce the Hamiltonian Differences, or to deduce the Hamiltonian Numbers, directly in a continued chain from one another without the use of any intermediate numbers Mr. JAMES HAMMOND has shown that it is possible, and has made the remarkable discovery that it is the Numbers of Hamilton, and not the Hypothenusal Numbers, which are subject to a very simple scale of relation These being found, of course the Differences become known. This is contrary to what one would have expected. *A priori*, one would have anticipated that the determination of the Hypothenusal Numbers would have preceded that of their sums.

I leave Mr. HAMMOND to give his own account of his mode of obtaining the wonderful formula of reduction, which, by a slight modification, I find, may be expressed as follows.—Using E_i to denote the $(i+1)^{\text{th}}$ Hamiltonian Number augmented by unity, so that $E_0 = 3$, $E_1 = 4$, $E_2 = 6$, $E_3 = 12$, $E_4 = 48$, ...; and $\beta_i m$ to signify the coefficient of t^i in $(1+t)^m$; then, for any value of i greater than unity,

$$\beta_0 E_i - \beta_1 E_{i-1} + \beta_2 E_{i-2} - \beta_3 E_{i-3} + \ldots + (-)^i \beta_i E_0 = 0$$

tant part in the systematic treatment of the problem, first appears in my memoir on the subject in the 100th volume of 'CRELLE'

. It is proper also again to notice that what I call the Numbers of HAMILTON (at all events those subsequent to the number 5) are not the smallest numbers requisite for fulfilling the condition above specified Smaller numbers will serve to satisfy that condition taken alone, but when such smaller numbers are substituted for HAMILTON'S the resolving equations will be less simple, inasmuch as they will contain a greater number of equations of the higher degrees than when the larger Hamiltonian numbers are employed This distinction will be found fully explained in the memoir cited, and the smallest numbers substitutable for HAMILTON'S are there actually determined for r equations of degrees extending from 1 to r for all values of r up to 8 inclusive.

I have added nothing (for there is nothing to be added) to the fundamental formula of HAMILTON expressed by the equation

$$[\lambda, \mu, \nu, \ldots \pi] = 1 + [\lambda - 1, \lambda + \mu, \lambda + \mu + \nu, \ldots, \lambda + \mu + \nu + \ldots + \pi],$$

where, supposing the letters $\lambda, \mu, \nu, \ldots \pi$, to be i in number, $[\lambda, \mu, \nu, \ldots \pi]$ means the number of letters required in order that it may be possible to satisfy, according to the process employed by HAMILTON (in conformity with a certain stipulation of JERRARD), a system of λ equations of degree i, μ equations of degree $i-1$, ν equations of degree $i-2$, ..., π equations of the degree 1, without solving any single equation of a degree higher than i This formula, applied λ times successively, will have the effect of abolishing λ and causing $[\lambda, \mu, \nu, \ldots \pi]$ to depend on $[\mu', \nu', \ldots \pi']$, where $\mu', \nu', \ldots \pi'$ are connected with $\lambda, \mu, \nu, \ldots \pi$ by means of the formulæ given at the commencement of the present paper, but where instead of the letters $\lambda, \mu, \nu, \ldots$ I have used the letters $p, q, r, \ldots$

It is presumable that the *reduced* Hamiltonian numbers would be found much less amenable to algebraical treatment than the Hamiltonian numbers proper; for numerical equalities and inequalities have to be taken account of, in determining them, which have no place in the determination of the latter numbers. HAMILTON, as already stated, expressly alludes to the reduction of 11 to 10, but with that exception has avoided the general question of finding the *absolutely* lowest number of letters required in order that a system of equations (expressed in terms of those letters) of given degrees may admit of being satisfied without the necessity arising to solve any equation of a higher degree than the highest of the given ones

Or in other words, writing $\beta_0 E_i = 1$, $\beta_1 E_{i-1} = E_{i-1}$, and replacing $i-1$ by i,

$$E_i = 1 + \beta_2 E_{i-1} - \beta_3 E_{i-2} + \ldots + (-)^{i+1} \beta_{i+1} E_0$$

for all values of i greater than zero.

This is eminently a practical formula, as all the numerical calculations made use of to obtain any E are available for finding the E which follows. Dispensing with the symbol β, we may deduce all the values of E successively from those that go before by means of the *equivalence*

$$S = (1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots \equiv 1 - 2t,$$

which, by equating the powers of t on the two sides of the equivalence, gives

$$E_0 = 3,$$

$$E_1 = 1 + \frac{3.2}{1\ 2} = 4,$$

$$E_2 = 1 + \frac{4\ 3}{1\ 2} - \frac{3\ 2.1}{1.2.3} = 6,$$

$$E_3 = 1 + \frac{6\ 5}{1.2} - \frac{4\ 3\ 2}{1.2\ 3} + \frac{3\ 2\ 1\ 0}{1.2\ 3.4} = 12,$$

and so on.

I use the term equivalence and its symbol in order to convey the necessary caution that the relation indicated is not one of quantitative equality; for, although the series on the left-hand side of the symbol converges for all positive values of t less than 2, it is never equal to the expression on the right-hand side except when $t = 0$. Thus, *e.g.*, when t is unity the two terms of the equivalence are 0 and -1, and when $t = \frac{1}{2}$ they are

$$2^{-E_0} + 2^{-E_1-1} + 2^{-E_2-2} + \ldots \text{ and } 0, \text{ respectively;}$$

and for all values of t within the limits of convergence the value of the left-hand side is in excess of the value of the right-hand side of the equivalence by a finite quantity which decreases continuously as t decreases from 2 to 0, and which vanishes when $t = 0$.*

In a word, the *generating equation* is not an equation in the usual sense of the term. Conceiving each term of the series S to be expanded in ascending powers of t, and like powers of t to be placed in columns under and above each other, the double

* Of the truth of the statement that the excess never changes sign, and continually decreases, I have scarcely a doubt, but it requires proof. Mr. HAMMOND remarks that

$$(1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots + t^n(1-t)^{E_n} = (1-2t) + t^2(1-t)^{E_n-2} F_n(t) - t^{n+1}(1-t)^{E_n-1},$$

where $F_n(t)$ is positive for all positive values of t. Probably a proof of the point in question might be deduced from this expression, but I have not thought it necessary to investigate the matter.

sum may be taken as a vertical sum of line-sums or as a horizontal sum of column-sums, and, although for licit values of t each sum has a finite value, the two finite values are not identical, just as a double definite integral may undergo a change of value when the order of its integrations is reversed *

I noticed at p. 478 of the 100th volume of 'CRELLE' that the value of any Hamiltonian Difference divided by the square of the preceding one was always greater than $\frac{1}{2}$, and stated as morally certain, but "awaiting exact proof," that this ratio ultimately becomes $\frac{1}{2}$ By aid of Mr. HAMMOND's formula for the numbers, I shall now be able to supply this proof, and at the same time to show that the ratio of a Hamiltonian *Number* to the square of its antecedent (which, of course, converges to the same asymptotic value $\frac{1}{2}$) is always *less* than that limit †

We must in the first place prove that in the series

$$\beta_2 E_{i-1} - \beta_3 E_{i-2} + \beta_4 E_{i-3} - \beta_5 E_{i-4} + \ldots$$

the absolute value of each term is greater than that of the one which follows it.

In proving this, I shall avail myself of the property of the Hypothenusal Numbers disclosed in the process of forming the triangle given at the outset of the memoir, viz, that $E_i - E_{i-1}$ is greater than $(E_{i-1} - E_{i-2})^2/2$

Let us suppose that the law to be established holds good for a certain value of i. For the sake of brevity, I denote E_i, E_{i-1}, E_{i-2}, E_{i-3}, ... by N, P, Q, R, ...

We have then

$$P - 1 = \frac{Q(Q-1)}{2} - \frac{R(R-1)(R-2)}{2.3} + \frac{S(S-1)(S-2)(S-3)}{2.3\ 4} - \ldots$$

$$N - 1 = \frac{P(P-1)}{2} - \frac{Q(Q-1)(Q-2)}{2.3} + \frac{R(R-1)(R-2)(R-3)}{2.3.4}$$
$$- \frac{S(S-1)(S-2)(S-3)(S-4)}{2.3.4.5} + \ldots$$

* Professor CAYLEY has brought under my notice a not altogether dissimilar, but perhaps less striking, phenomenon, pointed out by CAUCHY, that, although the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

is convergent, its square

$$u_0^2 + (2u_0u_1) + (2u_0u_2 + u_1^2) + \ldots,$$

i.e.,

$$1 - \sqrt{2} + \left(\frac{2}{\sqrt{3}} + \tfrac{1}{2}\right) - \ldots$$

is divergent.

† The fortunate circumstance of the two ratios in question being always respectively less and greater than the common asymptotic value of each of them enables us to find the value of the constant in the expression c^{2^x}, which is asymptotically equivalent to the half of the x^{th} Hamiltonian or Hypothenusal Number by a method exactly analogous to that of exhaustions for finding the Archimedian constant correct to any required number of decimal places. See end of this section (pp 298, 299).

If, then, the law to be proved is true for all the consecutive terms of the upper series it will obviously be true for the second series, *abstraction being made of its first term,* provided that no antecedent is less than its consequent in the series

$$\frac{Q-2}{3}, \quad \frac{R-3}{4}, \quad \frac{S-4}{5}, \ldots,$$

which is true *à fortiori* if

$$\frac{Q}{3}, \quad \frac{R}{4}, \quad \frac{S}{5}, \ldots$$

continually decrease, as is obviously the case, inasmuch as

$$Q, \quad R, \quad S, \ldots$$

form a descending series.

In order, then, to establish the necessary chain of induction, it only remains to show that

$$3P(P-1) - Q(Q-1)(Q-2)$$

is positive.

Now

$$(P-Q) - \frac{(Q-R)^2}{2} \left[\text{and } \textit{à fortiori } P - \frac{(Q-R)^2}{2}\right]$$

is positive for a reason previously given.

And, if in the series 3, 4, 6, 12, 48, 924, . . . we make exclusion of the first three terms, we have always

$$R = \text{or} < \frac{Q}{4},$$

and consequently

$$P > \frac{9Q^2}{32}.*$$

And, since under the same condition $(P-1)/(Q-1) > 4$, $3P(P-1) - Q^2(Q-1)$, and *à fortiori* $3P(P-1) - Q(Q-1)(Q-2)$, is positive if $12P - Q^2$ is positive, which is the case, since $P > 9Q^2/32$.

Hence, since the theorem to be proved is true for the several series

$$(1) \quad \frac{4.3}{1.2} - \frac{3.2.1}{1.2.3},$$

$$(2) \quad \frac{6.5}{1.2} - \frac{4.3.2}{1.2.3},$$

$$(3) \quad \frac{12.11}{1.2} - \frac{6.5.4}{1.2.3} + \frac{4.3.2.1}{1.2.3.4},$$

$$(4) \quad \frac{48.47}{1.2} - \frac{12.11.10}{1.2.3} + \frac{6.5.4.3}{1.2.3.4},$$

* The proof that the ratio of each term of the series 4, 6, 12, 48, 924, . . . to its antecedent continually increases is too easy and too tedious to be worth setting forth in the text.

it will be true universally, for in all the succeeding series the term we have called R will be higher than the term 6 in the scale 3, 4, 6, 12, 48, . .

Hence

$$P - 1 = \text{or} < \tfrac{1}{2}(Q^2 - Q).$$

For the initial values of Q, P, (viz., 3, 4,)

$$P - 1 = \tfrac{1}{2}(Q^2 - Q).$$

[When P represents any term beyond the first it is very easy to prove, but too tedious to set out the proof, that the sum of all the terms after the first in the series equated to $P - 1$ will be less than -2; so that, except in the case stated, $P < \frac{1}{2}(Q^2 - Q)$].

For the series 12, 48, 924, . . . we have seen that $P > 9Q^2/32$.

Hence, for the series 48, 924, . .,

$$Q > \frac{9R^2}{32} \quad \text{or} \quad R < \sqrt{\frac{32Q}{9}}$$

But

$$P > \frac{Q^2 - Q}{2} - \frac{R(R-1)(R-2)}{6},$$

$$> \frac{Q^2 - Q}{2} - \frac{R^3}{6}$$

Hence

$$P > \frac{Q^2 - Q}{2} - \tfrac{64\sqrt{2}}{81} Q^{\frac{3}{2}}, \quad \text{and} \quad P < \frac{Q^2 - Q}{2}.$$

Hence, when P, Q, are at an infinite distance from the origin,

$$\frac{P}{Q^2} = \tfrac{1}{2}.$$

Hence, also,

$$\frac{P - Q}{(Q - R)^2} \text{ ultimately} = \frac{P}{Q^2} = \tfrac{1}{2},$$

which proves the theorem left over for "exact proof" in the memoir referred to

It is convenient to deal with the halves of the *sharpened** Numbers of HAMILTON, which may be called the reduced Hamiltonian Numbers, and denoted by h with a subscript, or, when required, by $p, q, r, \ldots$ (the halves of P, Q, R, . . . respectively)

We have then

$$2p < \frac{4q^2 - 2q}{2},$$

* Numbers increased by unity may conveniently be denominated sharpened numbers, and numbers diminished by unity flattened numbers

or

$$p < q^2 - \frac{q}{2},$$

$$p > q^2 - \frac{q}{2} - \tfrac{128}{81} q^{\frac{3}{2}}$$

We may find a closer superior limit to p in terms of q as follows—

$$P - 1 = \text{or} < \frac{Q^2 - Q}{2} - \frac{R(R-1)(R-2)}{6} + \frac{S(S-1)(S-2)(S-3)}{24},$$

in which inequality it may be shown by inspection up to a certain point, and after that by demonstration, the tedium of writing out or reading which I spare my readers and myself, that P may be substituted for its flattened value $P - 1$.

We have then

$$P < \frac{Q^2 - Q}{2} - \frac{R^3 - 3R^2}{6} + \frac{S^4}{24}.$$

Let us suppose that S, R, are not lower in the scale of the E's than 12, 48, respectively; so that P is not lower than E_6, which is 409620.

Then, as we have previously shown,

$$Q^2 < \tfrac{32}{9} P, \qquad R^2 < \tfrac{32}{9} Q, \qquad S^2 < \tfrac{32}{9} R.$$

Moreover, we have

$$P < \tfrac{1}{2}(Q^2 - Q), \text{ whence it follows that } Q^2 > 2P + Q,$$

and, *à fortiori*,

$$Q^2 > 2P$$

Similarly

$$R^2 > 2Q,$$

and

$$S^2 > 2R$$

Now

$$\begin{aligned} P &< \frac{Q^2 - Q}{2} - \frac{R^3}{6} + \frac{R^2}{2} + \frac{S^4}{24} \\ &< \frac{Q^2 - Q}{2} - \tfrac{1}{6}(2Q)^{\frac{3}{2}} + \tfrac{1}{2}\left(\tfrac{32}{9} Q\right) + \tfrac{1}{24}\left(\tfrac{32}{9} R\right)^2 \\ &< \frac{Q^2}{2} - \tfrac{\sqrt{2}}{3} Q^{\frac{3}{2}} + \tfrac{23}{18} Q + \tfrac{1}{24}\left(\tfrac{32}{9}\right)^3 Q, \end{aligned}$$

i.e.,

$$P < \tfrac{1}{2} Q^2 - \tfrac{\sqrt{2}}{3} Q^{\frac{3}{2}} + \tfrac{13781}{4374} Q.$$

This result, expressed in terms of the reduced numbers p, q, takes the form

$$p < q^2 - \tfrac{2}{3} q^{\frac{3}{2}} + \tfrac{13781}{4374} q,$$

and we have previously shown that

$$p > q^2 - \tfrac{128}{81} q^{\frac{3}{2}} - \frac{q}{2},$$

at all events when P is not lower in the scale than E_6.

The fraction $\frac{128}{81}$ arises from our having substituted for R^3 the inferior value $(\frac{32}{9} Q)^{\frac{3}{2}}$; but, the higher we advance P in the scale, the nearer R approaches to 2Q, and is ultimately in a ratio of equality with it But, if we had written $(2Q)^{\frac{3}{2}}$ for R^3, the coefficient, which now stands at $-\frac{128}{81}$, would have been $-\frac{2}{3}$ In like manner, as P and Q are travelled on in the scale, R^2 and S^4 become indefinitely near to 2Q and $(2R)^2$, *i.e*, 8Q, so that the coefficient of Q in the superior limit approximates indefinitely near to

$$-\tfrac{1}{2} + 1 + \tfrac{1}{3}, \text{ i e., } \tfrac{5}{6},$$

and the two limits of p which have been obtained become

$$q^2 - \tfrac{2}{3} q^{\frac{3}{2}} + (\tfrac{5}{6} + \epsilon) q,$$
$$q^2 - (\tfrac{2}{3} + \eta) q^{\frac{3}{2}} - \tfrac{1}{2} q,$$

where ultimately ϵ and η are infinitesimals *

Hence it follows that the ultimate value of

$$(p - q^2) \div q^{\frac{3}{2}} \text{ is } -\tfrac{2}{3},$$

i.e.,

$$\frac{2E_i - E^2_{i-1}}{E^{\frac{3}{2}}_{i-1}} = -\sqrt{\tfrac{8}{9}} \text{ when } i = \infty.$$

Let $\lambda, \mu, \nu, \ldots$ represent the halves of the Hypothenusal Numbers in the triangle given at the commencement of the paper, *i e.*, the differences of the numbers which we have called $p, q, r, \ldots$

Since

$$p = q^2 - \tfrac{2}{3} q^{\frac{3}{2}} \qquad \text{and} \qquad q = r^2 - \tfrac{2}{3} r^{\frac{3}{2}},$$
$$p - q = q^2 - \tfrac{2}{8} q^{\frac{3}{2}} - q, \quad \text{and} \quad q - r = r^2 - \tfrac{2}{3} r^{\frac{3}{2}} - r.$$

Obviously, therefore, as a first approximation when λ, μ, are very advanced terms in the hypothenuse,

$$\lambda = \mu^2.$$

Let us write

$$\lambda = \mu^2 + \kappa\mu^a$$

for a second approximation.

* As a matter of fact, it will be found that, as soon as q and p attain the values 6, 24, $q^2 - \frac{2}{3} q^{\frac{3}{2}}$ may be taken as a superior limit. It may be noticed also, to prevent a wrong inference being drawn from the above expressions, that, as will hereafter appear, η is an infinitesimal of the order $1/q^{\frac{1}{2}}$, when q is infinite.

Then

$$q^2 - \tfrac{2}{3}q^{\frac{3}{2}} - q = (r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - r)^2 + \kappa(r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - r)^{\alpha},$$

or, *neglecting* terms of lower dimensions than r^3,

$$(r^2 - \tfrac{2}{3}r^{\frac{3}{2}})^2 - \tfrac{2}{3}r^3\left(1 - \frac{1}{r^{\frac{1}{2}}} + \frac{1}{6r} - \ldots\right) = (r^2 - \tfrac{2}{3}r^{\frac{3}{2}} - r)^2 + \kappa r^{2\alpha}$$

Therefore

$$-\tfrac{2}{3}r^3 = -2r^3 + \kappa r^{2\alpha}.$$

Consequently

$$\alpha = \tfrac{3}{2} \quad \text{and} \quad \kappa = \tfrac{4}{3}.$$

Thus, then, for the consecutive Hypothenusal Numbers λ, μ,

$$\lambda = \mu^2 + \tfrac{4}{3}\mu^{\frac{3}{2}} + \ldots$$

Let

$$\lambda = \mu^2 + \tfrac{4}{3}\mu^{\frac{3}{2}} + \theta\mu,$$

or say

$$\eta_{x+1} = \eta_x^2 + \tfrac{4}{3}\eta_x^{\frac{3}{2}} + \rho_x\eta_x,$$

where η_x is the x^{th} term in the series $\frac{1}{2}$, 1, 3, 18, . . .

The successive values of ρ_x and their differences are given in the annexed Table

x	η_x	ρ_x	$\Delta\rho_x$
1	·5	55719096	
2	1	·66666666	+ 10947570
3	3	69059893	+ 02393227
4	18	67647909	− 01411984
5	438	64334761	− 03313148
6	204348	61769722	− 02565039
7	41881398318	61139243	− 00630479
8	1754062953103547795958	61111171	− 00028072

The decimal figures following those given in ρ_8, required for ulterior purposes, being 5795.

An examination of the column of differences for $x = 5, 6, 7, 8$, shows that the ratios of each to the rest go on decreasing somewhat faster than their squares this makes it almost certain that $\rho_8 - \rho_9$ will be between the 400th and 500th part of ·000280, and that accordingly the value of ρ_9 will be ·6111111, &c. I believe it is beyond all moral doubt that the ultimate value of ρ is exactly $\frac{11}{18}$; and, indeed, it was the conviction I entertained of this being its true value, when I had calculated ρ_7,

that led me to undertake the very considerable labour of ascertaining the 10th Hamiltonian Number in order to deduce from it the value of ρ_8 This being taken for granted,* we may proceed to ascertain a further term in the asymptotic value of η_{x+1} expressed as a function of η_x.

For, calling

$$\rho_x - \tfrac{11}{18} = \delta_x \quad \text{and} \quad \sqrt[4]{\eta_x} = q_x,$$

we have

$$\begin{aligned} \delta_6 &= \cdot 00658611, \\ \delta_7 &= \cdot 00028132, \\ \delta_8 &= \cdot 0000006047, \end{aligned}$$

$$\left.\begin{aligned} q_6 &= 21, \\ q_7 &= 452, \\ q_8 &= 204649, \end{aligned}\right\} \text{neglecting decimals.}$$

Thus

$$\begin{aligned} (\delta q)_6 &= 1383, \\ (\delta q)_7 &= \cdot 1272, \\ (\delta q)_8 &= \cdot 12375. \end{aligned}$$

The value of

$$(\delta q)_6 - (\delta q)_7 \text{ being } 0111,$$

and of

$$(\delta q)_7 - (\delta q)_8 \qquad \cdot 0035,$$

we may feel tolerably certain, from the Law of Squares, that $(\delta q)_8 - (\delta q)_9$ will be somewhere in the neighbourhood of the tenth part of ·0035, and accordingly that $(\delta q)_9$ is about ·1234, so that the probable value of $(\delta q)_\infty$ is ·1234 .

Thus we have found

$$\eta_{x+1} = \eta_x^{2} + \tfrac{4}{3}\,\eta_x^{\frac{3}{2}} + \tfrac{11}{18}\,\eta_x + [\quad]\,\eta_x^{\frac{3}{4}} + \qquad ,$$

the only moral doubt being as to the degree of closeness of propinquity of the coefficient of $\eta_x^{\frac{3}{4}}$ to the decimal ·1234 . . . †

For the benefit of those who may wish to carry on the work, I give the following numerical results which have been employed in the preceding arithmetical determinations .—

* It is reduced to *certainty* in the supplemental 3rd section.

† The exact value of the coefficient of $\eta_x^{\frac{3}{4}}$, left blank in the text, is proved in section 3 to be $\frac{10}{81}$, i e , the recurring decimal $\dot{1}2345679\dot{0}$

$$\frac{E_8(E_8-1)}{1\ 2} = 6153473687194529702895764001115884685871706$$

$$\frac{E_7(E_7-1)(E_7-2)}{1.2\ 3} = 979509444484142161376072006375 20$$

$$\frac{E_6(E_6-1)(E_6-2)(E_6-3)}{1\ 2\ 3\ 4} = 117302430235229583 8445$$

$$\frac{E_5(E_5-1)(E_5-2)(E_5-3)(E_5-4)}{1.2\ 3\ 4\ 5} = 5552272910184$$

$$\frac{E_4(E_4-1)(E_4-2)(E_4-3)(E_4-4)(E_4-5)}{1.2\ 3\ 4.5\ 6} = 12271512$$

$$\frac{E_3(E_3-1)(E_3-2)(E_3-3)(E_3-4)(E_3-5)(E_3-6)}{1\ 2\ 3\ 4\ 5\ 6\ 7} = 792$$

$$\eta_5 \div \eta_4 = 24\ 33333333\ .$$
$$\eta_6 \div \eta_5 = 466{\cdot}54794520\ .\ \ .$$
$$\eta_7 \div \eta_6 = 204951{\cdot}34925714\ \ .$$
$$\eta_8 \div \eta_7 = 41881671184\ 54776412\ .\ .\ .$$
$$\eta_9 \div \eta_8 = 1754062953159389842293{\cdot}346657805\ .\ \ .$$

$$\sqrt{\eta_4} = 4\ 24264068\ \ .$$
$$\sqrt{\eta_5} = 20\ 92844819\ \ .$$
$$\sqrt{\eta_6} = 452\ 04866994\ .\ .$$
$$\sqrt{\eta_7} = 204649\ 45227877\ .\ .\ .$$
$$\sqrt{\eta_8} = 41881534751\ 051659567667\ .\ .\ .$$

Finally, it is interesting to find the asymptotic value of h_x and η_x (the halves of the sharpened Hamiltonian and of the Hypothenusal Numbers), which are ultimately in a ratio of equality to each other, in terms of x. Obviously each of these is ultimately in a ratio of equality with M^{2^x}, where M is a constant to be determined.

Let

$$M = 10^{2^a} \quad \text{and} \quad u_x = 10^{2^{x+a}}.$$

Then, for finite values of x, remembering that (in the preceding notation)

$$p < q^2 \quad \text{and} \quad \lambda > \mu^2,$$

between the corresponding terms of the two series

$$= \tfrac{1}{2},\ 1,\ 3,\ 18,\ 438,\ 204348,\ 41881398318,\ .\ .\ .\ .\ ,$$
$$= 2,\ 3,\ 6,\ 24,\ 462,\ 204810,\ 41881603128,\ .\ .\ \ .$$

By means of this formula, writing for u_x corresponding values of η and h, and retaining so much of the two corresponding determinations of α as is common to both, we can find α precisely to any desired number of places of decimals, as shown in the following Table, in which 18 and 24 are taken as the terms of place zero in the respective series.

$u_x =$ 18,	438,	204348,	41881398318,
$\alpha =$ ·32,	·401,	·4088,	·4089863 . .
$u_x =$ 24,	463,	204810,	41881603128,
$\alpha =$ ·46,	·413,	·4090,	·4089866 . .

Hence, if we now change the origin, taking $\frac{1}{2}$ and 2 as the zero terms, we have approximately

$$M^{2^{x+3}} = 10^{2^{x+\alpha}}$$

and

$$8 \log M = 2^{\cdot 408986},$$

which gives

$$M = 1{\cdot}4654433 \quad .^*$$

As a verification, since $2^3 = 8$, $(1{\cdot}46544)^8$ should lie between 18 and 24; and, as a matter of fact, a rough calculation gives

$$\begin{aligned}(1\ 46544)^2 &= 2\ 1473 \ . \quad ,\\ (2{\cdot}1473)^2 &= 4\ 608 \ \ldots,\\ (4{\cdot}608)^2 &= 21{\cdot}234 \ \ldots,\end{aligned}$$

which is about midway between the two limits.—J J. S.

§ 2.—*Proof of the Formula for the Successive Determination of each in turn of* HAMILTON'S *Numbers from its Antecedents.*

Let

$$\begin{aligned}1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots &= F_0(x),\\ 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \ldots &= F_1(x),\\ 6x^2 + 15x^3 + 29x^4 + 49x^5 + 76x^6 + \ldots &= F_2(x),\\ 36x^3 + 210x^4 + 804x^5 + 2449x^6 + \ldots &= F_3(x),\\ \ldots \quad \ldots \quad \ldots \quad \ldots &\ldots,\end{aligned}$$

where the coefficients of the various powers of x are the numbers set out in the triangular Table at the commencement of this paper.

If, in general, we write

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

the coefficients of $F_{n+1}(x)$, expressed in terms of those of $F_n(x)$, are as follows :—

* See Note 1, p 312

$$a_{n+1} = b_n + \frac{a_n(a_n+1)}{1.2}$$

$$b_{n+1} = c_n + a_n b_n + \frac{a_n(a_n+1)(2a_n+1)}{1.2.3}$$

$$c_{n+1} = d_n + a_n c_n + \frac{a_n(a_n+1)}{1.2} b_n + \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1.2.3.4}.$$

$$\cdots \quad \cdots \quad \cdots$$

Now

$$(1-x)^{-a_n} = 1 + a_n x + \frac{a_n(a_n+1)}{1.2}x^2 + \frac{a_n(a_n+1)(a_n+2)}{1.2.3}x^3 + \ldots,$$

when multiplied by

$$F_n(x) = a_n x^n + b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots,$$

gives

$$\begin{aligned}(1-x)^{-a_n}F_n(x) = a_n x^n &+ b_n x^{n+1} + c_n x^{n+2} + d_n x^{n+3} + \ldots \\ &+ a_n^2 x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ &+ \frac{a_n^2(a_n+1)}{1.2}x^{n+2} + \frac{a_n(a_n+1)}{1.2}b_n x^{n+3} + \ldots \\ &+ \frac{a_n^2(a_n+1)(a_n+2)}{1.2.3}x^{n+3} + \ldots \\ &+ \ldots\end{aligned}$$

Comparing this with

$$\begin{aligned}F_{n+1}(x) = b_n x^{n+1} &+ c_n x^{n+2} + d_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)}{1.2}x^{n+1} + a_n b_n x^{n+2} + a_n c_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)(2a_n+1)}{1.2.3}x^{n+2} + \frac{a_n(a_n+1)}{1.2}b_n x^{n+3} + \ldots \\ &+ \frac{a_n(a_n+1)(a_n+2)(3a_n+1)}{1.2.3.4}x^{n+3} + \ldots \\ &+ \ldots,\end{aligned}$$

we see that the difference of the two expressions is

$$a_n x^n + \frac{(a_n-1)a_n}{1.2}x^{n+1} + \frac{(a_n-1)a_n(a_n+1)}{1.2.3}x^{n+2} + \frac{(a_n-1)a_n(a_n+1)(a_n+2)}{1.2.3.4}x^{n+3} + \ldots$$

which is equal to

$$x^{n-1}(1-x)^{-(a_n-1)} - x^{n-1}(1-x).$$

Thus

$$F_{n+1}(x) = (1-x)^{-a_n}F_n(x) - x^{n-1}(1-x)^{-a_n+1} + x^{n-1}(1-x).*$$

Multiplying this equation by $(1-x)^{s_{n+1}}$, where

$$s_{n+1} = a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n,$$

* See Note 2, p. 312.

we obtain

$$(1-x)^{s_{n+1}} F_{n+1}(x) = (1-x)^{s_n} F_n(x) + x^{n-1}(1-x)^{s_{n+1}+1} - x^{n-1}(1-x)^{s_n+1},$$

which gives, when we write successively $n-1$, $n-2$, $n-3$, . . . 0 in the place of n,

$$(1-x)^{s_n} F_n(x) = (1-x)^{s_{n-1}} F_{n-1}(x) + x^{n-2}(1-x)^{s_n+1} - x^{n-2}(1-x)^{s_{n-1}+1},$$
$$(1-x)^{s_{n-1}} F_{n-1}(x) = (1-x)^{s_{n-2}} F_{n-2}(x) + x^{n-3}(1-x)^{s_{n-1}+1} - x^{n-3}(1-x)^{s_{n-2}+1};$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(1-x)^{s_1} F_1(x) = (1-x)^{s_0} F_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1}.$$

Hence, by addition of these n equations, we find

$$(1-x)^{s_n} F_n(x) = (1-x)^{s_0} F_0(x) + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)^{s_0+1}$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where it has been assumed that it is possible to assign to s_0 (previously undefined) such a value as will make the last of the above n equations, viz.,

$$(1-x)^{s_1} F_1(x) = (1-x)^{s_0} F_0(x) + x^{-1}(1-x)^{s_1+1} - x^{-1}(1-x)^{s_0+1},$$

identically true. That this can be done is obvious, for, if in that equation we write for $F_1(x)$, $F_0(x)$, and s_1 their values, viz.,

$$F_1(x) = (1-x)^{-2} - 1, \qquad F_0(x) = (1-x)^{-1}, \qquad \text{and } s_1 = a_0 = 1,$$

then, on making $s_0 = 0$, the equation becomes

$$(1-x)^{-1} - (1-x) = (1-x)^{-1} + x^{-1}(1-x)\,(1-x-1)$$

Thus the general value of $F_n(x)$ is given by the equation

$$(1-x)^{s_n} F_n(x) = (1-x)^{-1} + x^{n-2}(1-x)^{s_n+1} - x^{-1}(1-x)$$
$$+ x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

which is equivalent to

$$(1-x)^{s_n} F_n(x) - (1-x)^{-1} + x^{-1}(1-x) - x^{n-1}(1-x)^{s_n+1}$$
$$= x^{n-2}(1-x)^{s_n+2} + x^{n-3}(1-x)^{s_{n-1}+2} + x^{n-4}(1-x)^{s_{n-2}+2} + \ldots + x^{-1}(1-x)^{s_1+2},$$

where, a_0, a_1, a_2, a_3, . . . being the Hypothenusal Numbers 1, 2, 6, 36, . . . we have

$$s_1 = a_0 = 1,$$
$$s_2 = a_0 + a_1 = 3,$$
$$s_3 = a_0 + a_1 + a_2 = 9,$$
$$. \quad . \quad . \quad ,$$

i.e., the successive values of $s_n + 2$ are the Hamiltonian Numbers 3, 5, 11, 47 . . .

Now $F_n(x) = a_n x^n + \ldots$, so that the coefficient of x^n in $(1-x)^{s_n} F_n(x)$ is the same as the coefficient of x^n in $F_n(x)$, viz, a_n. Consequently, equating coefficients of x^n on each side of the equation just obtained, we find

$$a_n - 1 + (s_n + 1) = \frac{(s_n + 2)(s_n + 1)}{1\ 2} - \frac{(s_{n-1} + 2)(s_{n-1} + 1) s_{n-1}}{1\ 2\ 3}$$
$$+ \quad .$$
$$+ (-)^{n+1} \frac{(s_1 + 2)(s_1 + 1) \quad (s_1 + 2 - n)}{1.2\ 3 \quad (n+1)}$$

Remembering that

$$a_n + s_n = s_{n+1},$$

if we call the Hamiltonian Number $s_n + 2$, H_n, the above relation may be written thus:

$$H_{n+1} - 2 = \frac{H_n(H_n - 1)}{1.2} - \frac{H_{n-1}(H_{n-1} - 1)(H_{n-1} - 2)}{1\ 2\ 3}$$
$$+ \frac{H_{n-2}(H_{n-2} - 1)(H_{n-2} - 2)(H_{n-2} - 3)}{1\ 2\ 3\ 4}$$
$$- \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ (-)^{n+1} \frac{H_1(H_1 - 1)(H_1 - 2) \ldots (H_1 - n)}{1.2.3 \ldots (n+1)}.$$

To obtain Professor SYLVESTER's modification of this formula given in the preceding portion of this memoir, we multiply the equation from which it was obtained by $1 - x$ before proceeding to equate coefficients. Thus we have to equate coefficients of x^n on both sides of

$$(1-x)^{s_n+1} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{s_n+2}$$
$$= x^{n-2}(1-x)^{s_n+3} + x^{n-3}(1-x)^{s_{n-1}+3} + x^{n-4}(1-x)^{s_{n-2}+3} + \ldots + x^{-1}(1-x)^{s_1+3}.$$

Or, writing

$$s_n + 3 = E_n,$$

we equate coefficients on both sides of

$$(1-x)^{E_n-2} F_n(x) - 1 + x^{-1}(1-x)^2 - x^{n-1}(1-x)^{E_n-1}$$
$$= x^{n-2}(1-x)^{E_n} + x^{n-3}(1-x)^{E_{n-1}} + x^{n-4}(1-x)^{E_{n-2}} + \ldots x^{-1}(1-x)^{E_1}.$$

This equation is easily transformed into

$$\begin{aligned}(1-x)^{E_0}+x(1-x)^{E_1}+x^2(1-x)^{E_2}+\ \ldots+x^n(1-x)^{E_n}\\ =1-2x+x^2(1-x)^{E_n-2}F_n(x)-x^{n+1}(1-x)^{E_n-1},\end{aligned}$$

from which, as Professor SYLVESTER has pointed out in this memoir, by equating coefficients of all powers of x from 0 to n, we can obtain the successive values of E_n.

The general formula

$$1-E_{n-1}+\frac{E_{n-2}(E_{n-2}-1)}{1\ 2}-\ .\quad+(-)^n\frac{E_0(E_0-1)\ .\quad(E_0-n+1)}{1\ 2\ .\ \ .\ n}=0$$

arises from equating the coefficients of x^n.—J. H.*

§ 3.† *Sequel to the Asymptotic Theory contained in* § 1.

The relation

$$p=q^2-\tfrac{2}{3}q^{\frac{3}{2}},\text{ etc.}$$

previously obtained supplies only the two first terms of the remarkable asymptotic development

$$\frac{q^2-p}{q}=\tfrac{2}{3}\left(q^{\frac{1}{2}}+q^{\frac{1}{4}}+q^{\frac{1}{8}}+\quad .+q^{(\frac{1}{2})^i}\right)+\Xi,$$

where i is any assigned integer and Ξ is of a lower order of magnitude than the lowest power of q in the series which precedes it This may be easily established as follows :—

By the scale of relation proved in the preceding section we have

$$\begin{aligned}p&=q^2-\tfrac{2}{3}r^3+\frac{s^4}{3}+\ldots\\ &=q^2-\tfrac{2}{3}r^3+\text{ terms whose maximum order is that of } r^2.\end{aligned}$$

Let, now,

$$p=q^2-\tfrac{2}{3}q^{\frac{3}{2}}-\tfrac{2}{3}hq^{\alpha}-\tfrac{2}{3}kq^{\beta}-\tfrac{2}{3}lq^{\gamma}\ \ldots\ ;$$

therefore

$$q=r^2-\tfrac{2}{3}r^{\frac{3}{2}}-\tfrac{2}{3}hr^{\alpha}-\tfrac{2}{3}kr^{\beta}-\tfrac{2}{3}lr^{\gamma}\ .\quad .$$

and

$$\begin{aligned}p&=q^2-\tfrac{2}{3}r^3\left(1-r^{-\frac{1}{2}}-hr^{\alpha-2}-kr^{\beta-2}-lr^{\gamma-2}\ \ldots\right)+\ldots\\ &\qquad-\tfrac{2}{3}hr^{2\alpha}-\tfrac{2}{3}kr^{2\beta}-\tfrac{2}{3}lr^{2\gamma}\ \ldots\\ &=q^2-\tfrac{2}{3}r^3+\tfrac{2}{3}\left(r^{\frac{5}{2}}+hr^{\alpha+1}+kr^{\beta+1}+lr^{\gamma+1}+\ldots\ \right)+\ldots\\ &\qquad-\tfrac{2}{3}hr^{2\alpha}-\tfrac{2}{3}kr^{2\beta}-\tfrac{2}{3}lr^{2\gamma}-\quad\ldots\ .\end{aligned}$$

Therefore

$$h=1,\qquad k=1,\qquad l=1,\qquad m=1,\qquad\ldots$$

$$2\alpha=\tfrac{5}{2},\qquad 2\beta=1+\alpha,\qquad 2\gamma=1+\beta,\qquad 2\delta=1+\gamma,\qquad\ldots$$

i.e.,

$$\alpha=\tfrac{5}{4},\qquad\beta=\tfrac{9}{8},\qquad\gamma=\tfrac{17}{16},\qquad\delta=\tfrac{33}{32},\qquad\ldots$$

* See Note 3, p 312

† Received July 28, 1887

and thus

$$p = q^2 - \tfrac{2}{3}q\,(q^{\frac{1}{2}} + q^{\frac{1}{4}} + q^{\frac{1}{8}} + q^{\frac{1}{16}} + \qquad + q^{(\frac{1}{2})^i}) + \Xi,$$

as was to be shown.*

* This theorem may be rigorously demonstrated, and reduced to a more precise analytical form, as follows —

For the sake of brevity, we may call $-p/q + q$ the relative deficiency of p, and denote it by Δ

First it may be noticed that, if in the equation

$$\mathrm{F}(q) = \Sigma_0^\infty (q^{2^{-s}} - q^{-2^{-s}})$$

we write $\log q = k$,

$$\mathrm{F}(q) = 2\left(k + \frac{k^3}{1\ 2\ 3\,.\,7} + \frac{k^5}{1\ 2\ 3\,.\,4\ 5\ 31} + \frac{k^7}{1\ 2\ 3\ 4\ 5\ 6\ 7\,.\,127} + \,.\,.\right),$$

which is always convergent

Moreover, the value of $\mathrm{F}(q)$ may be calculated for any given value of q within close limits For, if we call U the right-hand branch of the series in q, beginning with $z - z^{-1}$, the terms of U will easily be seen to lie between those of two geometrical series of which $z - z^{-1}$ is the first term, and of one of which $\frac{1}{2}$, and of the other $(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^{-1}$, is the common ratio.

Hence U is intermediate between $2\,(z^2 - 1)/z$ and $(z^2 - 1)\ (z + 1)/z\,(z - z^{\frac{1}{2}} + 1)$

[The difference between these limits, it may be parenthetically observed, is

$$(z - z^{-1})\,\frac{(z^{\frac{1}{2}} - z^{-\frac{1}{2}})^2}{z^{\frac{1}{2}} - 1 + z^{-\frac{1}{2}}},$$

which, when z is nearly unity (the limit to which $q^{(\frac{1}{2})^r}$ converges), is nearly equal to $\frac{1}{16}(z - z^{-1})^3$, *i.e.*, if $z = 1 + \tau$, the difference between the limits (for τ small) is very near to $\tau^3/2$]

Now on p 309 (*post*) it is shown that $\sqrt{q} - r + s - \frac{2}{3}\sqrt{r} = \epsilon$, and that, when the rank of q is taken indefinitely great, ϵ converges to $\frac{1}{4}$. Hence ϵ always lies between finite limits

[For, in general, x being any one of a series of increasing numbers, and $\psi(x)$ a function of x which is always finite for finite values of x, but ultimately converges to c, by taking for x a value of L sufficiently great, we make the series of terms for $x > \mathrm{L}$ intermediate between $c + \delta$ and $c - \delta$, where δ is any assigned positive quantity, and consequently, if μ, ν, are the greatest and least values of $\psi(x)$ when x does not exceed L, the greater of the two values, $c + \delta$, μ, and the lesser of the two, $c - \delta$, ν, will be superior and inferior limits to the value of $\psi(x)$ for all values of x]

Hence, writing

$$\begin{aligned}
&\sqrt{p} - q + r - \tfrac{2}{3}\sqrt{q} = \epsilon_1,\\
&\sqrt{q} - r + s - \tfrac{2}{3}\sqrt{r} = \epsilon_2,\\
&\sqrt{r} - s + t - \tfrac{2}{3}\sqrt{s} = \epsilon_3,\\
&\quad .\quad .\quad .\quad .\quad .\quad ,\\
&\sqrt{6} - 3 + 2 - \tfrac{2}{3}\sqrt{3} = \epsilon_{x-1},
\end{aligned}$$

we obtain, by summation,

$$\sqrt{p} - q + \tfrac{1}{3}(\sqrt{q} + \sqrt{r} + \sqrt{s} + \ \ .\,. + \sqrt{6}) = \Sigma\epsilon - 2 + \tfrac{2}{3}\sqrt{3},$$

and, consequently,

$$\sqrt{p} - q + \tfrac{1}{3}(\sqrt{q} + \sqrt{r} + \sqrt{s} + .\,.\ \ + \sqrt{6}) = \rho x,$$

where ρ is always a finite quantity lying between determinable limits. But again (p. 307)—

$$p = (q - \theta\sqrt{q})^2,$$

where θ (whose ultimate value is $\frac{1}{3}$) is always a proper fraction. Hence

$$\frac{q^2 - p}{q} = 2\,(q - \sqrt{p}) - \theta^2$$

It is interesting to notice that the formula *apparently* remains arithmetically true for *finite* values of p and q, provided that q is not less than 24, when we replace each

Hence, from what has been shown above,

$$\frac{q^2-p}{q} = \tfrac{2}{3}(\sqrt{q}+\sqrt{r}+\sqrt{s}+\quad+\sqrt{6}) - 2\rho' x$$

In this equation we may write

$$\left.\begin{array}{l}\sqrt{r} = q^{\frac{1}{2}} + k_1 \\ \sqrt{s} = q^{\frac{1}{4}} + k_2 \\ \sqrt{t} = q^{\frac{1}{8}} + k_3 \\ \quad\cdot\ \cdot\end{array}\right\}(x-3 \text{ equations}),$$

where $k_1, k_2, k_3, \ldots$ are all of them finite (and, as a matter of fact, of no consequence for our immediate object, positive proper fractions) For, ultimately,

$$k_1 = \sqrt{r} - q^{\frac{1}{2}} = \frac{r - q^{\frac{1}{2}}}{r^{\frac{1}{2}} + q^{\frac{1}{2}}} = \frac{\theta_1 r^{\frac{1}{2}}}{r^{\frac{1}{2}} + q^{\frac{1}{2}}} = \tfrac{1}{2}\theta_1 = \tfrac{1}{6} \text{ (see p 307)},$$

and consequently the finiteness of each k is a direct inference from the general principle previously applied in the case of the ϵ's

Applying this result to the equation previously given, it follows that $q^{\frac{1}{2}} + q^{\frac{1}{4}} + \quad + q^{(\frac{1}{2})^{x-2}} = \frac{3}{2}\Delta - vx$ (where v is finite) $= \mathrm{F}(q) + (q^{-\frac{1}{2}} + q^{-\frac{1}{4}} + \quad + q^{-(\frac{1}{2})^{x-2}}) - \{(z - z^{-1}) + (z^{\frac{1}{2}} - z^{-\frac{1}{2}}) + (z^{\frac{1}{4}} - z^{-\frac{1}{4}}) + \quad\}$, where z lies between 1 and 2

The series of negative powers of q is obviously less than x, and the z-series, which follows it, is less than the finite quantity $2(z - 1/z)$, i e, $< 2(2 - \frac{1}{2})$ Hence $\frac{3}{2}\Delta = \mathrm{F}(q) + \Theta x$, where Θ is a number lying between fixed limits, and x, the rank of q, is of the same order of magnitude as log log q This equation contains as a consequence the asymptotic theorem to be proved, for, using i to denote any positive integer,

$$\tfrac{3}{2}\Delta - \Sigma_1^i\, q^{(\frac{1}{2})^i} = \mathrm{F}(q) - \Sigma_1^i\, q^{(\frac{1}{2})^i} - \Theta x = q^{(\frac{1}{2})^{i+1}} + \sum_{s=i+2}^{s=\infty}(q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}) - \Sigma_0^i\, 1/q^{(\frac{1}{2})^{i+1}} - \Theta x.$$

Hence, remembering that x is of the same order of magnitude as log log q, and that

$$\sum_{s=i+2}^{s=\infty}(q^{(\frac{1}{2})^s} - q^{-(\frac{1}{2})^s}) < 2(q^{(\frac{1}{2})^{i+2}} - q^{-(\frac{1}{2})^{i+2}}),$$

which is of a lower order of magnitude than $q^{(\frac{1}{2})^{i+1}}$, it follows that $\frac{3}{2}\Delta - \Sigma_1^i\, q^{(\frac{1}{2})^i}$ for all values of i is ultimately in a ratio of equality with $q^{(\frac{1}{2})^{i+1}}$, which is the theorem to be proved

We have thought it desirable to obtain the formula $\frac{3}{2}\Delta = \mathrm{F}q + \Theta x$ for its own sake, but, so far as regards the proof in question, that might be obtained more expeditiously from the expression given for $3\Delta/2 - vx$ without introducing the series $\mathrm{F}q$

It is easy to ascertain the ultimate value to which Θ converges In the first place, the series of fractions $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ to $x - 2$ terms (where x is the rank of q) may be shown to be always finite, and consequently, when divided by x, converges to zero.

For we know that $(p - q) > (q - r)^2 > (r - s)^4 \;\ldots\; > (6 - 3)^{2^{x-3}}$ Hence the last term of the series $q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}}\ldots$ (viz., $q^{(\frac{1}{2})^{x-2}}$) > 3 Hence the finite series $1/q^{\frac{1}{2}} + 1/q^{\frac{1}{4}} + 1/q^{\frac{1}{8}} + \ldots$ for a double *à fortiori* reason is less than the infinite geometrical series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots < \frac{1}{2}$.

[In fact, from § 1 (p. 299) it may easily be shown that the last term of the series $q^{\frac{1}{2}}, q^{\frac{1}{4}}, q^{\frac{1}{8}} \ldots > \mathrm{M}^4 > (1\,465)^4 > 4\,608$, so that the sum is really less than $\frac{1}{3\,608}$]

Hence, retracing the steps by which Θ has been obtained, and observing that ρ' differs from ρ by a finite multiple of $1/x$, we have ultimately $\Theta = v = k - 3\rho' = k - 3\rho = k - 3\epsilon = \frac{1}{6} - \frac{3}{4} = -\frac{7}{12}$. If, then (using u_x to denote the half of the sharpened x^{th} Hamiltonian number), we write $u_x - 1/u_x = v_x$, and understand by $\mathrm{G}(t - 1/t)$ the infinite series $(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + (t^{\frac{1}{4}} - t^{-\frac{1}{4}}) + (t^{\frac{1}{8}} - t^{-\frac{1}{8}}) + \ldots$, it is easily seen that the *principal* part of $\sqrt{(v_{x+1})}$, regarded as a function of v_x and x, is $v_x - \frac{1}{3}\mathrm{G}v_x + \frac{7}{36}x$.

term in the formula by its integer portion, and in the series on the right stop at the term immediately preceding the first term for which

$$\mathrm{E}q^{(\frac{1}{i})} = 1.$$

Thus, when

$$p = 462 \quad \text{and} \quad q = 24,$$

we have

$$\mathrm{E}\left(\frac{q^2 - p}{q}\right) = \mathrm{E}\left(\frac{576 - 462}{24}\right) = \mathrm{E}\left(\frac{114}{24}\right) = 4,$$

and

$$\mathrm{E}\left\{\tfrac{2}{3}(\mathrm{E}q^{\frac{1}{2}} + \mathrm{E}q^{\frac{1}{4}})\right\} = \mathrm{E}\left\{\tfrac{2}{3}(4 + 2)\right\} = 4.$$

So also, when

$$p = 41881603128, \qquad q = 204810,$$

$$\mathrm{E}\left(\frac{q^2 - p}{q}\right) = 319,$$

and

$$\mathrm{E}\left\{\tfrac{2}{3}(\mathrm{E}q^{\frac{1}{2}} + \mathrm{E}q^{\frac{1}{4}} + \mathrm{E}q^{\frac{1}{8}} + \mathrm{E}q^{\frac{1}{16}})\right\} = \mathrm{E}\left\{\tfrac{2}{3}(452 + 21 + 4 + 2)\right\} = \mathrm{E}\left(\tfrac{958}{3}\right) = 319.$$

But, if we had included the term $\mathrm{E}q^{\frac{1}{32}}$, the result would have been

$$\mathrm{E}\left\{\tfrac{2}{3}(452 + 21 + 4 + 2 + 1)\right\} = 320.$$

Again, when

$$p = 3076736843548289379224261404637538760216584,$$

$$q = 1754062953145429399086,$$

$$\mathrm{E}\left(\frac{q^2 - p}{q}\right) = 27921159919,$$

and

$$\mathrm{E}\left\{\tfrac{2}{3}(\mathrm{E}q^{\frac{1}{2}} + \mathrm{E}q^{\frac{1}{4}} + \mathrm{E}q^{\frac{1}{8}} + \mathrm{E}q^{\frac{1}{16}} + \mathrm{E}q^{\frac{1}{32}} + \mathrm{E}q^{\frac{1}{64}})\right\}$$
$$= \mathrm{E}\left\{\tfrac{2}{3}(41881534751 + 204649 + 452 + 21 + 4 + 2)\right\} = 27921159919.^*$$

We will now proceed to consider afresh the asymptotic development of any *Hypothenusal* Number $p - q$ in terms of its antecedent $q - r$, and to reduce to apodictic certainty results which in the first section were partly obtained by observation. It has already been shown in that section that

$$p > q^2 - \tfrac{128}{81} q^{\frac{3}{2}} - \frac{q}{2}$$

when p is not lower than 204810 in the scale 2, 3, 6, 24, 462, 204810, . . . , *i.e.*, when q is not less than 462.

Hence

$$p > q^2 - 2q^{\frac{3}{2}} + q + \left(\tfrac{34}{81} q^{\frac{3}{2}} - \tfrac{3}{2} q\right),$$

* [illegible] must be understood merely to affirm the *possibility* of the theorem being true, and to [illegible] opinion on the strength of the presumption raised that it is so.

or, since $\frac{84}{81} q^{\frac{3}{2}} - \frac{3}{2} q$ is a positive quantity,

$$p > (q - \sqrt{q})^2,$$

at all events when $q =$ or > 462.

It will be found also on trial that this formula remains true for all the values of q inferior to 462.

Thus

$$462 > (24 - \sqrt{24})^2,$$
$$24 > (\ 6 - \sqrt{6})^2,$$
$$6 > (\ 3 - \sqrt{3})^2,$$
$$3 > (\ 2 - \sqrt{2})^2.$$

Hence, universally,

$$p > (q - \sqrt{q})^2.*$$

But we know that

$$p < q^2$$

We may therefore write

$$p = (q - \theta \sqrt{q})^2,$$

where θ is some quantity between 0 and 1.

Similarly,

$$q = (r - \theta_1 \sqrt{r})^2,$$
$$r = (s - \theta_2 \sqrt{s})^2,$$
$$\cdot \quad \cdot \quad \quad \cdot \quad \cdot \quad \cdot,$$

where θ_1, θ_2, . . . are also positive fractions.

When p and q become infinite,

$$\frac{q^2 - p}{q^{\frac{3}{2}}} = \tfrac{2}{3} = 2\theta.$$

Hence the ultimate value of θ is $\frac{1}{3}$. Similarly, θ_1, θ_2, . . . all of them converge to the value $\frac{1}{3}$.

This agrees with the result previously demonstrated (p. 295), and is the starting point of all that follows.

We know that letters p, q, r, s, . . ., being used to denote the halves of the augmented Hamiltonian Numbers, they are connected by the scale of relation

$$p = \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2\,.\,3} + \mathrm{S} - \mathrm{T},$$

* Had this inequality been true only for values of q sufficiently great, it would have been enough for the purposes of the text.

where

$$S = \frac{s(2s-1)(2s-2)(2s-3)}{2\ 3.4}$$

and T stands for the remaining terms, involving

$$t,\ u,\ v,\ \ldots$$

Considering

$$q,\ r,\ s,\ t,\ \ldots$$

to be of the order

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \ldots,$$

we may reject the term $\frac{1}{2}$, which is of zero order, and write

$$p = q^2 - \tfrac{2}{3}r^3, \qquad -\frac{q}{2} + r^2 - \frac{r}{3} + S - T.$$

Hence, rejecting terms of order less than $\frac{3}{2}$ (which have, however, to be retained in obtaining the subsequent approximations),

$$\left.\begin{array}{r}(p-q)\\ -(q-r)^2\end{array}\right\} = \left\{\begin{array}{ll} q^2 - \tfrac{2}{3}r^3; & -\tfrac{3}{2}q + r^2 - \dfrac{r}{3} + S - T \\ -q^2 + 2qr; & -r^2 \end{array}\right.$$

$$= (2qr - \tfrac{2}{3}r^3);$$

i.e.,

$$(p-q) - (q-r)^2 = \tfrac{4}{3}q^{\frac{3}{2}}$$

when q is infinite.

Again, writing for S its expanded value, viz.,

$$\frac{s^4}{3} - s^3 + \tfrac{11}{12}s^2 - \frac{s}{4},$$

we have

$$\left.\begin{array}{r}(p-q)\\ -\ (q-r)^2\\ -\tfrac{4}{3}(q-r)^{\frac{3}{2}}\end{array}\right\} = \left\{\begin{array}{ll} 2qr - \tfrac{2}{3}r^3 - \tfrac{4}{3}q^{\frac{3}{2}} & \text{Order } \tfrac{3}{2}, \\ +\,2q^{\frac{1}{2}}r - \tfrac{3}{2}q + \tfrac{1}{3}s^4 & \quad ,,\quad 1, \\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2 - \dfrac{r}{3} - s^3 + \tfrac{11}{12}s^2 - \tfrac{1}{4}s - T & \quad ,,\quad < 1, \end{array}\right.$$

rejecting the terms $q^{-\frac{3}{2}}r^3$, $q^{-\frac{5}{2}}r^4$, ... in the expansion of $(q-r)^{\frac{3}{2}}$ because the order of none of them is superior to zero.

We now write

$$q = (r - \theta_1\sqrt{r})^2,$$

so that

$$2qr - \tfrac{2}{3}r^3 - \tfrac{4}{3}q^{\frac{3}{2}} = (2r^3 - 4\theta_1 r^{\frac{5}{2}} + 2\theta_1^{\,2}r^2) - \tfrac{2}{3}r^3 - (\tfrac{4}{3}r^3 - 4\theta_1 r^{\frac{5}{2}} + 4\theta_1^{\,2}r^2 - \tfrac{4}{3}\theta_1^{\,3}r^{\frac{3}{2}})$$
$$= -2\theta_1^{\,2}r^2 + \tfrac{4}{3}\theta_1^{\,3}r^{\frac{3}{2}}.$$

Hence

$$\left.\begin{array}{r}(p-q)\\ -(q-r)^2\\ -\tfrac{4}{3}(q-r)^{\frac{3}{2}}\end{array}\right\} = \left\{\begin{array}{ll} -2\theta_1^{\,2}r^2 + 2q^{\frac{1}{2}}r - \tfrac{3}{2}q + \tfrac{1}{3}s^4 & \text{Order } 1, \\ +\,\tfrac{4}{3}\theta_1^{\,3}r^{\frac{3}{2}} - s^3 & \quad ,,\quad \tfrac{3}{4}; \\ -\tfrac{1}{2}q^{-\frac{1}{2}}r^2 - \dfrac{r}{3} + \tfrac{11}{12}s^2 - \tfrac{1}{4}s - T & \quad ,,\quad < \tfrac{3}{4}. \end{array}\right.$$

Since

$$q = r^2 = s^4 \text{ (ultimately)},$$

the terms of Order 1 (which are the only ones with which we have to do at present) are ultimately equal to

$$(-2\theta_1^2 + 2 - \tfrac{3}{2} + \tfrac{1}{3})q,$$

or, giving θ_1 its ultimate value $\frac{1}{3}$, to $\frac{11}{18}q$, or to the same order of approximation to $\frac{11}{18}(q-r)$.

Hence, ultimately,

$$(p-q) = (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r).^*$$

We use this result to obtain a closer approximation to $\sqrt{q}$ than $r - \theta_1\sqrt{r}$, and to find the relation between the general values of θ_1 and θ_2.

Thus, assuming

$$\sqrt{q-r} = r - s + \tfrac{2}{3}\sqrt{r-s} + k,$$

we have, ultimately,

$$\begin{aligned} q - r &= (r-s)^2 + \tfrac{4}{3}(r-s)^{\frac{3}{2}} + (\tfrac{4}{9} + 2k)(r-s) \\ &= (r-s)^2 + \tfrac{4}{3}(r-s)^{\frac{3}{2}} + \tfrac{11}{18}(r-s). \end{aligned}$$

Consequently, as r becomes indefinitely great, k converges to the value $\frac{1}{2}(\frac{11}{18} - \frac{4}{9}) = \frac{1}{12}$.

Now

$$\sqrt{q-r} = \sqrt{q} - \tfrac{1}{2}\frac{r}{\sqrt{q}} \;\ldots = \sqrt{q} - \tfrac{1}{2} \text{ ultimately},$$

and similarly

$$\sqrt{r-s} = \sqrt{r} - \tfrac{1}{2} \text{ ultimately}.$$

Hence, ultimately,

$$\sqrt{q} = r - s + \tfrac{2}{3}\sqrt{r} + \tfrac{1}{12} + \tfrac{1}{2} - \tfrac{1}{3} = r - s + \tfrac{2}{3}\sqrt{r} + \tfrac{1}{4}.$$

We may therefore write

$$\sqrt{q} = r - s + \tfrac{2}{3}\sqrt{r} + \epsilon \text{ (where ultimately } \epsilon = \tfrac{1}{4}).$$

But

$$\sqrt{q} = r - \theta_1\sqrt{r},$$

and therefore

$$\theta_1\sqrt{r} = s - \tfrac{2}{3}\sqrt{r} - \epsilon.$$

Moreover

$$\sqrt{r} = s - \theta_2\sqrt{s},$$

whence it follows that

$$\theta_1\sqrt{r} = \tfrac{1}{3}s + \tfrac{2}{3}\theta_2\sqrt{s} - \epsilon \text{ (where } \epsilon = \tfrac{1}{4} \text{ ultimately)}.$$

* As previously obtained by observation in § 1 (pp. 296, 297) It will, of course, be understood that in the above and similar passages the sign $=$ is to be interpreted to mean "is in a ratio of equality with."

Resuming the development of $(p-q)$ in terms of $(q-r)$, we have

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\ \tfrac{4}{3}\,(q-r)^{\frac{3}{2}} \\ -\tfrac{11}{18}\,(q-r) \end{array}\right\} = \left\{\begin{array}{lll} -2\theta_1 r^2 + 2q^{\frac{1}{2}}r - \tfrac{19}{9}q + \dfrac{s^4}{3} & \text{Order} & 1. \\ +\tfrac{4}{3}\,\theta_1{}^3 r^{\frac{3}{2}} - s^3 & \quad ,, & \tfrac{3}{4}. \\ -\tfrac{1}{2}\,q^{-\frac{1}{2}}r^2 + \tfrac{5}{18}\,r + \tfrac{11}{12}\,s^2 - \dfrac{s}{4} - \mathrm{T} & \quad ,, & < \tfrac{3}{4}. \end{array}\right.$$

The terms of order inferior to $\frac{3}{4}$ are of no value for present purposes, and are only retained for the benefit of those who may wish to carry on the work.

To reduce the terms of Order 1, we write, in succession,

$$q = (r - \theta_1\sqrt{r})^2,$$
$$\theta_1\sqrt{r} = \tfrac{1}{3}s + \tfrac{2}{3}\,\theta_2\sqrt{s} - \epsilon,$$
$$r = (s - \theta_2\sqrt{s})^2.$$

Thus

$$\frac{s^4}{3} - 2\theta_1{}^2 r^2 + 2q^{\frac{1}{2}}r - \tfrac{19}{9}\,q$$

$$= \frac{s^4}{3} - 2\theta_1{}^2 r^2 + 2r^2 - \tfrac{19}{9}\,r^2;\ -2\theta_1 r^{\frac{3}{2}} + \tfrac{38}{9}\,\theta_1 r^{\frac{3}{2}};\ -\tfrac{19}{9}\,\theta_1{}^2 r$$

$$= \frac{s^4}{3} - \frac{r^2}{9} - 2\theta_1{}^2 r^2;\ +\tfrac{20}{9}\,\theta_1 r^{\frac{3}{2}};\ -\tfrac{19}{9}\,\theta_1{}^2 r$$

$$= \frac{s^4}{3} - \frac{r^2}{9} - 2r\left(\frac{s}{3} + \tfrac{2}{3}\,\theta_2\sqrt{s}\right)^2;\ +4\epsilon r\left(\frac{s}{3} + \tfrac{2}{3}\,\theta_2\sqrt{s}\right) + \tfrac{20}{9}\,\theta_1 r^{\frac{3}{2}};\ -2\epsilon^2 r - \tfrac{19}{9}\,\theta_1{}^2 r$$

$$= \frac{s^4}{3} - \tfrac{1}{9}\,(s^4 - 4\theta_2 s^{\frac{7}{2}} + 6\theta_2{}^2 s^3 - 4\theta_2{}^3 s^{\frac{5}{2}} + \theta_2{}^4 s^2) - \tfrac{2}{9}\,s^2\,(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s)$$
$$+\tfrac{4}{9}\,\theta_2 s^{\frac{5}{2}}\,(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s) - \tfrac{2}{9}\,\theta_2{}^2 s\,(s^2 + 4\theta_2 s^{\frac{3}{2}} + 4\theta_2{}^2 s),$$
$$+\tfrac{4}{3}\,\epsilon r s + \tfrac{20}{9}\,\theta_1 r^{\frac{3}{2}};\ +\tfrac{8}{3}\,\epsilon\theta_2 r\sqrt{s} - 2\epsilon^2 r - \tfrac{19}{9}\,\theta_1{}^2 r$$

$$\begin{array}{lll} = \tfrac{4}{3}\,\epsilon r s + \tfrac{20}{9}\,\theta_1 r^{\frac{3}{2}} & \text{Order} & \tfrac{3}{4} \\ + \tfrac{4}{3}\,\theta_2{}^3 s^{\frac{5}{2}} + \tfrac{8}{3}\,\epsilon\theta_2 r\sqrt{s} - \theta_2{}^4 s^2 - 2\epsilon^2 r - \tfrac{19}{9}\,\theta_1{}^2 r & \quad ,, & < \tfrac{3}{4}. \end{array}$$

Hence

$$\left.\begin{array}{r} (p-q) \\ -\quad (q-r)^2 \\ -\ \tfrac{4}{3}\,(q-r)^{\frac{3}{2}} \\ -\tfrac{11}{18}\,(q-r) \end{array}\right\} = \left\{\begin{array}{lll} \tfrac{4}{3}\,\theta_1{}^3 r^{\frac{3}{2}} - s^3 + \tfrac{4}{3}\,\epsilon r s + \tfrac{20}{9}\,\theta_1 r^{\frac{3}{2}} & \text{Order} & \tfrac{3}{4} \\ -\mathrm{T} + \tfrac{4}{3}\,\theta_2{}^3 s^{\frac{5}{2}} + \tfrac{8}{3}\,\epsilon\theta_2 r s^{\frac{1}{2}} & \quad ,, & \tfrac{5}{8} \\ -\tfrac{1}{2}\,q^{-\frac{1}{2}}r^2 + \tfrac{5}{18}\,r + \tfrac{11}{12}\,s^2 - \dfrac{s}{4} - \theta_2{}^4 s^2 - 2\epsilon^2 r - \tfrac{19}{9}\,\theta_1{}^2 r & \quad ,, & < \tfrac{5}{8}. \end{array}\right.$$

Here the terms of Order $\frac{3}{4}$ are ultimately equal to

$$\left(\tfrac{4}{3}\,\theta_1{}^3 - 1 + \tfrac{4}{3}\,\epsilon + \tfrac{20}{9}\,\theta_1\right) q^{\frac{3}{4}},$$

which, when θ_1 and ϵ receive their ultimate values, $\frac{1}{3}$ and $\frac{1}{4}$, becomes

$$\left(\tfrac{4}{81} - 1 + \tfrac{1}{3} + \tfrac{20}{27}\right) q^{\frac{3}{4}} = \tfrac{10}{81}\, q^{\frac{3}{4}}.^*$$

From this it follows immediately that (rejecting terms of an order of magnitude inferior to that $(q-r)^{\frac{3}{4}}$)

$$p - q = (q-r)^2 + \tfrac{4}{3}\,(q-r)^{\frac{3}{2}} + \tfrac{11}{18}\,(q-r) + \tfrac{10}{81}\,(q-r)^{\frac{3}{4}}.$$

The law of the indices in the complete development is easily deduced from the relation

$$p = \tfrac{1}{2} + \frac{q\,(2q-1)}{2} - \frac{r\,(2r-1)\,(2r-2)}{2\,.\,3} + \frac{s\,(2s-1)\,(2s-2)\,(2s-3)}{2\,.\,3\,.\,4} - \ldots$$

The terms carrying the arguments

$$q^2,\ q,\ r^3,\ r^2,\ r,\ s^4,\ s^3,\ s^2,\ s,\ t^5,\ \ldots$$

furnish the indices

$$2,\ 1,\ \tfrac{3}{2},\ 1,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{4},\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{5}{8},\ \ldots,$$

which, arranged in order of magnitude, become

$$2,\ \tfrac{3}{2},\ 1,\ \tfrac{3}{4},\ \tfrac{5}{8},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{1}{4},\ \ldots$$

Thus, calling $p-q$ and $q-r$ y and x respectively, the expansion for y in terms of x will be of the form

$$y = \Sigma \mathrm{A} x^{\frac{2m+1}{2^n}},$$

where n has all values from 0 to ∞, and $2m+1$ does not exceed $n+2$, *i.e.*, m has all positive values from 0 to $n/2$ or $\frac{1}{2}(n+1)$, according as n is even or odd.

But, besides this expressed portion of the development of a Hypothenusal Number, say η_{x+1}, as a function of its antecedent, η_x, there will be another portion, consisting of terms with zero and negative indices of η_x having functions of x for their coefficients, which observation is incompetent to reveal, and with the nature of which we are at present unacquainted. The study of HAMILTON'S Numbers, far from being exhausted, has, in leaving our hands, little more than reached its first stage, and it is believed will furnish a plentiful aftermath to those who may feel hereafter inclined to pursue to the end the thorny path we have here contented ourselves with indicating, which lies so remote from the beaten track of research, and offers an example and suggestion of infinite series (as far as we are aware) wholly unlike any which have previously engaged the attention of mathematicians.

J. J. S. and J. H.

* Agreeing closely with what had been previously found by observation in § 1 (p. 297).

NOTE 1, page 299.—(September 17, 1887.)

It is easy to see that, if δM and $\delta\alpha$ are corresponding errors in the values of M and α respectively,

$$\delta M = (M \log_e M \log_e 2)\, \delta\alpha = (\,38822 \ldots)\, \delta\alpha$$

(since $M = 1{\cdot}46544 \ldots$, $\log_e M = \;38220 \ldots$, and $\log_e 2 = {\cdot}69314 \ldots$).

Hence, $\delta\alpha$ being intermediate between ·0000003 and ·0000006,

δM lies between ·000000116 and ·000000233.

The value of M (the base of the Hamiltonian Numbers) is thus found to be 1·465443 . . . , correct to the last figure inclusive —J. J S.

NOTE 2, page 300.—(September 17, 1887.)

This equation may be obtained more simply from the *fundamental* formula of HAMILTON (middle of above note). It follows from the law of derivation there given that, if we write ${}^1F_n = (1-x)^{-1} F_n - x^n$, and, in general, ${}^{j+1}F_n = (1-x)^{-1}\,{}^jF_n - x^n$, then $F_{n+1} = {}^{a_n}F_n$; and, consequently,

$$F_{n+1} - (1-x)^{-a_n} F_n = -x^n\{1 + (1-x)^{-1} + (1-x)^{-2} + \ldots + (1-x)^{-a_n+1}\}$$
$$= x^{n-1}\{(1-x) - (1-x)^{-a_n+1}\}.\text{—J J. S}$$

NOTE 3, page 303.—(September 19, 1887.)

It is curious to notice the sort of affinity which exists between a form of writing the scale of relation for BERNOULLI's Numbers and that given at p. 289 for HAMILTON's.

If we write $G_0 = 1$, $G_1 = -1$, $G_2 = (-4) B_1$, $G_3 = 0$, $G_4 = (-4)^2 B_2$, $G_5 = 0$, $G_6 = (-4)^3 B_3, \ldots$ then, using β_κ in the same sense as at p. 289, we shall find the scale of relation between the B's (BERNOULLI's Numbers) is given by the equation

$$\sum_{\kappa=0}^{\kappa=i} (-)^\kappa \beta_\kappa i \,.\, G_{i-\kappa} = 0, \quad \textit{provided } i \textit{ is odd.}$$

On striking out the i which intervenes between β_κ and $G_{i-\kappa}$, so as to make the former operate on the latter, the equation becomes that given at p. 289 for the E's, the sharpened numbers of HAMILTON.—J. J. S.

XI. *On Evaporation and Dissociation.—Part V.* A Study of the Thermal Properties of Methyl-Alcohol*

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[PLATES 14–18.]

IN previous memoirs we have given the results of investigations of the thermal properties of ethyl-alcohol, acetic acid, and ethyl oxide (ether). The subject of the present paper is the vapour-pressures, vapour-densities, and expansion of methyl-alcohol; and from these results the heats of vaporisation have been deduced The range of temperature is from $-16°$ to the critical temperature 240°; and the range of pressure from 11 millims. to 60,000 millims.

Preparation of pure methyl-alcohol †—A finely crystallised sample of methyl oxalate was distilled with ammonia; the distillate was rectified, and when partially freed from water was distilled with quicklime. The distillate was again distilled from barium oxide, and then allowed to stand for some weeks over anhydrous copper sulphate; but the boiling-point was found to be by no means constant. It was then distilled six times over small quantities of sodium; and the rise of temperature during the last distillation was less than 0 1°. The boiling-point was 64 85° at 761·9 millims. A series of determinations of vapour-pressure at low temperatures was then made, and it was decided, before employing the alcohol for determinations of vapour-density, to re-distil it It boiled at 64·95°, under the same pressure, 761·9 millims. Preliminary experiments were then carried out, with a view to determining the critical temperature and pressure; but the volume-tube burst, and the experiments were delayed until a new volume-tube had been calibrated. As the boiling-point of the alcohol was not absolutely constant, it was repeatedly fractionated, and the

* Parts I and II. are published in the 'Philosophical Transactions' for 1886 (Part I.), Part III, *ibid.*, 1887; Part IV is published in the 'Transactions of the Chemical Society,' 1886, p 790, Part VI., in the 'Phil. Mag,' vol. 23, 1887, pp. 435–458; vol 24, 1887, pp. 196–212.

† See also Addendum at p. 330.

greater part was obtained, boiling at 64·7°, under 760 millims. pressure. The rise of temperature did not exceed 0·05° during the complete distillation.

The boiling-point of methyl-alcohol has been a subject of much dispute. KOPP found, with three separate samples, 65°, 65°–65 2°, 64·6°–65·2°, PIERRE, 66·3°; MENDELÉEFF, 66°; DELFFS, 66 5°; DUMAS and PELIGOT, 66·5°, LANDOLT, 66·3°; and DITTMAR and STEWART, 64·1°. PERKIN, who dried his alcohol with copper sulphate, found 65·8°–66°, but we found that several weeks' standing over copper sulphate produced little effect. REGNAULT, who dried his alcohol over lime, found 66·78°; and SCHIFF found 64·8° at 763 millims. pressure. There can be little doubt that the last is correct.

Apparatus employed.—The apparatus for determining vapour-pressures at low temperatures was that described in the 'Phil. Trans.,' 1884, p. 37; and in the 'Chem. Soc. Trans.,' 1885, p 42. The vapour-densities at low temperatures were determined by a modification of HOFMANN'S apparatus, whereby pressure, volume, and temperature could be altered at will. This apparatus has been described in our memoir on ether ('Phil. Trans,' 1887, A, p 59). The constants at high temperatures were determined by a modified ANDREWS' apparatus, of which a description is to be found in our paper on ether.

Experimental results.—Before giving experimental results, it should be noted that in every case the temperatures are those of an air thermometer, the pressures refer to the latitude of University College, Bristol (51° 27′ 25″ N.), and to 226 feet above sea-level; they are corrected according to AMAGAT'S experiments ('Comptes Rendus,' vol. 99, p. 1153), and are measured in actual millimetres of mercury.

Vapour-pressures at low temperatures.—The thermometer employed for temperatures below 40° was graduated in tenths of a degree. By reduction of pressure an apparent fall of temperature of 0·25° for 760 millims. was observed. This apparent fall is proportional to the pressure, and correction was applied accordingly. The thermometer was standardised by a determination of the vapour-pressures of water above 0°; and it was assumed that the graduation below 0° was equally regular

TABLE I

Series I				Series II		Series III.			
Pressure	Temperature	Pressure.	Temperature.	Pressure.	Temperature	Pressure	Temperature	Pressure	Temperature
mms.	°	mms	°	mms	°	mms.	°	mms.	°
11 15	−14 61	44 2	6 58	161 2	30 4	27 15	− 1 4	291 5	42 3
13 15	−12 35	55 0	10 16	189 7	33 7	29 6	0 0	334 05	45 5
13 7	−11 94	61 0	12 13	217 8·	36 6	40 2	+ 4·8	384 45	48 75
15 85	−10 02	88 4	18 64	257·1	39·8	41 0	5 0	436 55	51 45
17 05	− 8 70	105 0	21 77	299 1	43 2	52 65	9 3	484 8	53 75
19 05	− 6 89	125 5	25 19	344 3	46 3	68 4	13 95	537 6	56 4
24 05	− 3·19	147 35	28 32	357 2	47 2	82 8	17 45	588 5	58 7
26 3	− 1 98	173 4	31 63	390 5	48 9	103 9	21 4	645 8	60 9
29 85	+ 0 01	200 9	34 70	449 6	52 5	139 8	27 2	699 0	62 8
32 95	1 60	235 1	37 95	494 2	54 8	166 8	30 8	743 2	64 2
36 1	3 25			541 6	56 6	197 1	34 1		
40 65	5·06			590 6	58 7	239 65	38 2		
				640 9	60 9				
				698 4	62 95				
				761 8	65 0				

These results were plotted, and curves drawn through them, showing great regularity. The first two series were determined with substance boiling at 64 85°; and the third, with the alcohol after fractionation.

Vapour-densities at low temperatures.—The weight of the methyl-alcohol taken was not determined directly, but was ascertained from vapour-density determinations at the boiling-point of chlorobenzene under a pressure of 718·95 millims., corresponding to the temperature 130°.

TABLE II.

Temperature.	Pressure.	Volume	P V.	Mean P V
°	mms.	c cs.		
130	320 3	175	56,053	56,040
	339·9	165	56,083	
	373 2	150	55,980	
	406 1	138	56,042	
	466 8	120	56,016	

From these observations the weight was calculated to be 0·0716 gramme.

Temperature	Pressure.	Volume	P V.	Vapour-density. (H=1)
°	mms.	c.cs		
60	264 5	173 7	45,944	16 13
(Alcohol vapour, P = 350 3 mms)	265 7	172 4	45,807	16 17
	292 0	157 0	45,844	16 16
	317·5	144 25	45,800	16 18
	360 6	126 6	45,652	16 23
	408 0	111 05	45,309	16 35
	474 3	95 0	45,058	16 44
	557·8	80 1	44,680	16 58
	624 4	70 0	43,708	16 95
				(condensed ?)
40	243 8	174 0	42,421	16 42
(Alcohol vapour, P = 133 7 mms)	260 1	160 0	41,616	16·73

Condensed at smaller volumes.

The vapour-pressure at 60° is 625·1 millims.; and at 40°, 260·47 millims. The vapour-densities at 40° appear to be too high; this would be caused by the presence of dust particles, which may cause condensation at low temperatures and at pressures considerably below the vapour-pressure.

Constants at high temperatures.—Three different amounts of methyl-alcohol were employed for these experiments. The first quantity, which we shall call A, was comparatively large, and was employed for the determination of the orthobaric volumes * and of the vapour-pressures. Its volume was ascertained by direct measurement, and its weight from determinations of the specific gravity of methyl-alcohol at low temperatures. The weight of the second quantity, B, was calculated from a comparison of its volumes when gasified with those of the third quantity, under similar conditions of temperature and pressure. That of the third quantity, C, was ascertained from its density at 240°, at large volumes, when *p.v.* remained constant during considerable change of volume. In the course of the experiments the quantity C was slightly increased by rise of liquid from the lower portion of the tube. The increased weight, which we shall term C′, was ascertained by a comparison between the volumes of this portion and those given by the former.

A. *Results of experiments with large quantity. Specific gravity of methyl-alcohol.*—The specific gravity was determined by a SPRENGEL's tube of the form recommended by PERKIN ('Chem. Soc. Trans.,' 1884, p. 443).

Weight of methyl-alcohol at 22 94° . . .	16·59625 grammes.
Weight of water " . . .	20·9595 "
Specific gravity at 22·94° (water at 4° =1) .	0·78909 "
Volume of 1 gramme	1·26729 cub. centim.

(These weighings were reduced to a vacuum)

* We term "orthobaric volumes" the volumes occupied by 1 gramme of the liquid under the vapour-pressures corresponding to the temperatures of measurement.

The results of other observers* are :—

DUMAS and PELIGOT, at 20° .	0·798	KOPP (1847) . .	0·8180
REGNAULT, at 15–20° . . .	0·813	KOPP (1855) . . .	0·8142
DEVILLE, at 9°	0·807	PIERRE, at 0° . . .	0·8207
KOPP (1845), at 0°	0·8147	MENDELÉEFF . . .	0·8206

PERKIN, at 15°/15°, 0·79726; at 25°/25°, 0·78941.

PERKIN'S results give, when compared with water at 4°: at 15°, 0·79658; at 25°, 0·78714; they are in close accordance with ours.

The portion of methyl-alcohol A gave the following measurements :—

	Volume
At 18·2°	0·27622 c.c.
22·4°	0·27697 „
80·0°	0·29666 „

From these numbers a curve was constructed, and the volume at 22·94° was read; it was 0·27701 c.c. The specific gravity at 22·94° was found to be 0·78909; hence the weight of portion A was 0·21858 gramme.

The volume-tube was a new one, 35 centims. long, and about 1·1 millim. internal diameter; the external diameter was about 8 millims. It was carefully calibrated by weighing with mercury at a known temperature.

The following corrections were applied :—

For volume.—Meniscus of mercury and of liquid.

Expansion of glass by heat.

The expansion of the tube owing to internal pressure was not allowed for, as no data are available. It is probable that any correction would have been within the limits of error of reading.

For pressure.—A low-pressure and a high-pressure manometer were employed; where possible, readings on both were taken. The manometers were calibrated by weighing with mercury, and contained air dried over phosphorus pentoxide.

Corrections.—Meniscus of mercury.

Difference of levels of mercury in volume-tube and in pressure-gauges.

Difference in temperature of water-jacket at time of filling and time of reading.

Deviation of air from BOYLE'S Law, as determined by AMAGAT.

For temperature.—The data given by us in the 'Chem. Soc. Trans.,' 1885, p. 640,

* LOSSEN gives a tabulated statement of boiling-points and specific gravities in the 'Annalen der Physik und Chemie,' vol. 214, p. 104.

were employed. No correction was necessary except that the pressures under which the standard liquids boiled were read at the temperature of the room instead of at 0°, and altered accordingly. The temperatures are those of an air thermometer.

The vapour-pressures at each temperature were determined at widely different volumes. The extreme volumes are as a rule about 0·91 and 0·46 cub. centim.

A.

Jacketing vapour	Pressure.	Temperature	Volume of 1 gramme.	Specific gravity	Vapour-pressure.
	mms.	°	c cs		mms.
Chlorobenzene . .	144 8	80	1 3572	0 7368	1,340
					1,341
					1,341
					1,341
				Mean .	1,341
" . .	208 35	90	1 3814	0·7239	1,896
					1,898
					1,898
					1,896
				Mean . .	1,897
" . . .	292 75	100	1 4015	0 7135	2,621
					2,637
					2,627
					2,625
				Mean .	2,627
" .	402 55	110	1 4228	0 7028	3,554
					3,561
					3,565
					3,565
				Mean . .	3,561
" . . .	542·8	120	1 4495	0 6899	4,745
					4,764
					4,775
					4,770
				Mean . .	4,763
"	718 95	130	1 4755	0·6777	6,257
					6,245
					6,234
					6,224
				Mean	6,238

Jacketing vapour.	Pressure	Temperature.	Volume of 1 gramme	Specific gravity.	Vapour-pressure.
	mms.	°	c cs		mms
Bromobenzene	372 65	130	1 4764	0 6773	6,257 6,245
				Mean	6,251
„	495 8	140	1 5039	0 6649	8,052 8,060 8,092 8,076 8,078 8,066 8,080 8,077
				Mean	8,073
„ . .	649·05	150	1·5400	0 6494	10,309 10,308 10,297
				Mean	10,305
Aniline . .	283 7	150			10,355 10,370 10,358 10,352
				Mean .	10,359
„ . .	386·0	160	1 5821	0 6321	13,037 13,014 13,043 13,029
				Mean	13,031
„	515 6	170	1·6234	0·6160	16,295 16,297 16,292 16,286
				Mean .	16,292
„	677·15	180	1 6732	0 5977	20,044 20,067 20,069 20,070
				Mean	20,062
Methyl salicylate .	249·35	180	1 6758	0·5967	20,129 20,152 20,144 20,157
				Mean .	20,145

Jacketing vapour.	Pressure	Temperature	Volume of 1 gramme	Specific gravity	Vapour-pressure
	mms.	°	c cs		mms
Methyl salicylate .	330·85	190	1·7302	0·5780	24,556
					24,611
					24,634
					24,658
				Mean .	24,615
,, ,, . .	432·35	200	1·8098	0·5525	29,759
					29,776
					29,827
					29,833
				Mean .	29,799
,, ,,	557·5	210	1·9033	0·5254	35,739
					35,753
					35,779
					35,808
				Mean	35,770
,, ,, . .	710·1	220	2·0410	0·4900	42,678
					42,735
					42,739
				Mean .	42,713
Bromonaphthalene . .	181·75	220	. .	.	42,617
					42,670
				Mean	42,644
,, . .	207·35	225	2·1367	0·4680	46,320
					46,332
					46,350
				Mean	46,334
,, . . .	235·95	230	2·2633	0·4418	50,461
					50,421
				Mean .	50,441
,, . . .	248·3	232	2·3253	0·4301	52,227
					52,187
				Mean .	52,207
,, . .	261·2	234	2·4229	0·4127	53,988
,, . .	274·65	236	2·5300	0·3953	55,685
,, . . .	281·6	237	2·5868	0·3866	56,955
,, . .	288·7	238	2·6989	0·3705	57,707
,, . . .	292·3	238·5	2·7455	0·3642	58,329
,, . . .	295·95	239	.	. .	58,810
,, . . .	299·65	239·5	.	.	59,141
,, . . .	303·35	240	.	. .	59,661

No meniscus was visible at 240°. On lowering the temperature to 239·9° the meniscus appeared after a short time. The apparent critical temperature, therefore, lies between 239 9° and 240°.

Only one reading is given at high pressures. The reason for this is that the alteration of level of the mercury in the volume tube altered the level of the mercury in the gauge by only 0·1 mm.

B. The weight of this portion was ascertained by a comparison of its volumes in the gaseous state with those of C under similar conditions of temperature and pressure. It was found to be 0·02565 gramme. With this portion vapour-densities at high temperatures, and vapour-pressures, were determined

For this series the jacketing vapour was bromonaphthalene boiling under various pressures

Pressure of bromonaphthalene	Temperature	Volume of 1 gramme.	Pressure	Vapour-density
mms.	°	c cs.	mms.	
181 75	220	39 934	20,645	18 58
		29 364	26,109	19 98
		23 462	30,510	21 40
		17 610	36,058	24 18
		14 713	39,016	26 68
		13 271	40,590	28 43
		12 554	41,397	29 48
		12 125	41,829	30 21
		11·840	42,178	30 68
		11 696	42,231	31 02
		Vapour-pressure	= 42,319	
207 35	225	39 938	20,999	18·45
		29 368	26,647	19 77
		23 466	31,282	21 08
		17 614	37,146	23·65
		14 717	40,536	25 94
		13 275	42,269	27 58
		11 844	44,165	29 59
		11 123	45,168	30 80
		10·410	45,907	32 38
		10 269	46,012	32 75
		10 125	46,118	33 14
		Vapour-pressure	= 46,222	
235 95	230	39 94	21,337	18 34
		29 37	27,192	19 57
		23 47	31,980	20·83
		17 62	38,191	23 23
		14·72	41,914	25 34
		13·28	43,918	26 81
		11 84	46,011	28·69
		11·13	46,894	29·96
		10 41	47,801	31·40
		9 696	48,756	33 06
		8 982	49,737	34 99
		8 674	49,884	36 12
		Vapour-pressure	= 50,262	

Pressure of bromo-naphthalene	Temperature	Volume of 1 gramme	Pressure.	Vapour density
mms.	°	c cs	mms	
248 3	232	39 94	21,545	18 27
		29 37	27,428	19 48
		23 47	32,260	20 73
		17 62	38,753	22 99
		14 72	42,457	25 11
		13 28	44,718	26 43
		11 84	46,761	28 34
		10 41	48,829	30 86
		9 696	49,820	32 48
		8 982	50,851	34 36
		8 265	51,655	36 75
		Vapour-pressure	= 52,192	
261 2	234	39 94	21,697	18 18
		29 37	27,630	19 41
		23 47	32,541	20 63
		17 62	39,086	22·88
		14·72	43,094	24·84
		13 28	45,211	26·25
		11 84	47,443	28 04
		10 41	49,727	30 42
		8 982	51,969	33 75
		8·265	52,801	36 09
		7 552	53,741	38 81
		7 267	53,796	40 29
		Vapour-pressure	= 53,890	
274 65	236	39 95	21,856	18 11
		29 38	27,881	19 19
		23 48	32,901	20 48
		17·63	39,515	22 72
		14 73	43,669	24 60
		11 85	48,375	27 60
		10 42	50,496	30 08
		8 982	52,729	33 48
		8 265	53,967	35 45
		7 552	54,726	38 26
		6 838	55,414	41 73
		Vapour-pressure	= 55,563	
288 7	238	39 95	21,951	18 11
		29 38	28,078	19 25
		23 48	33,137	20 42
		17 63	39,869	22 60
		14 73	44,154	24 43
		11·85	48,865	27 43
		10 42	51,280	29 73
		8 982	53,822	32·84
		7 552	55,972	37 56
		6·838	56,853	40 83
		6 421	57,111	43 30
		Vapour-pressure	= 57,445	

Pressure of bromo-naphthalene	Temperature	Volume of 1 gramme	Pressure	Vapour density
mms	°	c cs	mms	
295 95	239	39 95	22,020	18 08
		29 38	28,243	19 17
		23 48	33,232	20 40
		17 63	40,209	22 46
		14 73	44,361	24 36
		11 85	49,432	27 17
		10 42	51,849	29 46
		8 982	54,282	32 63
		7 552	56,808	37 08
		6 838	57,307	40 59
		6 127	58,328	44 52
		5 696	58,568	47 67
		Vapour-pressure	= 58,672	
299·65	239 5	11 85	49,634	27 09
		10 42	52,301	29 24
		8 982	54,568	32 49
		7 552	56,992	36 99
		6 838	57,836	40 25
		6 127	58,703	44 27
		5 410	58,776	50 08
303 35	240	39 95	22,100	18 05
		29 38	28,361	18 72
		23 48	33,454	20 30
		17 63	40,651	22 26
		14 73	44,677	24 24
		11 85	49,761	27 04
		10 42	52,406	29 21
		8·982	54,993	32 27
		7 552	57,071	36 98
		6 127	58,751	44 28
		5 410	59,590	49 45
		4 697	59,775	56 77
		3 984	60,038	65 84

C. The weight of this portion was 0 00221 gramme. It was ascertained from measurements of the volume occupied by the vapour at 230° and 240°. At 230° the change of volume was from 1 0245 cub. centim. to 0·6019 cub centim ; the product of pressure and volume was constant, and gave for the weight 0·002211 gramme ; at 240°, with the same change of volume, the weight was 0·002206 gramme. The value taken was 0 00221 gramme The error can hardly be more than 0 5 per cent.

The following determinations were made :—

Pressure of bromo-naphthalene	Temperature	Volume of 1 gramme	Pressure	Vapour-density
mms.	°	c cs.	mms	
235 95	230	463 58	2,115	15 94
		340 90	2,862	16 02
		272 35	3,583	16 02
		204 44	4,747	16 11
		170 81	5,658	16 17
		137 44	7,021	16 20
		104 23	9,114	16 45
		87 65	10,754	16 59
		71 10	13,022	16 88
		54 50	16,472	17 41
		46 22	19,050	17 75
		37 90	22,421	18 40
		29 62	27,085	19 49
		21 35	33,884	21 61
		Vapour-pressure	= 50,511	
303 35	240	463 71	2,143	16 04
		428 48	2,326	16 00
		410 87	2,422	16 02
		375·72	2,650	16·01
		341 00	2,913	16 05
		272·43	3,647	16 05
		204·50	4,846	16 09
		170 86	5,775	16·16
		137 48	7,168	16 29
		104 26	9,328	16·39
		87·67	10,991	16 54
		71 12	13,350	16 79
		54 62	16,949	17 25
		46 23	19,639	17 56
		37·91	23,076	18 23
		29 63	28,068	19 17
		21 35	35,616	20 96
Methyl salicylate. C' 432 35	200	453 12	2,013	16·12
		333 20	2,722	16 21
		266 20	3,391	16 28
		199·82	4,478	16 43
		166 96	5,319	16 55
		134 34	6,552	16·70
		101 88	8,491	16 99
		85 67	9,969	17·21
		69 49	12,022	17 60
		53 27	15,042	18 34
		45 18	17,279	18 83
		37 04	20,200	19 65
		29 95	23,817	21·32
		20 87	28,876	24·40
		Vapour-pressure	= 29,739	
Aniline 677·15	180	101 82	8,057	17 16
		85 62	9,362	17·58
		69·45	11,207	18 09
		53 24	13,912	19 01
		45 15	15,696	19 87
		37·02	18,025	21·10
		32 17	19,674	22 24
		Vapour-pressure	= 19,970	

Pressure of aniline	Temperature	Volume of 1 gramme	Pressure	Vapour-density
mms	°	c cs	mms	
386 0	160	452 63	1,846	16 11
		332 85	2,484	16 28
		265 91	3,085	16 40
		199 61	4,049	16 65
		166 78	4,777	16 89
		134 20	5,855	17 13
		101 77	7,505	17 62
		85 58	8,707	18 06
		69 42	10,282	18 85
		61 29	11,299	19 43
		53 22	12,518	20 20
		49 97	12,965	20 77
		Vapour-pressure	= 13,011	
204 6	140	101 72	6,890	18 31
		85 53	7,878	19 05
		82 32	8,024	19 43
		77 45	8,046	20 60
		Vapour-pressure	= 8,056	
Chlorobenzene 542 8	120	452 17	1,658	16 29
		400 64	1,860	16 39
		332 51	2 212	16 61
		282 31	2,584	16 74
		249 01	2,900	16 91
		199 41	3,539	17 31
		166 61	4,120	17 79
		142 15	4,684	18 34
		Vapour-pressure	= 4,705	
292 75	100	451 95	1,560	16 44
		400 44	1,739	16 65
		332 34	2,065	16 89
		282 17	2,391	17 18
		265 52	2,496	17 49
		248 89	2,591	18 12
		Vapour-pressure	= 2,597	

Reduction and Arrangement of Results.

1. *Vapour-pressures.*—The vapour-pressures determined by us with quantities A, B, C, and C′, as well as the vapour-pressures calculated from these observations by the formula $\log p = a + b\alpha^t + c\beta^t$, and pressures calculated by Regnault, who also employed Biot's formula, are given in the following Table .—

Temperature.	Pressure								
	Still method.	A	B	C and C′	Mean	Calculated from formula	Difference in millims	Difference in degrees.	REGNAULT
°	mms.	mms.	mms.	mms.	mms				
−20				-		7 605			6 27
−10	15 5				15 5	15 402	− 0 1	+0 1	13 47
0	29·6		.		29 6	29 600	0	0	26 82
+10	54 7	.	.		54 7	54 224	− 0 48	+0 15	50 13
20	96 0				96 0	95 104	− 0 90	+0 17	88 67
30	160 0	..		.	160 0	160 28	+ 0 28	−0 03	149 99
40	260 5				260 5	260 47	− 0 03	±0	243 51
50	406 0			.	406 0	409 47	+ 3 47	−0 2	381 68
60	625 0	.	.		625 0	625 10	− 0 10	±0	579 93
70	.			.		926 60			857 10
80	.	1,341			1,341	1,340 3	− 0 7	+0 01	1238 47
90		1,897			1,897	1,894 6	− 2 4	+0 04	1741 67
100		2,627	.	2,597	2,621	2,622 5	+ 1 5	−0 02	2405 15
110		3,561	.	.	3,561	3,561 1	+ 0 1	0	3259·60
120		4,763		4,705	4,751	4,751 3	+ 0·3	0	4341 77
130	.	6,238 and 6,251	..		6,242	6,239 2	− 2 8	+0 02	5691 30
140	.	8,073	..	8,056	8,071	8,072 5	+ 1·5	−0·01	7337 10
150		10,305 and 10,359	..		10,336	10,306	− 30	+0 12	9361·35
160		13,031	..	13,011	13,027	12,999	− 28	+0 09	..
170	..	16,292	.	.	16,292	16,213	− 79	+0 23	..
180	..	20,062 and 20,145	..	19,970	20,089	20,016	− 73	+0 18	..
190		24,615	..		24,615	24,481	−134	+0·28	..
200	..	29,799		29,739	29,787	29,688	− 99	+0 18	..
210	..	35,770		..	35,770	35,722	− 48	+0·07	..
220	..	42,713 and 42,644	42,291	..	42,573	42,676	+103	−0 14	
225	..	46,334	46,222	..	46,297	46,530	+233	−0 29	..
230	.	50,441	50,262	50,511	50,414	50,651	+237	−0 28	..
232		52,207	52,192	..	52,202	52,378	+176	−0 20	..
234	..	53,988	53,890	.	53,939	54,151	+212	−0 24	..
236		55,685	55,563		55,624	55,971	+347	−0 38	..
237	..	56,955	..	..	56,955	56,900	− 55	+0 06	..
238		57,707	57,445	.	57,576	57,839	+263	−0 29	..
238 5		58,329		.	58,329	58,310	− 19	+0 02	..
239	.	58,810	58,672		58,741	58,790	− 49	+0 05	.
239 5	..	59,141	..		59,145	59,268	+127	−0 13	.
240		about 59,660		.	.	59,759	..	.	..

The number of observations with quantity A was large, and the mean pressure given lays due stress on this fact. As with alcohol and ether, the formula expresses the observations very closely; the greatest divergence expressed in temperature being only 0·38° at 236°. The constants employed were calculated from observations at 0°, 55°, 110°, 165°, and 220°. To obtain the best values at these temperatures, portions of the curve were smoothed by the method of ratios (see 'Phil. Mag,' vol. 20, 1885, p. 516 *et seq.*). The constants for the formula given were—

$$a = 22{\cdot}307096, \qquad \log \alpha = \bar{1}{\cdot}99988416,$$
$$\log b = 1{\cdot}2649587, \qquad \log \beta = \bar{1}{\cdot}99599796,$$
$$\log c = 0{\cdot}3855770$$

b and c are both negative.

It will be remarked that REGNAULT's vapour-pressures are in every case much lower than ours. On reference to his memoir ('Mémoires de l'Académie des Sciences,' vol 26, p 456), it is evident that his low results are to be accounted for by insufficient drying of his specimen; it was only rectified several times over quicklime He states, also, that the fluctuations of his thermometer when the dynamical method was employed were so great as to interfere with the accuracy of his results REGNAULT, moreover, did not think it worth while to convert the readings of his mercury thermometer into those of an air thermometer

The orthobaric volumes of 1 gramme of methyl-alcohol were read from a curve drawn so as to pass through the experimental points. We have not constructed a formula to represent these relations. They are given in the Table which follows :—

Orthobaric Volumes of 1 *gramme of Liquid*

Temperature	Volume	Specific gravity	Temperature	Volume.	Specific gravity	Temperature	Volume	Specific gravity
°			°			°		
20	1·265	0·7905	120	1·449	0·690	220	2·041	0·490
30	1·2765	0·783	130	1·477	0·677	225	2·139	0·4675
40	1·291	0·7745	140	1·506	0·664	230	2·268	0·441
50	1·307	0·765	150	1·540	0·6495	232	2·328	0·4295
60	1·324	0·7555	160	1·577	0·634	234	2·413	0·4145
70	1·3405	0·746	170	1·623	0·616	236	2·528	0·3955
80	1·360	0·7355	180	1·672	0·598	237	2·597	0·385
90	1·379	0·725	190	1·733	0·577	238	2·699	0·3705
100	1·401	0·714	200	1·808	0·553	238·5	2·751	0·3635
110	1·4245	0·702	210	1·903	0·5255			

Orthobaric Volumes of 1 *gramme of Vapour*

The following Table gives the volume of 1 gramme of saturated vapour at even temperatures —

Temperature	Volume.	Specific gravity	Vapour-density	Temperature	Volume	Specific gravity	Vapour-density.
°	c cs			°	c cs.		
0	17,803	0·0000562	16·10	160	50·16	0·01994	20·64
10	10,044	0·0000996	16·15	170	39·59	0·02526	21·45
20	5,899	0·0001695	16·23	180	31·40	0·03186	22·40
30	3,608	0·0002772	16·30	190	24·94	0·04010	23·57
40	2,276	0·0004394	16·41	200	19·70	0·05075	25·13
50	1,484	0·0006739	16·52	210	15·33	0·06521	27·40
60	993·8	0·001006	16·66	220	11·58	0·08635	31·00
70	682·7	0·001465	16·85	225	9·97	0·1003	33·50
80	479·8	0·002084	17·06	230	8·42	0·1187	36·80
90	344·0	0·002907	17·31	232	7·83	0·1277	38·40
100	251·0	0·003984	17·61	234	7·24	0·1381	40·30
110	186·0	0·005376	17·97	236	6·64	0·1505	42·75
120	140·0	0·007142	18·36	238	5·95	0·1681	46·25
130	106·6	0·009379	18·81	238·5	5·59	0·1789	48·70
140	82·25	0·01216	19·33	239	5·33	0·1878	50·80
150	64·00	0·01562	19·93				

The next Table gives the densities of the unsaturated vapour at even intervals of pressure and temperature. These were read from isotherms, and smoothed by constructing isobars from the isotherms, and, after joining the points to form a curve, re-transferring to isotherms.

Pressures.	Temperature.																				
	100°	120°	130°	140°	150°	160°	170°	180°	190°	200°	210°	220°	225°	230°	232°	234°	236°	238°	239°	239 5°	240°
mms.																					
2,000	16 800	16 455	16·35	16 30	16 26	16 24	16 22	16 20	16 16	16·12				16 01							16 00
2,500	17·325	16 70				16 35				16·19											
4,000		17 65	17 27	17 00	16 80	16 64	16 57	16 46	16 40	16 33				16 11							16 02
6,000			18 48	17 83	17 40	17·15	16 96	16 80	16 72	16 60				16 22							16 16
8,000				19 16	18 25	17 82	17 52	17 06	17 06	16 90				16 42							16 30
10,000					19 55	18 69	18 12	17 74	17 43	17 22				16 64							16 45
12,000						19 86	18 90	18·32	17 92	17 59				16 86							16 63
14,000							19 90	19 06	18 48	18 01				17 15							16 85
16,000							21 18	19 97	19 08	18 50	18 14	17 80	17 62	17 43	17 39	17 31	17 22	17 18			17 14
18,000								21 10	19 86	19 02	18 55	18 12	17 94	17 77	17 67	17 60	17 52	17 42			17 38
20,000								22 40	20 80	19 73	19 00	18·45	18 24	18 04	17 98	17 89	17 81	17 74			17 66
24,000									23 40	21 32	20 05	19 33	19 05	18 80	18 70	18 60	18 52	18 41			18 35
28,000										23 76	21 64	20 52	20 08	19 70	19 57	19 42	19 28	19 16			19 03
32,000											24 00	22 10	21 40	20 84	20 63	20 43	20 25	20 08			19 93
36,000											28 ?	24 18	23 10	22 26	21 98	21 72	21 46	21 25	21 15		21 04
40,000												27 60	25 43	24 06	23 63	23·24	22 90	22 60	22 44		22 30
42,000												30 10	27·10	25 36	24 80	24 30	23 85	23 48	23 28		23 10
44,000													29 26	26 81	26 10	25 48	24 90	24 40	24 15		23 93
46,000													32 47	28 70	27 70	26 86	26 12	25 45	25 18		24 85
48,000														31 60	29 86	28 65	27 60	26 72	26 30		25 90
50,000														35 53	32 70	30 87	29 40	28 20	27 66	27 41	27 20
52,000															37 50	34 38	32 20	30 40	29 60	29 20	28 82
54,000																40 80	36 00	33 10	31 99	31 42	30 98
56,000																		37 66	35 48	34 80	34 27
58,000																			42 40	40 90	40 00

Heats of Vaporisation.

From the well-known thermo-dynamic equation

$$\frac{L}{s_1 - s_2} = \frac{dp}{dt} \cdot \frac{t}{J}$$

the heats of vaporisation at definite intervals of temperature were calculated The values of dp/dt were calculated by means of the formula $\log p = a + b\alpha^t + c\beta^t$, pressures were calculated for one-tenth of a degree above and below the required temperature, and the difference was multiplied by 5 to obtain the value for 1°. The pressures were reduced to grammes per square centimetre, and the value of J was taken as 42,500.

Temperature		dp/dt		$s_1 - s_2$	L
° C	° Abs.	millims.	grammes	c cs.	calories.
0	273	1·861	2 530	17,802	289 17
10	283	3 160	4 296	10,043	287 36
20	293	5·150	7 002	5,898	284 54
30	303	8 067	10 978	3,607	282 07
40	313	12 202	16 590	2,275	277 78
50	323	17 890	24 323	1,483	274 14
60	333	25 473	34 633	993 5	269 41
70	343	35 381	48 104	681 4	264 51
80	353	47 91	65·139	478 4	258 96
90	363	63 54	86 330	342 6	252 76
100	373	82 57	112 26	249 6	246 01
110	383	105 81	143 86	184 6	239 27
120	393	133 10	180 96	138 6	232 00
130	403	165 41	224 89	105 1	224 07
140	413	202 52	275 35	80 75	216 12
150	423	243 95	331 67	62 46	206 13
160	433	294 70	400 67	48 58	198 34
170	443	349 95	475 80	37 97	188 25
180	453	411 15	559 00	29 73	177 16
190	463	482·0	655 33	23 21	165 64
200	473	560 85	762 54	17 89	151 84
210	483	649 75	883 41	13 43	134 78
220	493	747 9	1016 8	9 54	112 53
225	498	797 5	1084 3	7 83	99 50
230	503	851 9	1158 2	6 16	84 47
232	505	875 0	1189 7	5 50	77 73
234	507	897 5	1220 2	4 82	70 15
236	509	921 0	1252 2	4 11	61 66
238	511	945 5	1285 5	3 25	50 22
238 5	511·5	951 7	1294 0	2 84	44 23

The heat of vaporisation of methyl-alcohol has been determined by ANDREWS, and by FAVRE and SILBERMANN, at the boiling-point under ordinary pressure. The former found 263 7 calories, and the latter 263 86 calories. At the boiling-point our results give 266·7 calories; this is a satisfactory agreement.

From the foregoing account of our experiments with methyl-alcohol it is evident

that the views which we have already expressed regarding alcohol and ether are substantiated; there is no tendency towards the formation of complex molecules at low temperatures.

This research is not so complete as the former two. We have made no attempt to determine the compressibility of the liquid. The cause of this omission is the very high pressure exerted by the vapour of methyl-alcohol at high temperatures. During these experiments we lost a volume-tube and a gauge by bursting, and, as the labour involved in replacing them was very great, we did not choose to risk further breakage.

The approximate critical pressure of methyl-alcohol is 59,700, the approximate temperature 240°, and the approximate volume 3 683 cub centims. It is noticeable here that the difference in temperature between the boiling-points of methyl- and ethyl-alcohols is by no means the same as the difference between their critical temperatures.

In Plate 14 the relations between pressure, temperature, and volume for saturated and unsaturated vapour are shown, the orthothermic constants for the liquid are also given. The small diagram shows these relations between the pressures 1600 millims. and 60,000 millims. The critical isothermal is also given. The large diagram represents the higher temperatures observed; the circles are actual experimental points.

Plate 15 shows the densities, compared with hydrogen, of the saturated and unsaturated vapour mapped against pressure It is here noticeable that the density of the saturated vapour becomes normal at low pressures and corresponding low temperatures.

Plate 16 shows the orthobaric densities of the liquid and vapour mapped against temperature, and their approach towards identity at the critical point.

Plate 17 gives the vapour-densities of the saturated vapour, and the heats of vaporisation, mapped against temperature Again the rapid increase of density and the rapid decrease of latent heat as the critical point is approached are noticeable.

ADDENDUM.

(Added July 28, 1887)

The value of such work as the foregoing evidently depends on the purity of the substance employed; and, as it appears that our paper is deficient in this proof, we adduce further evidence of the purity of the sample of methyl-alcohol employed in experiments on its thermal constants.

The proofs submitted are :—

1 Constancy of boiling-point.

2. Agreement of statical and dynamical methods of measuring vapour-pressures (See REGNAULT, 'Mémoires de l'Académie des Sciences,' vol 26, p. 341.)
3 Comparison of results with those of most careful observers.

1. *Constancy of boiling-point.* (Extracts from Note-book.)

"Methyl-alcohol prepared from methyl oxalate by action of ammonia. dried with barium oxide; dried with anhydrous copper sulphate; stood for some weeks (see PERKIN, 'Chem. Soc Journ.,' vol. 45, p. 465). Boiling-point by no means constant

"Dried with sodium. A good deal boiled constantly at 65·55° (corr.). Bar = 761 9 mms at 0°, and then temperature rose.

"Dried again with sodium. A good deal boiled constantly at 65°, temp. rose slowly to 65·8°. Bar same.

"Dried again with sodium Began at 64·9°; most of it came over within a tenth Temp. rose finally to 65·3°. Bar. same.

"Dried again with sodium. Began at 64 85°; most of it came over within half-a-tenth Temp rose to 65 05°. Bar. same.

"Dried again with sodium. Began at 64 8° (or a trace above). Temp. rose only to 64·9° (rather below)."

(All these temperatures are those of REGNAULT's air thermometer)

Vapour-pressure determinations were then made by our method ('Chem Soc. Trans.,' 1885, January).

"Alcohol re-distilled with a little sodium; B.P. 64 95°. Bar. = 761·9 mms. (at 0°.)"

The specific gravity was then taken.

"Methyl-alcohol re-distilled and fractionated; B.P. 64·7° at 760 mms (at 0°) Range, 0·05°."

Experiments at high temperatures and pressures were then made, and, after these were completed,

"Fractionated alcohol was re-distilled; B.P 64 2° at 743·2* mms. (corr. to 0°). The vapour-pressures were again determined in our apparatus."

The accompanying curve on Plate 18 shows the $\left\{\begin{matrix}\text{boiling-points}\\\text{vapour-pressures}\end{matrix}\right\}$ of the two samples of methyl-alcohol

2. *Agreement of statical and dynamical methods of measuring vapour-pressures.*

a. At low temperatures. No special measurements were made at low temperatures by the statical method, but two observations were incidental during the determination of the volumes of 1 gramme by means of the "modified HOFMANN's apparatus." These are represented on the curve by large crossed circles. The actual numbers are :

* Value of dp/dt at 60°, 25 47; at 70°, 35 38 mms Sufficiently correct estimate at 65°, 30 mms. Add 0 55°. Corrected B P. at 760 mms., 64 75°.

	Calculated by formula.	Found by statical method
°	mms.	mms
T = 40	260 47	262 4
T = 60	625 10	624 4

b. The formula $\log p = a + b\alpha^t + c\beta^t$ is BIOT'S. The constants were calculated from all observations by dynamical method at low temperatures, and by statical method at high temperatures We reproduce the numbers showing the transition from the one method to the other.

Temperature	Observed.		Calculated
	Dynamical	Statical	
°	mms.	mms	mms
30	160 0		160 28
40	260 5	..	260 47
50	406 0		409 47
60	625 0	.	625·10
70	..		926 6
80		1341	1340·3
90	..	1897	1894·6
100	..	2621	2622·5
110	..	3561	3561 1

As regards the value of this method of determining the purity of a substance, we quote REGNAULT —

"J'ai déjà fait voir, dans mon Mémoire sur les forces élastiques de la vapeur aqueuse (t. 21, p. 524), que cette coincidence est parfaite pour l'eau; les deux méthodes, statique et dynamique, donnant des valeurs identiques aux mêmes températures. Je montrerai qu'il en est de même pour les autres liquides volatils, pourvu qu'ils soient à l'état de pureté parfaite. Mais lorsqu'un liquide renferme une portion, même extrêmement petite, d'une autre substance volatile, les deux méthodes donnent des valeurs différentes pour les forces élastiques de sa vapeur à la même température; et c'est un moyen extrêmement délicat pour juger de l'homogénéité d'une substance volatile." ('Mémoires de l'Académie des Sciences,' vol 26, p. 341.) Again, p. 390: "On peut déduire également de ces expériences sur l'éther que les forces élastiques de la vapeur, déterminées par la méthode statique et par la méthode dynamique, coïncident parfaitement lorsque la substance est homogène."

3 *Comparison of physical properties with those found by most careful observers.*

a Boiling-point Determined by numerous observers with most discordant results

×	PIERRE .	66 3°	at 759	(at 0° ?)—Sp gr reduced to 0°		0 8207
×	MENDELÉEFF	66 0°	?	..	„	0 8206
×	KOPP	64 9°	at 744	?	„	0 8180
×	DELFFS . .	60 5°	at 748	?	„	0 8138
×	DUMAS and PELIGOT	66 5°	at 761	?	„	0 8155
×	DEVILLE . .	..	..		„	0 8153
×	KOPP (1845) .	65 0°	at 752	..	„	0 8147
×	KOPP (1855)	64 6–65 2°	at 744	.	„	0 8142
×	LANDOLT	66 0°	at 753		„	0 8137
×	DUCLAUX	66 0°	?	..	„	0 8128
BaO	GRODZKI and KRAMER	65 75–66 25°	?	(uncor)	„	0 8130
Na	„ „	65 6–66 2°	at 764 8	(uncor)	„	0 8116
	(Specially remarked that Na is preferable to BaO)					
×	GRAHAM .	66 0°	?		„	0 8105
K_2CO_3	LINNEMANN	67 1°	?		„	0 8768
$CuSO_4$	PERKIN .	65 8–66°	? 760		$\frac{15°}{4}$	0 79658
					$\frac{25°}{4}$.	0 78714
CaO	REGNAULT	?		Thermometer varied at same pressure within several degrees Calculated 66 78°		
Na	SCHIFF (once) .	64 8°	at 763			

It is almost certain that every one of these observers has dried the sample of alcohol used over lime. Those marked with a × used either lime or gave no account of the method of drying, or we have not been able to verify the reference in Bristol.

GRODZKI and KRAMER specially state that sodium is preferable to barium oxide. Their results are close to ours (65 6 to 66·2 at 764 8 mms, not corrected to 0°, with the stem of their thermometer in vapour up to 30°)

SCHIFF distilled once over sodium, and obtained a number practically identical with ours (64·8 at 763 mms).

The other results are all higher, that of DELFFS is probably a misprint (Perhaps not . see KOPP.)

b. Specific gravity. As Dr. PERKIN has recently determined the specific gravity of methyl-alcohol, taking all precautions, and as the same method has been followed by ourselves, it is striking that the agreement is absolute.

PERKIN . . . . $\frac{22\ 94°}{4°}$ = 0·78908 (calculated).

RAMSAY and YOUNG . . . „ = 0 78909

Such a close coincidence, however, must be regarded as accidental

It is specially to be noticed that REGNAULT did not obtain coincident results by the statical and dynamical methods. He says (vol. 26, p. 461). "La courbe graphique que j'ai tracée sur la planche V, d'après les éléments de cette Table, représente parfaitement les observations de la Série 1, et celles de la Série 3, qui dépassent 100°.

Mais toutes les expériences par ébullition, qui correspondent à des températures inférieures à 100°, donnent des points qui sont très-inférieurs à la courbe, comme si la vapeur du liquide bouillant avec soubresauts était surêchauffée. Je n'ai pas jugé utile de calculer une seconde formule pour laquelle les températures seraient prises sur le thermomètre à air" He evidently did not consider his results very trustworthy, and this is borne out by the fact that he did not determine the specific heat of the liquid, or its heat of vaporisation.

In conclusion, we append a Table of comparative results, including recent determinations by DITTMAR, as well as those by PERKIN, REGNAULT, SCHIFF, and RAMSAY and YOUNG.

BOILING-POINT

REGNAULT	(calculated)	66°78
PERKIN	(observed)	65 8–66 0
SCHIFF	(observed)	64 8
DITTMAR	(calculated)	64 97
RAMSAY and YOUNG	(calculated)	64 90

VAPOUR-PRESSURES.

Temperature	REGNAULT.	DITTMAR.	RAMSAY and YOUNG
°	mms.	mms	mms
0	26 82	29 7	29 60
10	50 13	53 8	54 22
20	88 67	94 0	95 10
30	149 99	158 9	160 28
40	243 57	260 0	260·47
50	381 68	409 4	409 47
60	579 93	624 3	625 10
65	707 33	761·0	763 15

REGNAULT'S and DITTMAR'S results were obtained by the statical method; ours by the dynamical method. (See also our observations at 40° and 60° by statical method, p. 332.)

SPECIFIC gravity at 22·94°. (Water at 4° = 1 00000.)

DITTMAR	(calculated)	·78897.
PERKIN	(calculated)	·78908
RAMSAY and YOUNG	(observed)	78909

SPECIFIC gravity at 64·8°

DITTMAR	(calculated)	·74795
SCHIFF	(observed)	7476

XII. *Some Anomalies in the Winds of Northern India, and their Relation to the Distribution of Barometric Pressure.*

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Communicated by H. F. BLANFORD, *F.R S , Meteorological Reporter to the Government of India.*

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[PLATES 19–21]

IN the 13 years which have elapsed since Mr. BLANFORD published his paper on the Winds of Northern India,* very great additions have been made to our knowledge of the meteorology of the country. The carefully organised system of observations, commenced in Bengal and the North-Western Provinces, has been extended to include the whole of India, and placed under the direction of Mr. BLANFORD himself, aided by local officers in all the larger provinces. Verified instruments have been supplied to all the stations, and the elevations of these above sea-level have been determined by connecting them with the lines of spirit-levelling, carried inland from the coast, in various directions, by the officers of the Great Trigonometrical Survey; or, where this was impracticable, by spirit-levelling to some of the trigonometrical stations of the Survey. In this way, trustworthy and inter-comparable series of barometric observations, extending over ten years or more, have been obtained for all the more important stations. At the same time, the diurnal variations of the barometer at certain selected stations have been determined by long-continued series of hourly observations, with the object of enabling us to reduce the readings made in the ordinary way (usually at 10 A M. and 4 P.M.) to true daily means. Simultaneously with the collection of this immense quantity of accurate and reliable barometric data, observations have been made of temperature, humidity, cloud, wind, and rain. Latterly also barometric and wind charts of the Bay of Bengal have been prepared from observations made on board ships navigating those waters.

During these 13 years, the winds prevailing over the Indian continent and the Bay of Bengal, and their relations to the distribution of pressure at sea-level, have

* "The Winds of Northern India," by HENRY F BLANFORD, F.G.S , 'Phil. Trans.,' vol 164, 1874, Part II

been discussed from time to time, both in their normal aspects for each month or season and in their abnormal or disturbed conditions during the passage of storms. The latter conditions in particular have been very fully described by Mr. ELIOT in his numerous reports on cyclones in the Bay of Bengal, while the former have been noticed in the annual reports on the meteorology of India, in occasional papers appearing in the 'Indian Meteorological Memoirs,' * and latterly in a broad and general review in Mr. BLANFORD's great monograph on the Rainfall of India.†

There still remain several points, however, in connection with the winds of Northern and Central India, with regard to which our knowledge is very far from complete,—points in which the relation of wind direction and intensity to the distribution of pressure, as given by observation, is more or less directly opposed to what we should expect from theory, and the object of this paper is to try to find a key to the solution of such anomalies.

The chief anomalies to which I refer are the following:—

(1) The direction of the wind in the hot season, on the plains of Northern India, has often no relation to the baric gradient; instances not infrequently occurring in which the wind appears to blow directly in opposition to the gradient, *i.e.*, from a place of low pressure to one where the barometer stands higher.

(2) Over the plains of Northern India, the average velocity of the wind has little or no apparent relation to the pressure gradient, but a very obvious one to the temperature; being, on the whole, greatest when the temperature is highest, while, at this season, the local pressure gradients are extremely small.

(3) Mr. BLANFORD has shown ‡ that it is highly probable there is some connection of the nature of cause and effect between copious and late snowfall (producing low temperature) on the Himalayan ranges and the subsequent prevalence of dry winds from the north-west over northern and western India. On the strength of this relation he has for several years based forecasts of the character of the coming rainy monsoon, which have on the whole been amply verified. The anomaly here is that a low temperature to the north of India, producing high pressure, at sea-level, under the sub-Himalayan stations, gives rise to unusually strong westerly winds, whereas, according to the law usually quoted as "BUYS BALLOT's," which has been clearly deduced from the hydrodynamical principles by FERREL, COLDING, and others, such high pressure should give rise to easterly winds; the rule being that the highest pressure lies in the northern hemisphere to the right of a person travelling with the wind.

These three anomalous relations will be here discussed in the order mentioned.

* "On the Winds of Calcutta," by H. F BLANFORD, 'Indian Meteorological Memoirs,' vol. 1, page 1; "The Winds of Kurrachee," by FRED. CHAMBERS, *ibid*, page 249, "The Meteorology of the North-West Himálaya," by S. A. HILL, *ibid.*, page 377.

† 'Indian Meteorological Memoirs,' vol. 3

‡ 'Indian Meteorological Memoirs,' vol. 3, Part II., and 'Proceedings of the Royal Society,' vol 37, 1884.

Part I.—The Wind Direction Anomaly.

The hot winds of the Indian plains have been often described by travellers and writers on Meteorology, but the existence of such a condition as that mentioned above, viz., their blowing directly against the baric gradient, has been doubted by competent authorities, or at all events suspected to be only an erroneous deduction from a comparison of monthly mean pressures with the resultant wind directions. The daily weather telegraphic reports of the last few years, however, leave no room to doubt that, almost every year, there are several days in April, May, or June when this *primâ facie* impossible condition obtains over a large extent of country at 10 A.M., while, from the known character of the diurnal variations of pressure and wind direction, we may safely infer that the condition is much more frequent and more distinctly marked in the early hours of the afternoon, the diurnal fall of the barometer increasing slightly as we advance inland from Bengal, and the wind direction being steadiest from the north-west about 2 P.M.*

The following examples, selected from the 10 A.M. telegraphic reports, will prove the existence of this paradoxical condition of atmospheric circulation. The stations are enumerated in order from west to east, the general direction of the lines joining them being about W.N.W. to E.S.E. The pressures are reduced to sea-level.

TABLE I.—Instances of Wind blowing in opposition to Baric Gradient.

May 17, 1882.

Station.	Barometric pressure	Wind direction.
	Inches	
Meerut	29 753	NW
Bareilly	29 758	NW
Lucknow	29 759	W
Gorakhpur	29 785	WNW.

May 9, 1883.

Station.	Barometric pressure	Wind direction.
	Inches.	
Meerut	29 581	NW.
Bareilly	29 597	NW.
Lucknow	29 631	WNW.

* 'Indian Meteorological Memoirs,' vol. 1, page 353, and Plate XXX

May 15, 1884.

Station.	Barometric pressure.	Wind direction.
	Inches	
Roorkee	29·511	N W
Meerut	29·545	NW
Bareilly	29·561	NW
Lucknow	29·560	SW.
Allahabad	29·577	NW.

May 29, 1884.

Station.	Barometric pressure	Wind direction.
	Inches	
Roorkee	29·466	Calm
Meerut	29·485	NW.
Bareilly	29·484	SW
Lucknow	29·497	W.
Allahabad	29·520	NW.

May 24, 1885.

Station.	Barometric pressure.	Wind direction.
	Inches.	
Bareilly	29·825	NNW
Lucknow	29·845	NW.
Allahabad	29·842	NW.

May 27, 1885.

Station.	Barometric pressure.	Wind direction
	Inches.	
Bareilly	29·766	NW
Lucknow	29·793	NW.
Allahabad	29·805	NW.

May 20, 1886.

Station	Barometric pressure	Wind direction.
	Inches.	
Bareilly	29·802	SW.
Lucknow	29·829	WNW.
Gorakhpur	29·833	Calm.

These instances abundantly prove the frequent occurrence of the condition described —the wind blowing from a place of low pressure to one where the pressure is slightly higher; but it may be objected that the places enumerated probably happen to lie nearly on an isobaric line, while the wind blows almost parallel to it, so that, if the slight disturbances of the true wind direction due to local causes were eliminated, there would be no anomaly to explain. The answer to this is that, at the season under consideration, the stations enumerated lie near the middle of a very large area of uniform low pressure (see Plate 19), while the winds are the strongest and steadiest of the year. It may be added, also, that near the equator the wind does not blow parallel to the isobars, even over the sea, but makes an angle of two or three points with them; whilst at inland stations, where the friction coefficient is greater, the angle must be still greater, as FERREL has pointed out in his works on the movements of the atmosphere. The charts prepared from the daily telegraphic weather reports for India show an average angle of something like 45° between the wind directions and the isobars. The characteristic winds of the hot season, therefore, do really blow sometimes in opposition to the pressure gradient, and the cause which sets them in motion must be sought somewhere else than in differences of pressure at sea-level, or rather at the level of the plain over which they blow.

Since, in these winds, the air-particles are not urged from west to east by any considerable difference of pressure at the ground-level, whilst they are retarded by friction, and sometimes even by an increasing pressure, as they go eastwards, their velocity must be gradually diminished as they approach the Bay of Bengal. This fact is familiarly known to residents on the plains of the Ganges. The "hot winds," which are excessively dry as well as hot—and which, therefore, serve very effectually to refrigerate the interiors of dwelling-houses by blowing through wet *tatties*, or screens of fragrant grass, placed in the door and window openings on the windward side—seldom extend as far eastwards as Monghyr, in Behar,* and their extinction takes place very gradually. It is a matter of extreme difficulty to obtain truly comparable observations of wind velocity, owing in part to slight differences in the anemometers, but chiefly to the impossibility of ensuring uniform conditions of exposure; but the following sets of stations, where the surroundings are as nearly alike as possible, show clearly a gradually diminishing velocity as we go eastwards during the hot weather, while, at other seasons there is, in some cases, a tendency for the velocity to increase as we approach the sea. The figures are taken from the Report on the Meteorology of India for 1884. They represent the averages of many years.

* For a short time at the hottest hours of the day westerly winds are felt in Bengal, and even in Assam, but they do not possess the dryness and gusty character of the winds under discussion, though when they blow they are probably due to similar causes.

TABLE II.—Comparison of Mean Wind Velocities at Places on Lines running W N W to E S.E.

Station	Wind velocity in miles per diem			
	January	May	July	October
Jeypore	92	181	154	92
Agra	82	144	121	72
Benares	68	119	111	61
Patna	50	113	87	47
Neemuch	168	298	284	175
Sutna	94	194	223	103
Hazaribagh	114	208	204	120
Malegaon	96	257	286	111
Akola	81	206	200	78
Raipur	58	171	275	87

The only instance in the Table, where the velocity during May increases in going eastwards, is between Sutna and Hazaribagh. The latter station is, however, the higher by nearly 1000 feet, but, being situated on a broad plateau, not an isolated peak, this would probably not make much difference. At Hazaribagh, however, we are on the border of the region over which these hot westerly winds prevail, the excess of observations in the direction of the westerly resultant being, in May, only 15 per cent., and the increase of velocity at that station is probably the effect of the frequent incursion of sea winds from the south-east.

A suggestion regarding the origin of these hot winds may be derived from their excessive dryness, above referred to; in fact, it was from this circumstance that I was first led to think of the key to their explanation, which I believe will be found in the following pages. This dryness is such that the relative humidity of the air over the Bundelkhand plateau, south of the Ganges, frequently falls in the middle of the afternoon to 5 per cent. or less. Indeed, the humidity deduced by the usual Tables, (founded on AUGUST's formula with REGNAULT's constants) is occasionally zero or apparently negative.* On the Gangetic plain it never falls quite so low, but in all

* It must be recollected that the formula is founded on the convection theory of the psychrometer. The quantity to be deducted from the tabular pressure of saturated vapour, at the temperature of the wet bulb, in order to get the vapour-pressure actually present, is $\frac{s/d\ (t-t')\ h}{L}$. The true value of the mean s/d, specific heat of air divided by relative density of vapour, is about ·38, but, to suit the formula to the usual condition of nearly still air, REGNAULT empirically increased it to ·48. Hence, during a high wind, the quantity to be deducted is too great when REGNAULT's formula is employed, and if the actual humidity be very low the result may come out negative.

the drier months of the year there is a decided diminution of vapour pressure at the hottest hours of the day, when west winds, similar to, or identical with, the "hot winds," are blowing The following hourly values of the pressure of vapour at Allahabad will illustrate this. They are taken, with some corrections derived from subsequent observations, from a paper on the "Results of Meteorological Observations at Allahabad," published in 1881 in the first volume of the 'Indian Meteorological Memoirs.'

TABLE III.—Diurnal Variation of the Pressure of Vapour at Allahabad.

Month	Mid-night	2 h	4 h	6 h	8 h	10 h	12 h	14 h	16 h	18 h.	20 h	22 h
	″	″	″	″	″	″	″	″	″	″	″	″
January .	342	334	325	313	326	355	369	366	372	384	369	352
May	580	585	588	617	621	590	557	527	515	534	552	575
July .	928	925	923	919	943	955	958	964	956	949	940	935
October	656	·650	·648	655	683	672	652	646	654	671	666	662

It will be seen that, whereas, in the month of July, the quantity of vapour in the air increases, with slight irregularities, as the temperature rises from 6 A.M up to 2 P.M., and diminishes again steadily as the temperature falls after 2 P.M., this regular march is interrupted in the other months by a fall in the middle of the day. The fall is very slight in January, but considerable in October, making the afternoon minimum in that month slightly lower than that of the early morning. In May, however, and in the other dry hot months, the afternoon fall so completely masks the variation due to the diurnal changes of temperature, as seen in July, that the morning minimum is altogether obliterated.

Whence comes this excessive dryness in the afternoons, which is not merely a decrease of *relative* humidity, due to rise of temperature, but a diminution of the absolute quantity of vapour in the air near the ground, from an average pressure of 621 inch in May, at 8 o'clock in the morning, to ·515 inch at 4 in the afternoon? It may be caused by the wind coming from drier regions to the north-westward, but this can hardly be the true cause, for in May there is very little decrease in the mean pressure of vapour as we go north-westwards over the Gangetic plain, and there is still less in April. For example, we have for three selected stations on the plain of the Ganges, arranged in order from SE. to NW., the following values for the mean pressure of vapour.

	Allahabad.	Lucknow	Bareilly
	″	″	″
April	414	·431	421
May . .	553	565	549

On the plateau to the south-west of the Jumna, where the hot winds are even stronger and more persistent than over the great plain, there is a positive increase of vapour in the westward direction in the air stratum with which our observations deal. Thus we have the following mean values for May —

Sutna.	Jhansi	Jeypore
367″	448″	459″

The dryness characteristic of the hot winds cannot, therefore, be due to the eastward transference of air near the ground surface from the arid region of the Rajputana desert; for, if this were the cause, we should find a gradual decrease of moisture on going north-westwards, instead of which we find an increase

Taking the variation of humidity in the lower atmospheric strata in conjunction with the diurnal variation of cloud, which, as I have shown in the paper last quoted, attains its maximum about the time of highest temperature, as it does in most parts of the world, the idea of an interchange between the upper and lower strata by local convection currents is at once suggested. The westerly winds of the hot weather are so dry, perhaps because they descend from regions where the proportion of water vapour in the air is normally much less than at sea-level. The diurnal interchange between the lower atmospheric strata and those lying considerably higher was first suggested by Dr. KOPPEN,* to explain the daily inequalities of wind direction and velocity—inequalities which are opposite in phase over plains and on high mountain peaks. In the sequel it will be shown that it probably suffices to clear up all the anomalies observed in the wind system of India. The explanation of the dryness of these winds by the descent of air from a higher stratum was previously suggested by MUHRY, in a passage† quoted by BLANFORD at p. 614 of his paper on the "Winds of Northern India"; but this was under the mistaken notion of an aërial cascade pouring over the edge of the Himalaya from the elevated plains of Thibet, a notion which is quite inconsistent with the facts of observation.

A general idea of the average relations of pressure and wind direction over the Indian area will be obtained from Plate 19, on which have been charted the mean isobaric lines and the resultant wind directions at the more important observing stations, for the months of January, May, July, and October, typical respectively of the cold, the hot, and the rainy season, and the autumn transition period. The pressures used in drawing the charts are the means for ten years, published in the Report on the Meteorology of India for 1884. They have been reduced to sea-level

* 'Zeitschrift der Oesterreichischen Gesellschaft für Meteorologie,' vol. 14, 1879, p. 333. KÖPPEN's hypothesis has also been suggested by BLANFORD (*loc. cit.*) to account for the hot winds, and he adduces their dryness as a reason for believing it to be the true explanation

† 'Untersuchungen über die Theorie und das allgemeine Geographische System der Winde,' p. 99.

and to the value of gravity at latitude 45°. The isobars have been extended across the Bay of Bengal by means of observations extracted from ships' logs, recently reduced and tabulated in the Bengal Meteorological Office. The charts given in Plate 19 are, therefore, identical with those for the same months attached to Mr. BLANFORD's paper on the Rainfall of India,* except that the observations of one or two stations which seemed to me of doubtful validity have been rejected.

On the chart for January, the resultant wind directions, on the whole, conform to the distribution of pressure in accordance with BUYS BALLOT's law, but there are some exceptions For example, at Neemuch, in Southern Rajputana, south-westerly winds prevail where the law would indicate an excess of easterly components. The chief exception is in Northern Bengal, where the winds should be easterly or south-easterly according to the law, whereas there is a decided prevalence of north-westerly winds as far as Chittagong.

Elsewhere, except on the west coast, where the sea-breezes prevent observations, made only in the day time,† from giving the true resultant direction, the prevailing wind agrees closely with that given by the law; when it is borne in mind that, near the Equator, the angle between the wind direction and the isobars is larger than in higher latitudes.

In May, while there is close conformity to the law in Southern India, and near the coast generally, there is no apparent relation whatever between pressure distribution and wind direction over the very large area north of the parallel of 20° and west of the meridian of 86°. The greater portion of this area is enclosed by the very peculiarly curved isobar of 29·60″, within which are two distinct areas where the pressure is below 29 55″. One of these occupies the upper valley of the Mahanadi, in the eastern part of the Central Provinces, and the other and larger is in Upper Sindh and the Bikanir desert. Over these areas bounded by closed curves, and across the axis of low pressure connecting them, as well as across the isobar of 29·60″ which runs along the southern face of the Himalaya, the westerly winds blow without interruption as far as Behar and Chutia Nagpur, and, at a higher level, as far as Darjiling; whilst the easterly winds, which should prevail, according to the law, at all stations north of the low pressure axis, are only felt along the base of the Himalaya up to Gorakhpur. These west winds have a southerly component in the Indus Valley and Western Rajputana, as well as at most of the Himalayan stations of 6000 or 7000 feet elevation, whilst, over the rest of the region, they have usually a northerly component; but a glance at the chart for May will show that they are probably all stream lines of one continuous atmospheric flow.

The chart for July shows that the pressure gradients over the whole of India and the Bay are much steeper than in the other months, while the usual relation between the direction of the wind and the isobars holds good over nearly every part of the

* 'Indian Meteorological Memoirs,' vol. 3

† At most stations the observations are made only at 10 A.M. and 4 P.M.

country, the most important exceptions being at Agra, Jeypore, and Multan, where the winds have a northerly component, whereas a southerly one is indicated by the distribution of pressure. At the Himalayan stations on the outermost range, as at Darjiling, Naini Tal, and Mussoorie, there is usually an easterly component in the resultant wind; while at others, more in the interior, as at Ranikhet, Chakrata, Murree, and Leh, the resultant is westerly.

In October the pressure differences are very small, and the winds as a rule are light and irregular. The resultant directions, however, conform in most places to BUYS BALLOT's law, but the anticyclonic circulation of the air over South-eastern Rajputana and Central India States is apparently more regular and uniform than might be expected from the irregularly-shaped isobar of 29·85″.

Over Bengal also, from the Himalayas to the head of the Bay, the wind blows almost exactly at right angles to the isobars, instead of more or less inclined to them.

From the charts on Plate 19 it will, therefore, be seen that, when the air is moist, the sky cloudy, and the diurnal range of temperature small, the usual relation obtains between the distribution of pressure and the resultant direction of the wind. These conditions obtain over most parts of the country in July, and at many of the coast stations in the other months.

In January there are exceptions to the rule, the most important being in Bengal, which, at that season, is a region of moderately high temperature, clear skies, and a large daily range of the thermometer.

This will be evident from the following Table, in which certain meteorological conditions for January and July at several Bengal stations are contrasted:—

TABLE IV.—Contrast of Bengal Climates in January and July.

Station	January.			July.		
	Mean Temperature	Daily Range	Cloud Proportion.	Mean Temperature	Daily Range.	Cloud Proportion.
	°	°		°	°	
Purneah	61 9	27 9	1 2	84·0	12·9	7·1
Berhampur	64·5	25·3	2 2	83·2	10 7	8 8
Burdwan	66·0	24 4	1·7	83 5	10 9	8·1
Jessore	65·1	26 1	1·6	82 5	10·7	8·3
Dacca	66 6	22·9	1 6	83 7	9 5	8 0
Calcutta . .	67·3	22·4	1·3	83 1	9·6	8 0
Chittagong . . .	66·6	21 0	1·3	80 7	9 3	7·4

The skies in this region are usually almost cloudless in January, and the daily range of temperature is between 20 and 30 degrees; while in July the average cloud proportion is high, and the daily range of the thermometer only about 10 degrees. The disturbances of thermal equilibrium by the daily transit of the sun must, therefore, presumably be nearly three times as great in January as in July, and if the distribu-

tion of pressure, at high levels, be different from that observed near the ground surface, we may, on KOPPEN's theory, expect to find in January a greater departure of the actual wind direction, from that given by the usual law, than we find in July.

On the Upper Gangetic plains, and in the Punjab, the range of temperature in January is quite as high as in Bengal, though the skies are somewhat more cloudy. We do not there, however, find departures from BUYS BALLOT's law to occur so frequently in this month, probably because the disturbances of thermal equilibrium, causing convection currents, are less intense. For this there are two reasons. In the first place, whilst the diurnal range of the thermometer is but little greater on the plains than it is in Bengal, the range at the nearest hill stations is considerably greater than at Darjiling, so that the difference between high and low stations, at the hottest hour of the day, is not so considerable in North-western India as it is for the same difference of altitude in Bengal. Again, the temperature probably diminishes less rapidly with height at the North-western stations during January than it does in Bengal. Putting these two causes together, the effect is that the rate of upward decrease of temperature in the North-west is not by any means so rapid at the hottest time of the day as it is in Bengal. In the following Table, three pairs of stations are compared, so as to show the diurnal variation of the rate of temperature decrement with height during the month of January.

TABLE V.—Temperature Decrements at different times of the day during January.

	Darjiling—Goalpara * 6555 Feet	Chakrata—Roorkee. 6165 Feet.	Murree—Rawal Pindee 4692 Feet.
	°	°	°
Difference of Means	21·5	13·4	8·4
Rate per 1000 feet	3·3	2·2	1·8
Difference of Maxima	29·5	19·2	15·2
Rate per 1000 feet	4·5	3·1	3·2
Difference of Minima	13·5	7·6	1·6
Rate per 1000 feet	2·0	1·2	0·4

The vertical decrement of temperature in North Bengal at the hottest time of the day is, therefore, 43 per cent. more rapid than at north-western stations, namely, 4·5° F. per 1000 feet, instead of 3·1° or 3·2°, according to the Table. These rates of decrement are not those which actually obtain in the free atmosphere, since they include a certain variation with latitude; and, moreover, there is reason to believe that, on mountain slopes, the decrease is considerably less rapid than that given by balloon ascents. The ratio between the rates for the two regions compared is, how-

* This column is derived from BLANFORD's 'Indian Meteorologist's Vade Mecum,' page 57. The others are from my paper on the Temperature of North-Western India, 'Indian Meteorological Memoirs,' vol. 2.

ever, probably nearly correct, since the disturbing circumstances are similar; and this ratio shows that the diurnal convection currents, which tend to mix together atmospheric strata belonging to different altitudes, must be more active during the month of January in North Bengal than they are in the upper parts of the great plain—a relation which it is necessary to prove in order that the observed deviations of the resultant winds from their theoretical directions, in the two regions compared, may be explicable by KOPPEN'S hypothesis

The following considerations will enable us to discuss this point a little more fully. At page 236 of his valuable work on the Meteorology of the Bombay Presidency, Mr C CHAMBERS, F.R.S., has shown that rapid convective movements of the atmosphere, of the kind he calls "topsy-turvy," probably extend in the day time up to the level at which the temperature is the same as that given by the law of convective equilibrium, which may be proved from thermodynamical principles to be, for dry air, a decrease of 1° F. in 183 feet. There is no reason, however, to believe that all interchange between successive layers ceases at this level, for the molecular theory of gases indicates that the ultimate distribution of temperature would be uniform in a vertical column of the atmosphere free from all disturbing causes.

It is difficult to estimate, with any exactitude, the height to which such rapid convective movements here extend, for we do not know the rate of temperature decrement in the open air over the plains of India; all our observations at high levels having been made at mountain stations, where the temperature, at least in the day time, is higher than it would be in the free atmosphere. The results of GLAISHER'S balloon observations may help us, however, to obtain some idea of this height. When slightly smoothed, they may be thrown into the form of Table VI., which gives a rapid decrement of temperature in the first 2000 feet, especially with clear sky.

TABLE VI.—Rates of Temperature Decrement from GLAISHER'S Observations :—

Height.	Clear sky	Cloudy sky
	°	°
Sea-level to 1,000 feet	6·2	4·5
1000 „ 2,000 „	4·7	3·8
2000 „ 3,000 „	3·8	3·5
3000 „ 4,000 „	3·3	3·3
4000 „ 5,000 „	3·0	3·1
5000 „ 6,000 „	2·8	3·0
6000 „ 7,000 „	2·6	2·8
7000 „ 8,000 „	2·6	2·6
8000 „ 9,000 „	2·5	2·4
9000 „ 10,000 „	2·4	2·2
Sea-level to 10,000 „	33·9	31·2

This Table is represented graphically on Plate 21, which also shows the line of uniform decrement under the law of adiabatic convection. It will be seen that,

according to the principles above discussed, the topsy-turvy movements extend, when the sky is clear, to an average height (in summer in England) of about 2100 feet, whilst in cloudy weather they can hardly take place at all. In India there is, in all probability, a still more rapid decrement of temperature near the ground, when the sky is free from cloud From four years' observations made at Alipore, near Calcutta,* I find that at the hottest time of the day, about 2 P.M. during the months of greatest serenity, the temperature decreases 1·4° F. between 4 and 40 feet above the ground, or at the rate of 38 9° per 1000 feet. This we may take to be the most probable initial rate of decrease for clear weather. In the cloudy weather of July the rate is only 27 8° per 1000 feet, or 1·0° between 4 and 40 feet. Both these are very much greater than the initial rates given by GLAISHER's observations for England, which, by drawing tangents at the starting point of the curves on Plate 21, or differentiating the formulæ representing them, are found to be 7·7° and 5 2° per 1000 feet respectively; and we may infer that convective action is consequently much more energetic in India Whilst observations made at mountain stations thus give rates of decrement which are probably different from those obtaining in the free atmosphere, at all events for the layers of the atmosphere nearest the ground, we may still, however, get an idea of the *relative* intensities of diurnal convection currents over different parts of the Indian plain by a comparison like that made in Table V.

On the Central Indian plateau, and in South Rajputana and Sindh, where there is usually but little cloud in January, and the range of temperature is very great, such convection movements doubtless take place much more frequently, and extend to a greater height than they do in the Punjab and North-Western Provinces. To the interchange thus effected may probably be attributed the anomalous wind direction observed at Neemuch.

During the hot weather, vertical convection currents are extremely active over Northern and Central India, as testified by the constant occurrence of dust-whirls, familiarly known as "devils," as well as by the frequent piling up of cumulus clouds in the afternoons and the occasional occurrence of thunderstorms. Another evidence of such action is the gradual charging of the air with dust up to a height of 8000 feet or more; so that, just before the rains set in, it may be likened in appearance to muddy water or pea-soup At this season, the mean temperature of the 24 hours decreases rapidly on ascending, the mean rates on mountain slopes, when variations in latitude and longitude are eliminated, being the following † in the month of May.—

* 'Indian Meteorological Memoirs,' vol. 2, page 450 *et seq.*

† See 'Indian Meteorological Memoirs,' vol. 2, pages 132 to 136.

	North-Western Himalaya	Central India and Rajputana
	°	°
Sea-level to 1000 feet	4 11	2 84
1000 „ 2000 „	3 96	3 25
2000 „ 3000 „	3 79	3 66
3000 „ 4000 „	3 61	4 06

When allowance is made for the decrease of diurnal range on ascending, these rates become, for the maxima—

	North-Western Himalaya	Central India and Rajputana
	°	°
Sea-level to 1000 feet	6 5	6 2
1000 „ 2000 „	5 0	6 1
2000 „ 3000 „	4 0	6 0
3000 „ 4000 „	3 5	5 8

Such very rapid rates for Central India are partly confirmed by observations made during the present year (1886) in a tower at Allahabad, which for April and May give initial rates of decrement of 40° and 37·5° per thousand feet respectively at the time of diurnal maximum.

According to the law of decrement deduced from the last Table, the topsy-turvy movements described by CHAMBERS would, at the hottest time of the day, ascend the slopes of the North-Western Himalaya to an elevation of some 2600 feet; while, over the mountain tops of Central India and Rajputana, they would rise to above 11,000 feet, if the same law of temperature decrement held good *

The heights thus computed, of course, have no pretension to exactness; in fact, the former is undoubtedly too low, for the temperature decreases much less rapidly on a broad mountain zone heated by the sun than it does in the free atmosphere They suffice, however, to show that about midday, in the hot season, convective currents are very active up to an altitude of several thousand feet over the drier part of Northern and Central India. It is probable, therefore, that the wind direction anomalies, which are so striking at this season, may be explained by the descent, from a considerable height, of atmospheric strata which retain, for a time, the velocity acquired by them under the pressure differences prevailing at the level from which they descend.

* The temperature of the Himalayan slope may be represented by the formula $t = T - 7{\cdot}5\,h + 1{\cdot}0\,h^2 - 0{\cdot}08\,h^3$, while according to the ultimate law of convection $t = T - 5\,46\,h$; T being the sea-level temperature and h the height expressed in thousands of feet. Equating these, we get $h = 2560$ feet. For Central India and Rajputana, the formula best expressing the results in the Table is $t = T - 6{\cdot}3\,h + 0{\cdot}$[illegible] h^2, from which we find the height, where the temperature would be the same as that given by the law of convective equilibrium, to be 11,330 feet.

In Northern Bengal and Assam, at this season, the decrease of maximum temperature with height is so slow in the first three or four thousand feet, that no rapid convective action during sunshine can take place. The observations at Goalpara, 286 feet, and Darbhanga, 166 feet; at Shillong, 4794 feet, and Kathmandu, 4388 feet; and at Darjiling, 7421 feet,* give a combination from which variations in latitude and longitude are almost completely eliminated, and from the maximum temperatures of these places in May we get the following rates of decrement —

	°
Sea-level to 1000 feet	3·0
1000 „ 2000 „	3·3
2000 „ 3000 „	3·6
3000 „ 4000 „	3·9
4000 „ 5000 „	4·2
5000 „ 6000 „	4·5

These are so much slower than those for North-Western and Central India, that there is, comparatively speaking, no disturbance of the kind above discussed, and accordingly we find that the anomalous hot winds do not usually penetrate into Bengal, or become greatly modified in character by mixing with the lower atmospheric strata, if they do.

During the earlier part of October, south-west monsoon conditions prevail over Bengal, and the wind directions are those given by the local pressure gradients, but, towards the end of the month, convection currents, induced by the high range of temperature, become active, the day winds at the same time backing from east to north and north-west, thus causing the resultant for the month to be almost due north. In Central and North-Western India, where this month is characterised by clear skies and a large temperature range, there is probably a good deal of convective action, as may also be inferred from the afternoon fall of vapour-pressure in this month (see Table III.).

In this region, the mean temperature in October decreases with height above sea-level at the following rates.—

	°
Sea-level to 1000 feet	3·91
1000 „ 2000 „	3·69
2000 „ 3000 „	3·47
3000 „ 4000 „	3·25

When these are corrected for the diminution of daily range, which is extremely rapid—the decrease, for example, between 1500 feet (the Rajputana plain) and 4000 feet (Mount Abu) being from about 30° to 15°—the figures become for the daily maxima :—

* Formerly 6941 feet, but the observatory was removed to a new site in 1882.

		°
Sea-level to 1000 feet	. .	6 9
1000 „ 2000 „	. . .	6 6
2000 „ 3000 „	. .	6 3
3000 „ 4000 „	. . . .	6 0

In this region, therefore, we should expect the convective action to be even more energetic after the close of the rainy season than in the dry hot weather, though the rate of decrement falls off so rapidly that probably the action does not extend to so great a height.

If, therefore, we find, as we do, that the wind directions in October conform more closely than in May to the normal directions inferred from the distribution of pressure at sea-level, it is probably because there is less difference in the distribution of pressure at low and high levels in October than in May.* This will be investigated in the next section.

Part II.—The Velocity Anomaly.

In no part of the world is the diurnal variation of the wind velocity better marked than on the plains of Northern India. In the rainy season there is more or less wind both night and day, though the velocity at night is, as a rule, considerably less than in the day time; but during the dry season, from October to May, the nights are almost always perfectly calm, the only occasions when there is any wind at night occurring during the showery weather sometimes observed in January or February, and in occasional evening dust storms towards the end of the dry hot weather. At night, as the observations made at Alipore prove, the temperature increases for some distance with height above the ground, and under ordinary circumstances, therefore, there can be no convective interchange between upper and lower atmospheric strata. The diagrams on Plate 29 of the second volume of the 'Indian Meteorological Memoirs' show that from about 6.30 P.M. to 8.15 A.M., on the average of the year, the temperature increases with height. Between these hours, therefore, the wind movement is that due to the local pressure gradients at sea-level, whilst in the day-time, when convection takes place, the velocity is greater, owing to the descent of air from regions where the retardation by friction is much less than near the ground. This diurnal variation may be illustrated by the results of the Calcutta anemograms published at p. 23 of the first volume of the 'Indian Meteorological Memoirs,' and by three-hourly readings of a common anemometer made at Agra on four days each month for eight years, both of which are given in Table VII. For the last few years, valuable anemographic traces

* Another reason is that this rapid diminution of temperature in October obtains only during a short time each day. The temperature on the plains rises and falls so rapidly that, as will be shown in the next section, the mean rate of decrement during the six hours from 10 A.M. to 4 P.M. is only about half as great as that at the hottest time of the day.

have been obtained from three stations in Upper India—Jeypore, Roorkee, and Lucknow—but they have not been discussed yet.

TABLE VII.—DIURNAL VARIATION OF WIND VELOCITY.

Hours.	Calcutta.	Agra
	Miles per hour	Miles per hour
Midnight to 3 hours	3 28	2 87
3 " 6 "	2 96	3 33
6 " 9 "	4 51	4 25
9 " 12 "	6·12	5 71
12 " 15 "	6 55	6·82
15 " 18 "	5 65	5 19
18 " 21 "	4·36	3 49
21 " 24 "	3 87	2 95
Mean .	4 69	4 33

The figures in the Table represent annual means. They prove that the velocity attains its maximum about the hottest time of the day, and its minimum about midnight at Agra and 4 A M at Calcutta: the exact instants of the turning points, found by taking differences, being 3 h. 31 m and 12 h. 58 m. at Calcutta, and 0 h. 27 m. and 13 h. 13 m at Agra. The anemograms for several years at Karachi have been discussed by Mr. F. CHAMBERS and published in vol. 1. of the 'Meteorological Memoirs,' but, as this station is on the sea-coast, where there is little frictional retardation, the hourly means do not exhibit anything like such a large variation in proportion to the mean velocity, which, at that station, is nearly 17 miles per hour The extreme hourly distances traversed are 14 4 miles between 5½ h. and 6½ h., and 20·9 miles between 14½ h. and 15½ h.

At all the stations in North India, for which the hourly variations of the wind resultants have been computed* a double diurnal oscillation, related to that of barometric pressure, has been more or less distinctly observed, but the amplitude of this double oscillation, which may be taken to depend on the diurnal variation of pressure differences near sea-level, is extremely small in comparison with that of the single oscillation due to diurnal heating of the ground and consequent convective action. We have already seen that, in the month of May, the pressure differences over the upper Gangetic plains are often evanescent; and it will be shown below that, in this region, the differences of the mean pressures are very small at all times of the year. It must frequently happen, therefore, that any east and west pressure gradient, which may exist, is due solely to the difference in phase of the diurnal variation owing to difference of longitude; the epochs of maximum and minimum, and the range of this variation being very nearly constant through many degrees of longitude. Thus, taking two points 15 degrees apart on the parallel of Allahabad, we should find the

* 'Indian Meteorological Memoirs,' vol 1 (Papers 1, 9, and 10)

following pressure differences on days in May when the mean pressures at sea-level are alike :—

TABLE VIII.—Pressure Differences in May dependent on Diurnal Variation only.

Hour	Difference in 15 deg	Hour	Difference in 15 deg	Hour	Difference in 15 deg.	Hour	Difference in 15 deg
	"		"		"		"
0 to 1	+ 0112	6 to 7	− 0153	12 to 13	+ 0246	18 to 19	− 0133
1 „ 2	+ 0069	7 „ 8	− 0103	13 „ 14	+ 0275	19 „ 20	− 0193
2 „ 3	− 0030	8 „ 9	− 0051	14 „ 15	+ 0248	20 „ 21	− 0191
3 „ 4	− 0130	9 „ 10	+ 0009	15 „ 16	+ 0176	21 „ 22	− 0127
4 „ 5	− 0190	10 „ 11	+ 0088	16 „ 17	+ 0074	22 „ 23	− 0020
5 „ 6	− 0190	11 „ 12	+ 0173	17 „ 18	− 0036	23 „ 24	+ 0077

These differences, which are taken from the 'Indian Meteorological Memoirs,' vol. 1, page 326, would produce N.W. winds, indicated by the positive sign, from about 9 h. 21 m to 17 h. 7 m., and again from 22 h. 42 m. to 2 h. 12 m, the remaining hours having gradients for S.E winds, indicated by the negative sign. Now, from the paper last referred to, we learn that, in the hot season, the resultant wind at Allahabad blows at all hours from points between S. 73° W. and N. 5° W., the double oscillation suggested by the Table being combined with and masked by a prevalent N.W. direction, though it is plainly indicated in the diagram formed by joining the ends of the lines representing the resultants for the several hours. The differences in the Table, in fact, are so small, not exceeding ·0275-inch in about 813 geographical miles, or ·002-inch per degree of the earth's surface, that near the ground, where the friction coefficient is large, the winds due to them would be so light as to be almost imperceptible, and, accordingly, we find that the double gyration of the wind vane, which would be produced by this cause, is almost entirely hidden by the influence of the more powerful currents of the higher atmosphere which descend in the middle of the day. The N.W. winds due to the diurnal inequality of pressure are, however, probably more than twice as strong in the afternoons as in the early mornings, the maximum pressure differences being in the ratio of 275 to 112, so that this cause combines with convective interchange between the lower and higher strata to make the velocity greatest about the hottest time of the day.

On comparing the mean pressures for each month at several stations on the Upper Gangetic plains, we shall find, as has been stated in the introductory paragraphs, that their differences have no apparent relation to the mean wind velocity; but the latter has, in the annual as in the diurnal period, a very distinct relation to the temperature. In the North-Western Provinces and Oudh there are four stations, for which we possess registers extending, with few interruptions, over nearly 18 years. When the monthly mean pressures of these are reduced to sea-level and the value of gravity at latitude 45°, we get the figures given in Table IX., the differences of which from place to place are strikingly small.

TABLE IX.—Mean Pressures at Places in the North-Western Provinces and Oudh, reduced to Sea-level and Latitude 45°.

Month.	Roorkee 17–18 Years	Agra 16–18 Years	Lucknow 16–18 Years.	Benares 16–18 Years	Jhansi 15–18 Years
	"	"	"	"	"
January .	30 030	30 032	30 030	30 016	30 023
February	29 949	29 957	29 949	29 943	29 951
March	29 842	29 842	29 835	29 826	29 844
April	29 707	29 704	29 698	29 690	29 709
May .	29 577	29 578	29 593	29 587	29 586
June	29 440	29 450	29 465	29 463	29 460
July .	29 465	29 464	29 475	29 467	29 470
August	29 527	29 531	29 541	29 537	29 540
September	29 639	29 637	29 638	29 633	29 633
October	29 824	29 822	29 823	29 814	29 821
November	29 976	29 976	29 972	29 954	29 956
December . .	30 042	30 042	30 044	30 027	30 026
Year .	29 750	29 753	29 755	29 746	29 754

If we take the central station of the group—Lucknow—and measure the gradient from it towards each of the others without reference to the direction of the slope, we get the mean and extreme gradients per 60 geographical miles, which are compared in Table X. with the mean wind velocity at the same four stations, and with the mean temperature and the diurnal temperature range.

TABLE X.—Wind Velocity in the North-Western Provinces compared with Pressure Gradient and Temperature.

Month.	Mean Gradient.	Maximum Gradient.	Wind Velocity per Diem	Mean Temperature	Diurnal Range.
	"	"	Miles	°	°
January	0020	0029	58 2	59 2	27 5
February	0012	0032	73 7	64 9	27 4
March	0027	0036	87 2	74 6	30 0
April	·0021	0038	98 0	85 4	30 7
May .	0037	·0038	114 4	91 1	27 1
June .	0031	0060	120 3	91 5	21 3
July .	0028	0044	95 8	85 0	13 9
August	0024	0034	79 6	83 7	13 1
September	0010	0020	75 0	83 0	16 0
October .	0011	0036	52 6	77 5	25 3
November .	0035	·0072	40 6	67 7	30 7
December .	0033	0068	45 6	60 0	28·9

This Table fully justifies the statement above made as far as the monthly means are concerned. The smallest values of the gradient, both mean and extreme, are found in September, when the wind velocity does not differ much from the average. The mean gradient is steepest in May and November, the one month having a high wind velocity, and the other the lowest of the year, while the maximum value of the steepest gradient is also in November, the month of least wind movement. The suggestion,

however, arises that the pressure differences of any two months may have the same small mean value, while in one month the barometer stood at a uniform height all over the country, and in the other there may have been a series of more or less violent disturbances causing strong winds, that, in short, it is useless to compare mean values instead of actual observations. In North India, however, no feature of the meteorology is more remarkable than the simultaneity of barometric movements over a large extent of country; all the usual fluctuations, which are very numerous, though of small range, even in the most settled weather, occurring with the most absolute uniformity over an extent of country often larger than France, Germany, and Austria. Small disturbances, of the nature of cyclonic storms, do occur occasionally, more especially during the rainy season and in the cold weather, during the month of January or about the end of December; but, while they, doubtless, make the mean wind velocity, for the months mentioned, somewhat greater than it would otherwise be, they do not perceptibly interfere with the regular march of its annual variation

Table X. also shows, contrary to what might be expected from the preceding part of this Paper, that the velocity is much more distinctly related to the actual temperature of each month than to the daily range of the thermometer. The explanation of this probably is that in the interior of Northern India the daily range at all seasons is sufficient to set up convection currents, while the velocity, of the winds which descend from a considerable elevation over the plains is dependent upon the pressure gradients prevailing at high levels. Now these gradients are doubtless subject to a distinct annual range, depending on the temperature; for, if the plains be heated more than usual relatively to the Himalayan slope, the result will be to raise the planes of equal pressure and make them incline towards the mountains, thus giving rise to westerly upper currents, the velocity of which will, *cæteris paribus*, be proportional to the temperature difference between plains and mountains.

Now, if we compare the mean temperature for stations on the plains of the North-Western Provinces and Oudh, given in Table X., with that of the mountain slope at a height of about 5700 feet, as computed from the observations of Chakrata, Ranikhet, Pithoragarh, and Kathmandu, we find the following differences, which are subject to a very distinct annual variation, nearly coincident in phase with that of wind velocity on the plains :—

Month :—	January.	February.	March	April	May	June
Temperature difference . . .	13°4	17°1	18°·1	22°·1	24°2	21°3
Month :—	**July**	**August.**	**September**	**October**	**November**	**December**
Temperature difference .	16°1	15°3	15°·4	15°3	12°·0	10°·0

A similar relation obtains to the greatest elevation at which observations have been made on the mountains The temperature difference, between Leh (11,503 feet) and the Punjab plains, is greatest in May and June and least in November, so that, if we may assume the temperature decrement in the free atmosphere over the plains to be less variable than on the mountain slope, as seems *à priori* probable, there must be a greater pressure gradient for westerly winds at high levels in May and June than in November.

The proper way to verify this and various other deductions, which have been made from the hypothesis that convective interchange is the cause of the observed wind anomalies, is to find the distribution of pressure over India at a considerable height in the atmosphere, say 10,000 feet. For this purpose I have selected some 40 stations out of the large number in India and Ceylon for which 10-year averages of pressure are published in the Meteorological Report for 1884, and have tried to reduce their mean pressures for January, May, July, and October to the proposed elevation. The majority of stations selected are those on mountain tops or high plateaux, but, in order to complete the maps on Plate 20, fourteen stations situated at levels below 1500 feet have had to be added.

Table XI. gives the stations, with their elevations and the mean pressures of the months selected. Two of the stations mentioned, Dodabetta and Shillong, are not now on the list of meteorological observatories, but their pressures and temperatures have been taken from old reports. In the case of Dodabetta, the published barometric heights had first of all to be corrected for the temperature of the mercury. The elevations given have been found by levelling to Great Trigonometrical Survey stations, or other datum levels, in every case except Pithoragarh and Shillong. The elevation of the last-mentioned has been found by comparing barometric observations at the station with those made simultaneously at Darjiling and Silchar, which places are on opposite sides of Shillong, and are respectively considerably higher and lower than that station. The error in the determination cannot amount to more than two or three feet The elevation of Pithoragarh has been computed by comparing, month by month, the barometric observations of four years at that station with those of Ranikhet and Kathmandu, and taking the mean of the results. The error of this mean must be very small.

The pressure observations of two stations in Southern India, Wellington and Mercara, have had to be rejected, as, when reduced to a common level, they did not agree with those of neighbouring stations, such as Coimbatore, Dodabetta, and Bangalore, while the latter agreed very fairly with one another. The assigned elevation of Wellington appears to be about 200 feet too high, and that of Mercara 135 feet too low. These elevations have been deduced trigonometrically or by spirit-levelling from points fixed in the early days of the Survey, when the uncertainties of terrestrial refraction were not sufficiently allowed for.

The observations made at Simla have also been rejected, as there is some uncertainty about the true elevation of the barometer.

TABLE XI.—Observed Pressures used in constructing Table XIV.

Station	Elevation in feet	Pressure (corrected for temperature, &c)			
		January	May	July	October.
Newara Eliya	6,240	24 010	23 994	23 984	23 996
Kandy	1,696	28·226	28 137	28 165	28 187
Coimbatore	1,348	28 609	28 451	28 466	28 523
Dodabetta	8,644	22 045	22 018	21 946	22 019
Bangalore	2,981	27 013	26 863	26 857	26 931
Bellary	1,455	28 505	28 285	28 295	28 400
Belgaum	2,550	27 437	27 285	27 255	27 362
Secunderabad .	1,787	28 192	27 959	27 914	28 080
Poona	1,849	28 131	27 939	27·872	28 046
Chikalda	3,656	26 433	26 217	26 109	26 345
Pachmarhi	3,528	26 533	26 319	26 193	26·461
Raipur .	960	29·043	28 679	28 601	28 897
Indore	1,823	28 168	27 896	27 791	28 074
Hazaribagh . .	2,007	27 984	26 683	27 568	27 869
Sutna . .	1,040	28 977	28 593	28 484	28 823
Nimach . .	1,639	28 367	28 054	27 937	28 258
Mount Abu	3,945	26 146	25·961	25·818	26 092
Ajmere .	1,611	28 412	28 075	27 954	28 294
Quetta	5,501	24 684	24 546	24 397	24 688
Peshawar . . .	1,110	28 955	28 547	28 366	28 783
Leh	11,503	19 646	19 689	19·604	19 727
Murree .	6,344	23 882	23 789	23 701	23 918
Chakrata	7,052	23 263	23 196	23 108	23 302
Ranikhet .	6,069	24 113	24 013	23 928	24 113
Pithoragarh .	5,363	24 758	24 624	24 534	24 729
Kathmandu	4,388	25 714	25 492	25 383	25 633
Darjiling .	7,421	22 964	22 915	22 859	23 018
Shillong .	4,794	25 272	25 118	25 058	25 255
Sibsagar	333	29 739	29 430	29 281	29 570
Dacca . . .	35	30 015	29 698	29 571	29 845
Akyab . . .	20	30 005	29 763	29 666	29 856
Diamond Island	41	29 972	29·772	29 727	29 845
False Point	21	30 031	29 684	29 558	29 848
Vizagapatam	31	30 005	29 688	29 596	29 824
Madras .	22	30 000	29 733	29 728	29 850
Bhuj . .	395	29 639	29 311	29 162	29 491
Karachi . .	49	30 016	29 646	29 490	29 862
Multan . .	420	29 639	29 189	29·030	29 444
Delhi	718	29 316	28 904	28 780	29 143
Lucknow . .	369	29 674	29·264	29 152	29·408

In the formula connecting barometric pressure with height above sea-level, the most important variable is the temperature of the air. The chief object of this paper being to find out the cause of the anomalous *day* winds, the temperature of the air, in the day time when these winds blow, is that which should be adopted in the reduction to a height of 10,000 feet, which is now attempted For the several observing stations this is obtained with sufficient accuracy by combining the maximum temperature with that of the usual hours of observation, 10 A.M. and 4 P.M., or where observations have not been made at these hours, as at the Ceylon stations, by applying to the daily

mean a correction derived from some similarly situated station which is not too far off, this correction being made proportional to the daily range. These day temperatures are given in the 2nd, 3rd, 4th, and 5th columns of Table XII.

TABLE XII.—Temperatures used in constructing Table XIV.

Station	Temperature at Station				Temperature at 10,000 Feet			
	January	May	July	October	January	May	July	October.
	°	°	°	°	°	°	°	°
Newara Eliya	65 6	68 8	63 6	65 2	58 2	61 4	56 2	57 8
Kandy	79 9	85 1	81 3	81 3	55 8	61 0	57 2	57 2
Coimbatore	82 1	88 1	82 3	82 5	56 1	62 6	56 8	57 0
Dodabetta*	57 5	62 9	55 6	57 4	54 3	59 9	52 6	54 4
Bangalore	75 5	86 6	77 7	77 9	56 3	67 4	58 3	58 5
Bellary	82 8	97 3	87 1	85 1	57 5	72 0	62 3	60 3
Belgaum	79 5	87 2	73 2	79 0	58 5	66 2	52 2	58 0
Secunderabad	78 7	97 5	81 9	81 8	54 7	73 5	58 2	58 1
Poona	79 9	94 6	78 5	82 8	56 2	70 9	55 0	59 1
Chikalda	69 9	88 7	72 3	74 9	52 9	71 7	55·1	57 9
Pachmarhi	67 9	90 1	73 4	74 3	50 4	72 6	55 7	56 8
Raipur	77 7	105 0	83 1	84 3	49 8	77 1	56 2	56 4
Hazaribagh	70 3	94 2	82 2	80 7	47 3	71 2	59 3	57 8
Sutna	71 0	100 5	86 4	86 1	43 3	72 8	59 7	58 4
Indore	76 3	96 9	81 0	85 6	52 6	73 2	57 4	61 9
Nimach	72 2	98 1	83 4	87 4	47 9	73 8	59 3	63 1
Mount Abu	65·1	85 1	74 9	75 7	49 0	69 0	58 6	59 6
Ajmere	69 3	98 2	87 5	89 7	46 1	75 0	64 5	64 3
Quetta	47 2	78 4	87 3	71 3	35 8	66 3	75 2	59 2
Peshawar	60 1	91 8	96 2	82 0	33 8	64 5	68 9	54 7
Leh	25 7	58 1	70 0	50 7	31 8	60 5	70 9	54 9
Murree	45 0	71 8	73 9	66 7	36 0	62 6	64 9	57 5
Chakrata	47 7	70 7	67 5	63·4	40 3	63 3	60 4	56 0
Ranikhet	51 5	73 5	71 1	66 9	41 5	63 5	61 3	56 9
Pithoragarh	55 3	75 5	74 9	70 6	43 3	63 5	62 9	58 6
Kathmandu	58 1	75 0	78 5	74 3	43 4	60 1	63 6	59 6
Darjiling	42 6	59 9	64 5	58·5	36·5	53 8	58·4	52 4
Shillong	56 5	72 4	73 1	67 9	43 0	58 8	59 5	54 4
Sibsagar	68 0	81 9	87 5	82 4	38 3	52 2	57 8	52 7
Dacca	74 6	88 2	86 7	85 7	40 9	57 1	53 6	52 0
Akyab	76 2	87 4	82·1	85 2	42·5	56·3	51·0	51 5
False Point	75 5	90 2	87 2	88 4	42 8	56 5	56·1	54 7
Vizagapatam	81 9	93 5	80 6	88 8	48 2	59 8	58 5	55 1
Madras	81 9	93 9	90 7	85 7	48 2	60 2	59 6	54 6
Diamond Island	79 8	87 0	83 0	84 4	46 1	55·9	51 9	53 3
Bhuj	75 6	96 0	87 8	90 1	44·2	64 6	58 4	60 7
Karachi	71 8	89 0	87 6	87 6	38 2	58 0	56 6	54 0
Multan	64 0	99·5	98 7	88 4	32 6	68 1	67 3	57 0
Delhi	65 4	98 4	90 2	86 9	35 8	68 8	62 2	57 3
Lucknow	68 3	100 1	90 3	87 8	36 9	68 7	60 9	56 4

* The temperature of this station appears to be about 3° too low in comparison with Coimbatore, near which it lies. Coimbatore, is, however, in the dry region to the lee of the mountains, and when Dodabetta is compared with the west coast stations it appears warm in comparison. Thus the mean day temperatures of Cochin and Mangalore when reduced to 10,000 feet are:—

January.	May.	July	October.
51 4°	55·6°	49 8°	51·6°

The pressures given in Table XI. do not need any special correction for daily range, because in every part of India the mean height of the barometer between 10 and 16 hours does not differ by more than a few thousandths of an inch from the mean of the twenty-four hours, and it will be seen immediately that, with the unavoidable uncertainty there is as to temperature, it is unnecessary to take such small differences into account.

It has already been pointed out, in discussing the probable height to which convection currents extend at the hottest time of the day, that nothing accurate is known regarding the decrease of the day temperatures with height above the sea in the free atmosphere. But, by taking the differences between the day temperatures given in Table XII for stations in North-Western and Central India and the sea-level values of the mean temperatures of the same places, published in vol. 2 of the 'Indian Meteorological Memoirs,' pages 137–141, and combining the results into groups for stations below 2000 feet, between 2000 and 5000 feet, between 5000 and 8000 feet, and above 8000 feet, we can construct a set of curves giving the variations of temperature on mountain slopes, unaffected by changes of latitude or longitude. By graphic interpolation we can, then, get from these the rates of decrement. The results for the four months in the Table are shown in Table XIII. They exhibit a considerable diversity of character, and a tendency in the summer to an increase of temperature with height at the greatest elevations, owing to the greater diurnal range of the thermometer at Leh than at the stations on the outer ranges about 7000 feet elevation, whilst it is practically certain that there is no such tendency in the free atmosphere.

TABLE XIII.—Vertical Decrement of Day Temperature on the Mountains of Central and North-Western India.

Height.	January	May	July	October
	°	°	°	°
0 to 1,000 feet . . .	4·4	7·3	6·0	4·6
1,000 „ 2,000 „ .	3·4	6·1	5·3	4·1
2,000 „ 3,000 „ . .	2·7	5·1	4·6	3·8
3,000 „ 4,000 „ .	2·2	4·1	4·0	2·8
4,000 „ 5,000 „ .	1·8	3·2	3·4	2·4
5,000 „ 6,000 „ . .	1·6	2·3	2·8	1·7
6,000 „ 7,000 „ . .	1·6	1·5	2·3	1·5
7,000 „ 8,000 „	1·7	0·8	1·9	1·4
8,000 „ 9,000 „ . . .	1·9	0·1	1·5	1·5
9,000 „ 10,000 „ .	2·1	0·5	1·1	1·6
0 to 10,000 feet . .	23·4	30·0	32·9	25·4

The total decrement between sea-level and 10,000 feet is less in every month than that found by GLAISHER in the free atmosphere over England and France, as was to

be expected. The difference is not very great in the summer months, but amounts to 10° or more in January. At this season, however, the decrement in Bengal is, as we have seen, much more rapid, and such appears also to be the case in Southern India, when we compare the temperature of Dodabetta with that of Coimbatore and the nearest stations on the west coast,

For the whole country, therefore, after due consideration, I have decided to adopt the rates of decrement found by GLAISHER, and given above in Table VI. These include an allowance for diminishing diurnal range, since the observations were made in the day time only. Though considerably more rapid for some months than those in the last Table, they are, perhaps, too slow on the whole, the decrement in India being probably greater near the ground than in England, where insolation is not so powerful. The last four columns of Table XII give the probable temperatures at 10,000 feet, deduced by means of Table VI., the rate for cloudy skies being used when the average cloud proportion for the month amounts to half the expanse or more, and the rate for clear skies in the other months.

The formula adopted for the barometric reductions is the simple one.—

$$\log p = \log \mathrm{P} - \frac{h}{60{,}360\left(1 + \frac{\mathrm{T} + t - 64}{986}\right)}.$$

With the values of T and t in Table XII., and the mean value of p, about 21 inches, it can be shown, by differentiating, that for a difference of elevation of 10,000 feet, an error of 1° in estimating t would give an error in the resulting value of p lying between ·008 and ·010 inch. As the margin of possible error in many of the values of t may amount to several degrees, there is evidently nothing to be gained by using a more complicated formula, in which the variations of gravity and of density, owing to the presence of more or less water vapour, are taken into account.

The computed results, which have been reduced to the standard value of gravity, are given in Table XIV. Considering the uncertainty of the adopted rates of decrease of temperature, and the widely different altitudes of the base stations, these results are remarkably consistent. Even apparent exceptions to this consistency, such as the low pressure over Dodabetta compared with the neighbouring station Coimbatore, or of Chikalda compared to Poona, or Mount Abu to Neemuch, serve to confirm the probable correctness of the rates of temperature decrement adopted; for these are instances of the mid-day distribution of pressure which gives rise to the diurnal mountain winds, observed in all the warmer regions of the world.

TABLE XIV.—Mid-day Pressures at 10,000 Feet (corrected to Gravity of Latitude 45°).

Station.	Pressure			
	January	May	July	October
	"	"	"	"
Newara Eliya	20 922	20 924	20 888	20·906
Kandy	20 953	20 950	20 882	20 947
Coimbatore	20 996	20 958	20 901	20 945
Dodabetta	20 934	20 919	20 836	20 909
Bangalore	20 979	20 969	20 901	20·959
Bellary	21 014	21 021	20 886	20 966
Belgaum	21 017	20 978	20 809	20 955
Secunderabad	20 992	21 031	20 825	20 949
Poona	21 039	21 035	20 806	20 977
Chikalda	20 997	20 988	20 757	20 972
Pachmarhi	20 959	20 993	20 740	20 962
Raipur	20 947	21·037	20 719	20 946
Hazaribagh	20·925	20·958	20 746	20 956
Sutna	20 894	20 982	20 740	20 981
Indore	20·985	21 012	20 755	21 018
Neemuch	20 941	21 009	20·758	21 067
Mount Abu	20 951	20·964	20 764	20·999
Ajmere	20 925	21 013	20·805	21 070
Quetta	20 837	20·919	20 844	20 993
Peshawar	20 815	20 914	20 835	20 970
Leh	20 792	20·767	20 654	20 820
Murree	20 800	20 863	20 799	20 950
Chakrata	20 818	20 858	20 766	20 922
Ranikhet	20 816	20 856	20 768	20 906
Pithoragarh	20 839	20 863	20 784	20 920
Kathmandu	20 889	20 846	20 784	20 958
Darjiling	20 811	20 835	20 801	20 923
Shillong	20 832	20 822	20·779	20 902
Sibsagar	20 847	20 825	20 773	20 931
Dacca	20 882	20 865	20 741	20 920
Akyab	20 880	20 888	20 743	20 904
False Point	20·892	20 854	20 737	20 941
Vizagapatam	20 965	20 902	20 803	20 936
Madras	20 940	20 929	20 899	20 950
Diamond Island	20 923	20 903	20 815	20 919
Bhuj	20 917	20 973	20·754	21 022
Karachi	20 842	20 852	20 722	20 970
Multan	20 794	20·955	20 828	20 985
Delhi	20·811	20·947	20 764	20 975
Lucknow	20 822	20 964	20 767	20 967

The pressures given in this Table are charted on Plate 20, which is headed "Mid-day Pressures at 10,000 Feet." These charts deserve some study in detail.

In January, the region of lowest pressure, bounded by the isobar of 20 80 inches, covers a considerable part of Afghanistan and the Punjab, and extends through Kashmir into Ladakh and Thibet. The north-east corner of the Punjab and the region of the Hindu Kush have probably a slightly higher pressure. The isobar of

20·80 inches sends a long loop down the valley of the Ganges to the north of Delhi. The next two isobars are very similarly curved, but they extend further to the east over known regions, and are seen to curve southward again over Bengal and the head of the Bay. The others probably form closed curves of a roughly triangular form round a centre lying in the western Deccan near the town of Sholapur.

The system of winds, pertaining to this distribution of pressure at a height where the friction coefficient is very small, would be S.W. over the Bombay coast, Sindh, and Rajputana, S. or S.W. at most of the Himalayan stations, N. or N.E. at Murree, S.E. along the foot of the Central Himalaya, N.W. in Bengal, W. over Central India and the Gangetic plains, N.E. over nearly the whole of the Madras Presidency, and E. or S.E. in Travancore and Cochin. Now in almost every instance where the observed prevailing wind direction for this month is inconsistent with the distribution of pressure at sea-level it is in accordance with this upper system of wind currents.

A possible solution of another question of great interest is suggested by the chart. The cause of the disturbances which produce the winter rainfall of the Punjab, North-Western Provinces, and Rajputana, and the snows of the North-west Himalaya, is not yet fully understood. Mr. F. CHAMBERS has suggested that these precipitations may be due to disturbances of the same nature as the winter storms of high latitudes entering India from the west,* while Mr. BLANFORD attributes their origin to the disturbance of atmospheric equilibrium by the upward diffusion and condensation of water vapour formed locally.† It is difficult to understand how a cyclonic storm, if it be of the same nature as those formed in the Bay of Bengal at the turn of the seasons, can cross a mountainous country like Afghanistan; but that whirling storms do cross the Rocky Mountain region is a common experience in America.

Without presuming to dogmatise, therefore, about a subject which I have not sufficiently studied, I would suggest that it is possible such storms may be formed in more than one way, and that some of them at least may travel for long distances if they do not originate in a region of relatively low pressure in some upper stratum of the atmosphere. Such whirling storms, crossing the Indus in the month of January, might travel along the axis of lowest pressure directly into Kashmir, or they might pass along the V-shaped depression in the Valley of the Ganges as far as Behar; but, owing to the higher pressure in West Bengal and Nepal, they could seldom penetrate to Assam, except by the roundabout way of Thibet, in which case their chance of survival after twice crossing the Himalayas would be infinitesimal. Now these possible paths suggested by the form of the isobars are precisely those most frequently taken by the disturbances which bring the winter rains.

The chart for May shows that the distribution of pressure at 10,000 feet elevation during this month is extremely simple if we leave out of consideration local minima

* 'Nature,' vol. 23, page 400.

† 'Journal of the Asiatic Society of Bengal,' vol. 52, Part 2, and 'Indian Meteorological Memoirs,' vol. 3, page 93.

in the Nilgiris and Satpuras, such as cause the diurnal mountain winds above referred to. The high pressure region in the Deccan is much enlarged, and the centre shifted northwards to Berar. Round this the isobars curve in a nearly triangular form, following closely the outline of the country as bounded by the sea, the Hála and Suleiman ranges and the Himalayas; the most important exception to this rule being a recurvation of the lines of equal pressure round a low-pressure area extending obliquely across Bengal from Balasore to Assam. The wind system, induced by this distribution of pressure, would be S. or S.W. in the valley of the Indus and Rajputana, W. or N.W. over the rest of Northern India, as far east as Lower Bengal and as far south as Berar, S.W. in Eastern Bengal and Burma, N.E. along the eastern side of the Peninsula from Orissa to Negapatam, and S.E. or S. on the Malabar coast. That is to say, over the interior of Northern and Central India the upper currents in this month have exactly the same direction as the anomalous hot winds which we have reason to suppose may be caused by their descent under the convective action set up by the daily passage of the sun.

The steepest gradient on the map is that to the west of the Orissa coast. Disturbances originating at the head of the Bay would therefore, according to the usual experience that the line of least resistance is that of lowest pressure, experience much difficulty in proceeding westwards, but would meet with less resistance in the northward direction over Bengal. As a matter of fact, the majority of such storms as are formed in this region during May do proceed northwards,* and those which turn westwards across the plain of Orissa do not survive to reach the Central Provinces. Mr. ELIOT attributes their rapid break-up to the comparatively small height to which they extend in the atmosphere, and the resistance they experience from the hills west of Orissa, but, while this may be one reason for their disappearance, the rapidly increasing pressure in a westward direction at moderately great elevations is doubtless another.

The low-pressure area extending obliquely across Bengal and Assam is the scene of frequent local storms, known as "Nor'-Westers," in the hot season, and the area of maximum "spring storm rainfall" on Mr. BLANFORD's charts† coincides very closely with it. At page 105 of the work cited, Mr. BLANFORD gives a clear and interesting description of these storms and their origin, which he attributes to the mixing, by convective action, of the southerly sea winds with the north-west winds, which blow over the Chutia Nagpur plateau, and probably continue their course eastwards over the low plains of Bengal at nearly the same level. The line of maximum rainfall in May, according to Mr. BLANFORD's Table, passes through or near the following places, the rainfall gradually increasing as the distance from the sea increases and the Khasia Hills are approached:—

* "Account of S.W. Monsoon Storms generated in the Bay of Bengal during the years 1877 to 1881," by J. ELIOT, 'Indian Meteorological Memoirs,' vol. 2.

† "The Rainfall of India," 'Indian Meteorological Memoirs,' vol. 3.

Balasore	4·78 inches
Midnapore	5·08 "
Krishnaghar	6·62 "
Jessore	7·56 "
Faridpur	8·33 "
Dacca	9·26 "
Mymensingh	11·74 "
Sylhet	21·64 "
Cherra Punji	51·46 "

The line joining these places coincides almost exactly with the axis of lowest pressure at 10,000 feet on the chart for May.

In July the triangular area of high pressure in the Deccan has retreated to the extreme south-east of the Peninsula, and occupies those parts of the Carnatic where the rainfall at this time of the year is very light.

Along the foot of the Himalayas there is a zone of relatively high pressure, 20·80 inches, or slightly less, which widens out in the west, so as to include the western Punjab and Afghanistan. Parallel to this, and south of it, from the head of the Bay of Bengal to Sindh, extends a region of lower pressure, below 20·75 inches; which is, however, interrupted along the line of the Aravalis by a belt of higher pressure, exceeding 20·80 inches in the vicinity of Ajmere.

As regards the country south of the parallel of 24°, there can, therefore, be but little difference in direction between the winds prevailing at high and low levels, and, except for the diminution of frictional retardation on ascending, the upper currents have probably no greater velocity than those at the ground surface, since the gradients of 10,000 feet are no steeper than those at sea-level. In Bengal the winds at this elevation should be easterly or north-easterly, while those deduced from the distribution of pressure at sea-level by BUYS BALLOT's Law would be southerly. The prevailing winds are south-easterly, or in Cachar north-easterly, the average angle between the wind and the isobars being about 60°, an effect which is possibly due in part to convective interchange; though, as we have already seen, the activity of such interchange at this season is not great. The northerly winds at Multan, Jeypore, and Agra, to which attention was directed in the first part of this Paper, are evidently the effects of the high pressure in the upper strata over Afghanistan and the Aravali region respectively, these stations being all in the drier part of North-western India, where clear skies and a high temperature range, with their accompaniment of active convection currents, are not infrequent even in July.

The extensive belt of low pressure, at high altitudes, stretching across the Central Provinces and the head of the Bay, is doubtless connected with the fact, brought prominently to notice in Mr. ELIOT's paper last referred to, that cyclonic storms formed during the prevalence of the S.W. monsoon usually pass inland across the Orissa coast, meeting with little obstruction from the ranges of low hills they have to

surmount,* and often passing right across India to Sindh, the Rajputana desert, or the western Punjab, while those formed at the spring and autumn transition periods, which either do not extend to so great a height or meet with increasing pressures at moderate elevations as they go westwards, very seldom succeed in crossing from Orissa to the Central Provinces.

From the charts attached to Mr. ELIOT's paper it appears that, during the five years 1877–1881, inclusive, 30 cyclonic storms crossed the coast near the head of the Bay in the months of June, July, August, and September. Of these, 23 proceeded westwards at first, and the remaining seven went northward into Bengal. Only five storms occurred in the month of July, all of which crossed the Orissa coast in the westward direction. The majority of such storms either proceed almost due west towards Gujerat and Sindh, or turn off northwards to the valley of the Ganges, doubtless owing to the obstruction presented by the high-pressure region in Eastern Rajputana Of the 23 above enumerated, 9 turned off towards the Ganges Valley, 2 succeeded in crossing the Aravali range near Ajmere and ultimately reached the Indus Valley, and the remainder either proceeded *viâ* the Narbada Valley to Gujerat and Sindh, or died out during the trans-continental passage. To the frequent passage of such storms over the Central Provinces and Malwa is doubtless to be attributed the fact of these regions having a heavier and less variable rainfall than any other part of the interior of the Indian Peninsula.

In October, the highest pressure is over Rajputana and Malwa, and the lowest in Arakan and Upper Burma, and in the Trans-Himalayan regions of Ladakh and Thibet. The general appearance of the isobars is that of a system of more or less wavy loops surrounding the high-pressure centre in Rajputana, and extending to an unknown distance westwards over the Arabian Sea. The system of winds due to this distribution of pressure must be E. or S E. in the extreme south of the Peninsula, S.W. in the Indus Valley as well as in Lower Burma and Tenasserim, W. in the Punjab, N.W. in the Ganges Valley, N. over the Bengal delta, and N.E. over the rest of the country. Except in Bengal and on the western side of the Indus Valley, this is not dissimilar to the system due to the pressure distribution at sea-level, and accordingly, we find it is only in these parts of the country that there are any marked anomalies in the direction of the wind observed on the plains.

In confirmation of the general direction of the circulation of the atmosphere at 10,000 feet elevation, we have the evidence of the winds at the hill-stations in the Himalayas and in Southern India. On the whole, though these stations, with the exception of Leh, are only at elevations between 6,000 and 7,000 feet, the accordance of the actual wind directions with those given by the distribution of

* Instances have been known to occur in which such storms, after crossing the Orissa coast, gave little or no indication of their existence on the charts of sea-level isobars, but after a day or two reappeared in the Narbada valley. This seems to prove that, unlike the cylones of October, they are high-level disturbances.

pressure at 10,000 feet is very fairly exact, as may be seen by comparing the wind directions on Plate 19 with the isobars on Plate 20. For greater elevations I have only the evidence of the Allahabad nephescope, by means of which the directions of movement of the cirrus clouds has been observed for eight years. These observations were recorded and originally tabulated to 16 points, but in giving them here I have reduced the number of points to 8. The total number of observations, during the eight years 1878–1885, inclusive, with the resultants computed by LAMBERT's formula, are given in Table XV.

TABLE XV.—Observed Movements of Cirrus Clouds at Allahabad during the years 1878–85.

Month	N	N E	E.	S E	S	S W	W	N W	Resultant	
									Direction	Percentage
January . .	1			.	..	27	44	20	S 87° W	84
February . .	2	.	1		3	18	55	9	S 85° W	83
March	1		2	3	3	28	58	8	S 77° W	79
April .	2			1	1	10	31	15	N 86° W	80
May . . .	..					8	22	7	S 89° W	88
June .		6	6	4	2	15	24	7	S 77° W	43
July .		4	10	3	4	6	8	3	S 6° E	14
August .	2	6	10	2	1	7	9	4	N 33° W	5
September .	2	4	11	5	3	6	9	3	S 28° E	8
October . .		1	1	1		7	5	1	S 63° W	58
November. .	1		.		1	4	6	2	S 82° W	73
December . .			.		1	21	17	3	S 68° W	88

The resultants for January, May, and October are consistent with the distribution of pressure at 10,000 feet, but that for July has too large a southerly component. The very small percentages of steadiness for the three rainy months of July, August, and September, however, indicate that the pressure gradients of the rainy monsoon are evanescent at the level of the cirrus clouds.

As regards the country north of the parallel of 24° and west of the meridian of 86° E. longitude, the distribution of pressure in October is strikingly similar to that which obtains in May, but the gradients are much less steep. Therefore, although on account of the large diurnal range of temperature, convective interchange may be quite as active in October as in May, the winds which blow in the afternoons are not nearly so strong. In Table XVI., the maximum gradient through Lucknow, at 10,000 feet above sea-level, is compared with the gradient at sea-level and with the mean wind velocity at North-Western Provinces stations, for each of the four months under review.

TABLE XVI.—Comparison of Wind Velocities with Pressure Gradients over the Gangetic Plains.

Month	Velocity in miles per diem	Maximum gradient for 60 geographical miles	
		At sea-level	At 10,000 feet
	Miles	Inch	Inch
January .	58·2	·0029	·028
May .	114·4	·0038	·035
July . .	95·8	·0044	·012
October	52·6	·0036	·019

It seems probable from this Table that the high velocity for July may be attributed, partly at least, to the higher pressure gradient at sea-level in that month than in any of the others, and that the lower velocity in October than in May is the result of the lower gradient prevailing at high levels. The relation between the wind velocity and the gradients at high and low levels may be investigated as follows ;—

If v be the velocity of the wind near sea-level in miles per hour, and v' that prevailing in a higher stratum where the gradients are the same as at 10,000 feet in the Table, and if x be the number of hours during which the wind from this high level descends to the earth's surface, then the total distance traversed daily will be $v'x + v\,(24 - x)$. Should the wind not descend from a level where the gradients are so high as those given in Table XVI., x will not represent the true time during which the upper current descends, but will be longer or shorter in inverse proportion to the gradient.

From FERREL's formula for the relation between the wind velocity and barometric gradient, given in his work on the Movements of the Atmosphere, we get, on reducing to English measure, and taking the unit of distance to be one degree of the earth's surface, $\Delta B = \cdot 00607\,(525 \sin\phi + u)\,v$; or, taking ϕ equal to the mean latitude of the Gangetic plain (about 26°), and neglecting the angular velocity u, we get $v = 714\,\Delta B$. In arriving at this result, however, frictional retardation has been neglected; but, if the ratio between the actual and the theoretical velocity for a given gradient be called f, we may put for the actual velocity $v = 714\,f\,\Delta B$. Substituting for v and v' in the previous formula their values thus expressed in terms of the barometric gradients for each of the four typical months, and assuming the probable values $f = \cdot 4$ and $f' = \cdot 7$ since the equations are indeterminate, we get the following results :—

Month.	x.
January	2·9 hours.
May	5·4 "
July	13·9 "
October	3·3 "

These results, for the three dry months, are consistent with the variations of the length of the day, and the probable rates of decrease of temperature near the ground; but the result for July is much too large, and implies either that the convective action does not extend to so great a height as in the other months, or that, owing to the frequency of small cyclonic disturbances in July, the mean of the gradients actually obtaining from day to day is considerably steeper than that deduced from the chart of monthly mean pressures. Both reasons, doubtless, concur in producing the result just found.

The hypothesis of diurnal interchange between atmospheric strata lying at low and high levels, brought about by convection currents during the hours when the sun is shining, having thus been shown to account for the observed anomalies of the winds of Northern and Central India, both in direction and velocity, I shall now proceed to enquire whether the same hypothesis will suffice to explain the effect of unusual snowfall on the Himalaya in producing extraordinarily powerful and persistent N W. winds over the Indian plains.

Part III.—The Anomalous Effect of Heavy Snow in the Himalayan Region.

In the year 1877, remarkable for the almost total failure of the summer rains, I pointed out, in a letter to the Government of the North-Western Provinces, the probability of a rule that when the winter rains of Northern India are light those of the summer are heavy, and *vice versâ*. About the same time Mr. E. D ARCHIBALD arrived independently at the same empirical law. The investigation which led up to this law was continued and extended, and two years afterwards I published a paper on the subject in the 'Indian Meteorological Memoirs,' vol. 1. At page 209 of that volume it is shown that out of a total number of 34 years of which the rainfall statistics are discussed 25 tell in favour of the rule that, in the North-Western Provinces, the winter rainfall and that of the succeeding summer vary in inverse directions, and only nine are against it. Mr. BLANFORD has since investigated the subject further, and shown that if, instead of the rainfall during the winter and spring months over the plains, we take the precipitation on the North-West Himalaya (which usually, but not always, varies, *pari passu,* with the rainfall on the plains), and compare it with the rainfall of the succeeding summer monsoon, the unfavourable instances, with one or two exceptions, due to special temporary causes, all disappear.

The way, in which unusually heavy and late snowfall on the mountains exercises a retarding and weakening influence on the summer monsoon, is doubtless by keeping down the temperature of those regions, which constitute one of the goals towards which the monsoon winds blow. In such dry years, the westerly winds, which usually prevail over northern and western India during April and May, are strongly reinforced, and continue to blow far into June or July; or even, as in 1868 and 1877, right on through the months of the summer monsoon until September or October. In his

earliest writings on this subject Mr. BLANFORD seemed to incline to the notion that these north-west winds were a nearly direct effect of the cold to the north of India, their immediate cause being the (apparent) high pressure of the mountain region when the actual pressures observed were reduced to sea-level. It has been pointed out in the introductory paragraph, however, that the result of such a distribution of actual pressure would be to cause easterly instead of westerly winds, in accordance with the usual deflecting influence of the earth's rotation In his last paper ('Proceedings of the Royal Society,' vol. 37, 1884) this idea that the north-west winds are a direct effect of Himalayan cold has been dropped, as will be seen from the following quotation, and he anticipates the explanation on the hypothesis of convective interchange, which I now attempt to give. At page 18 he says. "The question then presents itself, 'What is the origin of the dry westerly current?' The supposition that the indraught from the south-west furnishes more than a small portion of the stream is at once negatived by the fact that, even at Karachi and Bhuj, southerly winds do not preponderate over northerly until May, and even then almost inappreciably—at Rajkot not before June, and the very fact of the great dryness of the west and north-west winds militates against the idea that any considerable portion of their air-mass can be drawn from the sea. Neither is it derived to any considerable extent from the valleys and lower slopes of the surrounding hills. There is no permanent drainage of air from these hill-slopes, and strong winds blowing outwards from the larger valleys, like the *dadu* of Hurdwar, are local and exceptional phenomena, restricted to certain hours of the day. At all the hill-stations of the outer North-West Himalaya, as far as the existing registers show, southerly winds preponderate over northerly all through the year, and, although this is probably due in some measure to the fact that the night winds have not hitherto been registered, it suffices to show that, up to a level of 7000 feet, there is no steady outflow of air from the hills to the plains

"There remains then only the supposition that these winds are fed by the descent of air from an upper stratum, viz., from a current moving at a considerable elevation from west to east. And that this is their true explanation several facts seem to testify. In the first place, they are characteristically winds of the day time, their movement being at a minimum (almost or quite a calm) in the morning hours, and indeed up to 9 or 10 o'clock in the forenoon—then increasing with temperature, and falling again towards evening; and, secondly, such observations as have been made on the decrease of temperature with elevation show that, in the dry weather, the vertical decrement is such as is incompatible with the vertical equilibrium of an air column, being considerably more than 1° in 183 feet. The diurnal variation of the movement is then probably to be accounted for on KOPPEN'S hypothesis, viz., the interchange of the higher and lower air strata, by convective movements which do not affect the existing horizontal movement of the higher atmosphere; so that the air of the latter, after its descent, preserves for a time its original eastward motion. The hypothesis of convective interchange receives further support from the character of the diurnal

variation curve of vapour-tension in a dry atmosphere near the earth's surface, which is the same in all parts of India. This shows a rapid fall of the absolute humidity of the air after 8 or 9 o'clock in the forenoon, reaching its minimum about the time of greatest heat, and a more or less sudden rise before sunset, which it is difficult to account for on any other supposition than that it coincides with the cessation of the convective movement." Having given thus clearly and concisely the evidence that the explanation of the diurnal hot winds is probably to be found in KOPPEN's hypothesis, Mr. BLANFORD does not go on to explain how the abundance or paucity of the snowfall on the mountains will affect their intensity. From what has gone before, however, it will be evident that their intensity is likely to be affected only by two causes (1) the intensity of the convective action which brings about an interchange between the higher and lower atmospheric strata; and (2) the pressure gradients and consequent intensity of the winds prevailing at high levels. The former, which depends on the rate of variation of temperature with elevation, is doubtless affected by rainfall over the plains in what is ordinarily the dry season; for such rainfall invariably lowers the temperature at the ground surface, though, from theoretical reasons, it would appear to raise it by the liberation of latent heat at the level where precipitation occurs, the double effect being a decrease in the rate of vertical decrement. The cooling effect of occasional showers over the plains is, however, a transient phenomenon, compared with that of the snows on the mountain slopes, hence it is probably by an increase of the baric gradient at high levels, owing to the sinking of the planes of equal pressure over the cold mountain zone, that the westerly winds are intensified to the extent observed in dry years.

A thorough investigation of this part of the subject would demand the preparation of charts of the distribution of pressure at a high level for each of a considerable number of years, a comparison of the gradients on which, month by month, with the rainfall and the prevailing wind directions over the plains, would enable us to completely verify or disprove the hypothesis. It is to be hoped that, in future, such high-level* charts may be prepared for the monthly mean pressures, as regularly as the charts at sea-level now published in the Annual Reports on the Meteorology of India. Such charts could not fail to be instructive. The labour of preparing such a series would, however, be very great, and all that can be attempted now is to ascertain as nearly as possible what were the high-level gradients prevailing over the Gangetic valley in the typical months of a few years of marked characteristics as regards rainfall.

During the last eight years for which the Meteorological Reports have been published, the rainfall of the Punjab, Rajputana, Central India, the North-West Provinces,

* The elevation, 10,000 feet, adopted in the charts attached to this paper, has been chosen only because it is expressed by a convenient round number, and because it lies above all the mountains in India except the snowy ranges. It may be found in practice more convenient to adopt some other level

and Oudh—a region comprising between one-third and one-half of the plains of India—has varied from the average as follows —

Year —	1877	1878	1879	1880	1881.	1882.	1883.	1884
Rainfall variation per cent	− 35	− 2	+ 5	− 20	+ 2	+ 4	− 17	+ 16

These are the variations computed on the total rainfall of the year, which averages 31·26 inches. Owing to the opposition in sign of the variations for the winter and summer months, the summer rainfall alone would give larger departures from the average, the defect in 1877,* for example, being 55 instead of 35 per cent. The totals for the separate seasons are not given in the summaries† from which the above figures are taken; but, as those of the summer nearly always vary in the same direction as the annual totals, the latter will suffice for our purpose, which for the present has reference only to direction and not to extent of variation.

Leaving out of account the two years 1878 and 1881, for which the variations lie within the limits of possible error, and also 1882, the barometric data for which at one of the stations given below are incomplete, there remain five years of distinctly marked positive or negative variations in the rainfall. Table XVII. gives the observed pressures and day temperatures, for the four typical months of these years, at three stations on the outer Himalayas, and three others in a nearly parallel line on the plateau to the south of the Ganges valley; and in Table XVIII. are given the computed pressures at an elevation of 10,000 feet. In making the computations for this Table, the same rates of temperature decrement have been assumed as in constructing the Table of normal values, though there is reason to believe, as above stated, that these rates may be different for dry and showery months of the winter and spring; but, then, we have at present no means of estimating exactly what the variations from month to month of this element of our calculations may be. Some of the apparently anomalous results worked out below would, doubtless, disappear if the true rates of decrement were known and used.

* Owing to the heavy rainfall of the winter and spring months of 1877, the total rainfall of the N.W. Punjab that year was considerably over the average

† Near the end of the letter-press of each Annual Report on the Meteorology of India.

TABLE XVII.—Pressures and Temperatures at North Indian Stations in Certain Years.

Year	Station.	Elevation in feet	Observed mean pressure				Day temperature.			
			January	May	July	October	January	May.	July	October
			″	″	″	″	°	°	°	°
1877	Ajmere	1632	28 444	28 111	27 992	28 305	69 9	97 4	94 9	89 0
	Sutna	1040	29 022	28 654	28 523	28 857	69 0	98 1	91·4	86·2
	Hazaribagh	2010	28 042	27 737	27 606	27 942	68 0	91 2	84 5	81 7
	Chakrata	7052	23 305	23 221	23 117	23 334	44 4	70·3	68 8	61 6
	Ranikhet	6069	24 167	24 035	23 926	24·153	47 2	71 2	72 7	64 5
	Darjiling	6912	23 503	23 361	23 291	23·506	46 8	63 0	66 9	61 5
1879	Ajmere	1632	28 366	28 013	27 931	28 252	74 6	105 3	91 6	88 9
	Sutna	1040	28 951	28 546	28 506	28 806	74 4	104 4	83 1	83 0
	Hazaribagh	2010	27 970	27 648	27 600	27 851	72 4	98 5	82 1	79 0
	Chakrata	7052	23 261	23 119	23 088	23 292	49 7	75 2	65 9	63 3
	Ranikhet	6069	24 116	24 006	23 905	24 108	53 3	79 7	68 6	65 2
	Darjiling	6912	23 426	23 349	23 288	23 460	47 8	65 4	66 5	63 8
1880	Ajmere	1611	28 373	28 069	27 969	28·301	73 4	97 7	85 5	88 8
	Sutna	1040	28 916	28 584	28 501	28 837	72 9	101 6	84 6	88 8
	Hazaribagh	2007	27 920	27 674	27 576	27 889	71 1	90 6	82 2	81 3
	Chakrata	7052	23·245	23 196	23 090	23 321	51 7	68 0	65 3	64 1
	Ranikhet	6069	24 077	24 000	23 912	24·130	54 0	71 4	69 6	67 0
	Darjiling	6912	23 389	23 345	23 301	23 475	47 9	62 6	65 3	60 4
1883	Ajmere	1611	28 382	28 058	27 961	28 334	68 8	96 7	85 3	87 1
	Sutna	1040	28 975	28 561	28 477	28 869	69 9	103 2	85 1	84 1
	Hazaribagh	2007	27 987	27 633	27 588	27 911	67 9	98 4	81 0	82 2
	Chakrata	7052	23 226	23 115	23 088	23 322	43 9	73 1	67 7	63 8
	Ranikhet	6069	24 089	23 975	23 908	24·134	48·5	74 3	71·7	66 8
	Darjiling	7421	22 974	22 895	22 860	23 048	40 9	61 1	65·4	58 1
1884	Ajmere	1611	28 462	28 084	27 953	28 338	71 2	99 7	89 3	84·0
	Sutna	1040	29 015	28 589	28 493	28 878	71 8	103 0	83 7	81 8
	Hazaribagh	2007	28·014	27 660	27 559	27·902	69 8	96 0	81 3	73·8
	Chakrata	7052	23 290	23 175	23·100	23·311	45 6	73 9	67 3	60 1
	Ranikhet	6069	24 138	23 997	23 928	24 134	52 5	75 4	71·1	63 1
	Darjiling	7421	23 025	22 903	22 869	23 057	45·1	58 8	64 6	57 2

TABLE XVIII.—Mid-day Pressures for certain Years, reduced to 10,000 feet and Gravity of lat 45°.

Station.	1877				1879.			
	January	May	July	October	January	May	July	October
	″	″	″	″	″	″	″	″
Ajmere . .	20 977	21 077	20 933	21 105	20 964	21 058	20 848	21 059
Sutna	20 900	21 003	20·829	21 006	20 921	20 994	20 719	20 932
Hazaribagh	20 939	20 974	20 799	21 019	20 938	20 981	20·767	20 974
Chakrata .	20 840	20 878	20 780	20 944	20 825	20 879	20·742	20 913
Ranikhet .	20 836	20 861	20 777	20 927	20 831	20 935	20 734	20 891
Darjiling	20 918	20 865	20 820	20 988	20 852	20 865	20 815	20 956

Station	1880				1883.			
	January	May.	July	October	January	May	July	October
	″	″	″	″	″	″	″	″
Ajmere .	20 951	21 003	20 793	21 075	20 899	20 938	20·789	21 084
Sutna . .	20 876	20 993	20 728	21 022	20 880	20 994	20·716	20 983
Hazaribagh	20 881	20·914	20 747	20 977	20 895	20 968	20 745	20 999
Chakrata .	20·820	20 847	20 739	20 943	20·713	20·795	20·748	20 942
Ranikhet	20·800	20·833	20 746	20·921	20 777	20 828	20 756	20 925
Darjiling .	20 821	20 848	20 820	20 956	20 814	20·820	20 805	20 948

Station	1884.			
	January.	May	July	October.
	″	″	″	″
Ajmere .	20 989	21 039	20 829	21 052
Sutna .	20·935	21 009	20 709	20 964
Hazaribagh . .	20 939	20 962	20 728	20 902
Chakrata	20 833	20 851	20 759	20·965
Ranikhet .	20 844	20 855	20 768	20 902
Darjiling . .	20 877	20 820	20 810	20·953

These Tables suffice to prove that, even at a height of 10,000 feet, the pressure is subject to very considerable variations from year to year, both on the mountains and over the plains. The actual pressures observed were most variable at Ajmere, (allowance being made for the change in the elevation of the barometer), whilst on the mountains, as a rule, the variations were less than on the plains; but the great variations at Ajmere, especially in January, are partly discounted by reduction to 10,000 feet; high pressure at low levels being usually accompanied by, and doubtless the result of, low temperature. Accordingly, we find that the greatest range of the pressure at 10,000 feet, in January, is shown by Chakrata, and the next greatest by Darjiling; and this indicates that if, as we have seen good reason to believe, the

direction of the wind is powerfully influenced by the distribution of pressure at high levels, it is more important, as far as the weather of the winter months is concerned, to know the variations of pressure from the normal at the highest hill-stations than at any place on the plains. In May there are considerable fluctuations over both regions, the greatest range being shown by Ajmere and the next greatest by Ranikhet; but in July the variations over the mountains, which are always more or less moist and equable in temperature at this season, are very small, the ranges not exceeding ·043 inch in the five years compared, while at Ajmere and Sutna they amount to ·144 and ·120 inch. In October, as we should expect from the fact of the mean temperature of the month being about the average of the year, and the normal distribution of pressure very uniform at all levels, the range of the variation is less than in any of the other three, and is greatest at Hazaribagh, which, during the first half of the month, is on the border of the region where the rains still penetrate after they have ceased in Upper India, while it also lies near the track of some of the October cyclones.

The great variability of pressure at 10,000 feet over Central India and Rajputana during the rainy season, which depends chiefly upon the still greater variability of temperature in the same region, is a good instance of the general tendency for any established set of meteorological conditions to persist by the interaction of cause and effect. An unusually high pressure in the upper atmosphere over this region in the months of May and June probably produces, as we have seen, westerly and north-westerly winds; these, being dry winds, neither cool the earth's surface by precipitation and subsequent evaporation, nor, by the interposition of a screen of clouds, prevent its temperature from rising high in the day-time; then, in consequence of the high day temperature in this region as compared with the Himalayas, which are more or less cloudy and moist, the pressure at 10,000 feet or other high levels remains relatively excessive; and again, in consequence of the high baric gradient thus conserved, the westerly winds continue to blow on, until some disturbance originating at a distance supervenes, or, as in 1877, until after the autumnal equinox, when the night begins to exceed the day in length, and the loss of heat by radiation to the clear sky over-balances the gain.

The mean gradients or pressure differences, measured more or less directly across the Gangetic plain, in each of the months compared in Table XVIII., are given in Table XIX., the figures representing thousandths of an inch, and the positive sign indicating a gradient for westerly winds, *i.e.*, meaning that the higher pressures belong to the southern stations, such being the most frequent condition.

TABLE XIX.—Pressure Differences over the Gangetic Valley in Thousandths of an Inch.

Year	Ajmere—Chakrata				Sutna—Ranikhet.			
	January.	May.	July	October	January	May.	July.	October.
1877	+137	+199	+153	+ 61	+ 64	+142	+52	+ 79
1879 .	+149	+179	+106	+146	+ 90	+ 59	−25	+ 31
1880 .	+131	+156	+ 54	+132	+ 76	+160	−18	+101
1883 .	+186	+143	+ 41	+142	+103	+166	−40	+ 58
1884 . .	+156	+188	+ 70	+ 87	+ 91	+144	−59	+ 62

Year	Hazaribagh—Darjiling.				Mean of the three Pairs of Stations.			
	January	May.	July.	October	January	May.	July.	October.
1877 .	+ 21	+109	−21	+31	+ 74	+150	+61	+57
1879 . .	+ 86	+116	−48	+18	+109	+118	+11	+65
1880 .	+ 60	+ 66	−73	+21	+ 89	+127	+12	+85
1883 . .	+ 81	+148	−60	+51	+123	+152	−20	+84
1884 .	+162	+142	−88	−51	+136	+158	−26	+33

On comparing this Table with the figures given above for the variations of rainfall, it will be seen that, in the worst years, 1877 and 1880, the mean gradients for January were least, indicating the probable existence, for a considerable number of days in the month, of moist south-easterly currents, bringing the snowfall, which, according to Mr BLANFORD's theory, was the cause of the subsequent drought. These moist winds were observed in January, 1877, on the plains; but in 1880, as Mr. BLANFORD has pointed out in the paper above quoted, they were confined chiefly to the higher valleys and slopes of the Himalaya. In May, the variations of the pressure differences are less evidently connected by any rule, and in July there apparently is in the mean of the three pairs of stations a nearly uniform decrease in the gradients for westerly winds or an increase in those for easterly ones during the eight years from 1877 to 1884.

It does not seem right, however, to group the Hazaribagh-Darjiling pair with the others, as the character of the rainfall of any season in Bengal is usually different from and often opposite to that which obtains in Upper India. Thus, in 1883, in which the gradients over West Bengal appeared rather favourable for rainy winds, the defect of rainfall in that province was only 4 per cent., as against 17 per cent. in the upper provinces, and this small defect may possibly be explicable by other causes;* whilst in 1880, during which the gradient between Hazaribagh and Dar-

* For example, the diminished absolute humidity of the air, owing to the temperature being below the average. The mean temperature of the whole of India in 1883 was 0·48° below the average, and the vapour tension was ·013 inch in defect.

jiling was still more favourable for easterly winds, there was an actual excess of rainfall over the Bengal provinces, amounting to 10 per cent., against a defect of 20 per cent. in Upper India.

The character of the Ajmere observations for accuracy, moreover, is not nearly so high as that of the work of the Sutna observer, and the barometer at Ajmere has been several times changed, while there has also been a change in its position during the years compared Hence, though every care has been taken to correct the observations for such changes, it is possible that there may be some residual errors undetected, which, though negligible, or nearly so, in the mean of a long term of years, many of which were anterior to the changes mentioned, may introduce some confusion in comparing the observations of single years. For these reasons, it is probable that the best criterion of the pressure gradients, prevailing over Bundelkhand, Rajputana, and the North-West Provinces, will be a comparison of the observations of Sutna and Ranikhet.

Thus estimated, the gradient for westerly winds was relatively high in May, July, and October, 1877. It was low in May and October, 1879, and in July there was a considerable gradient for easterly winds. In 1880, the gradient for westerly winds was high in May and October, and there was a small gradient for easterly ones in July ; thus, on the whole, the upper winds were such as to make the rainfall less than usual. In the dry year, 1883, the gradients for westerly winds were greater, or for easterly ones less, than in the wet year 1884, both in May and July; but in July, 1883, there was nevertheless a rather high gradient for easterly winds, contrary to what might be expected. It was in August, 1883, however, not in July, that the principal deficiency of rain occurred.

The verification of theory by means of the deductions from observations embodied in Table XIX. being thus less definite than is desirable, it will be worth while to work out the pressure gradients between Sutna and Ranikhet for the other months of the rainy season in each of the years compared. Table XX. gives the pressures at 10,000 feet for the months of June, August, and September.

TABLE XX.—Pressures at 10,000 feet for the other Months of the Rainy Season in certain Years

Year.	Sutna.			Ranikhet		
	June.	August	September	June	August.	September.
	″	″	″	″	″	″
1877 .	20 907	20 836	20 968	20 832	20 810	20 892
1879 .	20 857	20 730	20 824	20·772	20 718	20·829
1880 . .	20 881	20 811	20 911	20·765	20 781	20 842
1883 . . .	20·875	20 801	20 858	20 782	20·785	20 840
1884 .	20 903	20 752	20 828	20 793	20 779	20 851

Combining the differences of these figures with those for the same pair of stations in Table XIX., we get the following results for the whole summer half-year, that is to say, for the six months May to October inclusive, during which almost the whole rain of the year falls on the Gangetic plain.

TABLE XXI.—Pressure Differences between Sutna and Ranikhet for the Summer Half-Year.

Year	May	June	July	August	September	October	Mean.
1877 . .	+142	+ 75	+52	+26	+76	+ 79	+75
1879	+ 59	+ 85	−25	+12	− 5	+ 31	+26
1880	+160	+116	−18	+30	+69	+101	+76
1883	+166	+ 93	−40	+16	+18	+ 58	+52
1884 .	+144	+110	−59	−27	−23	+ 62	+34

In Table XXI. the relation between the pressure differences and the rainfall variation becomes quite clear. The driest years, 1877 and 1880, had the largest gradients for westerly winds, that is to say, in those years, easterly winds at high levels over the plains of Northern India were probably infrequent, or restricted to an unusually narrow belt along the Himalayas; on the other hand, the wet years, 1884 and 1879, had the smallest gradients, or in those years the easterly upper currents were steadiest, and reached their widest extension; while in the moderately dry year, 1883, there was a considerable gradient for westerly winds, but nothing like so great as in 1877 and 1880. If the month of May be left out in striking the mean for the season, the relation indicated becomes still more distinct; thus—

Year	1877	1879	1880	1883	1884.
Mean gradient for 5 months .	+62	+20	+60	+29	+13
Rainfall variation per cent .	−35	+ 5	−20	−17	+16

It appears, therefore, from this analysis of the observations, that the theoretical deductions from KOPPEN's convection hypothesis are fairly substantiated by experience; that in years of unusually heavy snowfall, and consequent cold, over the North-West Himalaya the pressure gradients producing westerly winds in the upper atmosphere are intensified, and that in consequence of this the easterly winds, which prevail as far south as the Ganges up to the level of the lower cirrus clouds in the rainy season of average years, are greatly restricted in horizontal extent, and doubtless also in height as well as in time, the result being a diminution of the rainfall, sometimes to a disastrous degree.

[illegible] these conclusions are substantiated by the average of the observations of [illegible] months, they are not completely borne out by those of any single month,

except September, and from Table XXI. it would appear impossible to predict in May or June what will be the prevailing pressure difference later on in the year. The irregularities in the pressure differences deduced from the observations of single months may, however, as already pointed out, be due in great part to the uncertainty of the assumed rate of decrease of temperature, which, while probably nearly true, or at all events giving consistent results when applied to the average of many years' observations for any calendar month, or even to the mean for five or six consecutive months of a single year, may be very considerably in error when applied to the observations of one month standing by itself.

On making a retrospective survey of all the evidence put forward in the preceding pages, it will, I think, be generally admitted that the hot winds of Northern and Central India cannot be satisfactorily explained by the distribution of pressure at the earth's surface, and that KOPPEN's hypothesis of convective interchange between the upper and lower strata is probably the true explanation of them, (1) because the vertical distribution of temperature is such that convective action must take place; (2) because the diurnal variation of the intensity of these winds and their characteristic dryness suggest such an origin for them, and (3) because the distribution of pressure at 10,000 feet above sea-level in May is such as to produce winds of the observed direction.

The distribution of the upper currents as suggested by the charts also elucidates many other obscure points in the wind system of India, and the same hypothesis of the origin of the westerly winds gives a rational explanation of the law worked out by Mr BLANFORD connecting the spring snowfall of the North-west Himalaya with the rainfall over the plains during the succeeding summer monsoon.

If these conclusions have not been all completely established, as I am by no means desirous of asserting, they have, I think, been shown to be sufficiently probable to warrant a more complete examination in detail by the determination, if possible, of more trustworthy rates of temperature decrement with height than those assumed in this paper, and by the systematic preparation of high-level pressure charts for the whole of India, month by month.

DESCRIPTION OF THE PLATES.

Plate 19 shows the isobars or lines of equal pressure at sea-level over India and the Bay of Bengal for the months of January, May, July, and October; typical respectively of the cold, the hot, and the rainy season, and the autumn transition period. The pressure, indicated by each line, is represented by the figures attached to it, those for successive lines differing by ·05 inch. The pressures have all been corrected for variations of gravity with latitude. The charts also give the prevalent wind directions at the more important stations, these "resultant" directions having been computed by means of LAMBERT's formula from the observations of many years.

Plate 20 gives the isobars for a stratum of the atmosphere 10,000 feet above sea-level.

The pressures represented are not, however, mean pressures, but those prevailing in the middle of the day, between 10 A M and 4 P M. They have been computed from the observations made at 40 stations (including nearly all those above 1500 feet) by assuming the temperature to decrease at the rates represented on Plate 21, and applying a Table computed by Major ALLAN CUNNINGHAM, R.E., on the basis of Professor RANKINE's modification of LAPLACE's formula

Plate 21 shows the probable rate of decrease of temperature, on ascending, in the day time, through the atmosphere over a plain; the form of the curves, which are slightly different for clear and cloudy skies, being deduced from Mr. GLAISHER's observations made during balloon ascents. The straight line shows the theoretical rate of temperature decrement in a mass of air carried upwards without gain or loss of heat.

XIII. *On Figures of Equilibrium of Rotating Masses of Fluid.*

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[PLATES 22, 23]

IN a previous paper* I remarked that there might be reason to suppose that the earliest form of a satellite might not be annular Whether or not the present investigation does actually help us to understand the working of the nebular hypothesis, the idea there alluded to was the existence of a dumb-bell shaped figure of equilibrium, such as is shown in the figures at the end of this paper. These figures were already drawn when a paper by M. POINCARÉ appeared, in which, amongst other things, a similar conclusion was arrived at. My paper was accordingly kept back in order that an attempt might be made to apply the important principles enounced by him to this mode of treatment of the problem. The results of that attempt are, for reasons explained below, given in the Appendix.

The subject of figures of equilibrium of rotating masses of fluid is here considered from a point of view so wholly different from that of M. POINCARÉ that, notwithstanding his priority and the greater completeness of his work, it still appears worth while to present this paper.

The method of treatment here employed is simple of conception; but it is unfortunate that, to carry out the idea, a very formidable array of analysis is necessary.

In the last section a summary will be found of the principal conclusions, in which analysis is avoided.

§ 1. *Formulæ of Spherical Harmonic Analysis.*

Let there be two sets of rectangular axes, as shown in fig. 1; and let z be measured from o to O, whilst Z is measured from O to o; let $r^2 = x^2 + y^2 + z^2$, $R^2 = X^2 + Y^2 + Z^2$, and let $c = oO$.

Then

$$x + X = 0, \qquad y + Y = 0, \qquad z + Z = c. \quad . \quad . \quad . \quad . \quad (1)$$

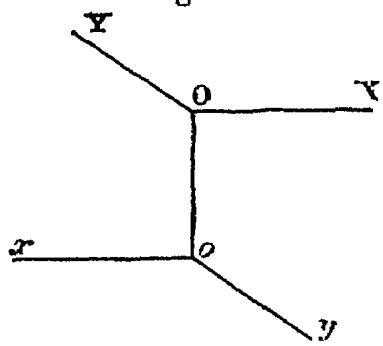
Let w_i, W_i, denote the solid zonal harmonics of degree i of the coordinates x, y, z, and X, Y, Z, respectively.

Now we shall require to express the solid zonal and certain tesseral harmonics of

* 'Phil Trans.,' Part II, 1881, p. 534

negative degrees with respect to the origin O as solid zonal and tesseral harmonics of positive degrees with respect to the origin o, and *vice versâ;* moreover, the results will have to be applied to a sphere of radius a with centre o, and to a sphere of radius A with centre O. This last clause is introduced in order to explain the introduction of the symbols a, A, in this place.

Fig 1.



The formulæ required will be called "transference formulæ," because they are to be used in shifting the origin from one point to the other.

The obvious symmetry of our axes is such that every transference formula from O to o has its exact counterpart for transference from o to O; thus a second symmetrical formula with capital and small letters interchanged will generally be left unwritten. When necessary, θ, ϕ, will be written for co-latitude and longitude with regard to x, y, z; and Θ, Φ, for the same with respect to X, Y, Z.

Then, since

$$R^2 = r^2 + c^2 - 2rc\cos\theta,$$

we have the usual expansion in zonal harmonics

$$\frac{c}{R} = \sum_{k=0}^{k=\infty} \frac{w_k}{c^k}. \qquad \qquad (2)$$

The usual formula for the derivation of the zonal harmonic of negative degree $i+1$ from $1/R$ is

$$\frac{(-)^i}{i!}\frac{d^i}{dZ^i}\frac{1}{R} = \frac{W_i}{R^{2i+1}}. \qquad \qquad (3)$$

Hence, on differentiating (2) i times with respect to Z, or, which is the same thing, with respect to $-z$, we have, from (3),

$$c\frac{W_i}{R^{2i+1}} = \frac{1}{i!}\sum_{k=0}^{k=\infty}\frac{d^i}{dz^i}\frac{w_k}{c^k}.$$

But

$$\frac{d^i}{dz^i}w_k = k(k-1)\ldots(k-i+1)\,w_{k-i} = \frac{k!}{k-i!}w_{k-i}.$$

Hence

$$c^i\frac{W_i}{R^{2i+1}} = \frac{1}{c}\sum_{k=0}^{k=\infty}\frac{k!}{i!\,k-i!}\frac{w_{k-i}}{c^{k-i}}.$$

In interpreting this formula, it will be noted that, if i is less than k, the term vanishes hence the summation runs from $k=\infty$ to $k=i$, it is therefore better to write $k+i$ for k, and we thus obtain

$$\frac{c^i W_i}{R^{2i+1}} = \frac{1}{c} \sum_{k=0}^{k=\infty} \frac{k+i!}{i!\,k!} \left(\frac{a}{c}\right)^k \frac{u_k}{a^k}. \qquad (4)$$

This is the first transference formula by which the solid zonal harmonic of degree $-i-1$ with respect to O is expressed as a series of solid harmonics of positive degree with respect to o The formula (4) includes (2) as the particular case where $i=0$. The right-hand side of (4) is convergent for r less than a A similar formula, convergent for r greater than a, is easily obtainable, but with this we shall not concern ourselves

It remains to find the transference formula for certain tesseral harmonics.

If we put

$$\rho = \tfrac{1}{4}(x^2+y^2), \qquad (5)$$

the general expression for the zonal harmonic is

$$w_i = \Sigma (-)^k \frac{i!}{k!^2\; i-2k!} z^{i-2k} \rho^k, \qquad (6)$$

where the summation extends from $k=0$ to $k=\tfrac{1}{2}i$ or $\tfrac{1}{2}(i-1)$

From (6) we have

$$\frac{dw_i}{d\rho} = \Sigma (-)^k \frac{i!}{k!^2 . i-2k!} k z^{i-2k} \rho^{k-1} \qquad (7)$$

Now, since $r^2 = z^2 + 4\rho$, we have

$$r^2 \frac{dw_i}{d\rho} = \Sigma (-)^k i! \left[\frac{-(k+1)}{k+1!^2\; i-2k-2!} + \frac{4k}{k!^2\; i-2k!}\right] z^{i-2k} \rho^k$$

$$= \Sigma (-)^k i! \frac{[4k(k+1) - (i-2k)(i-2k-1)]}{(k+1).k!^2 . i-2k!} z^{i-2k} \rho^k. \qquad (8)$$

Also

$$2(2i+1) w_i = \Sigma (-)^k i! \frac{2(2i+1)(k+1)}{(k+1)\; k!^2\; i-2k!} z^{i-2k} \rho^k. \qquad (9)$$

Subtracting (9) from (8), and simplifying the difference, we have

$$r^2 \frac{dw_i}{d\rho} - 2(2i+1) w_i = \Sigma (-)^{k+1} \frac{(i+1)(i+2)\; i!}{(k+1)\; k!^2 . i-2k!} z^{i-2k} \rho^k$$

$$= \Sigma (-)^{k+1} \frac{i+2!}{k+1!^2 . i+2-2k-2!} (k+1) z^{i+2-2k-2} \rho^{k+1-1}$$

$$= \frac{d}{d\rho} w_{i+2}; \qquad (10)$$

the last transformation being derived from (7) with $i+2$ in place of i, and $k+1$ in place of k.

Differentiate (10) with respect to ρ, and notice that $dr^2/d\rho = 4$, and we have

$$r^2 \frac{d^2 w_i}{d\rho^2} - 2(2i-1)\frac{dw_i}{d\rho} = \frac{d^2 w_{i+2}}{d\rho^2}.$$

Then, with $i+2$ in place of i,

$$r^2 \frac{d^2 w_{i+2}}{d\rho^2} - 2(2i+3)\frac{dw_{i+2}}{d\rho} = \frac{d^2 w_{i+4}}{d\rho^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now

$$\frac{d}{d\rho}\frac{w_i}{r^{2i+1}} = \frac{1}{r^{2i+3}}\left\{r^2\frac{dw_i}{d\rho} - 2(2i+1)w_i\right\}$$

$$= \frac{1}{r^{2i+3}}\frac{dw_{i+2}}{d\rho} \text{ by (10).}$$

Differentiating again,

$$\frac{d^2}{d\rho^2}\frac{w_i}{r^{2i+1}} = \frac{1}{r^{2i+5}}\left\{r^2\frac{d^2 w_{i+2}}{d\rho^2} - 2(2i+3)\frac{dw_{i+2}}{d\rho}\right\}$$

$$= \frac{1}{r^{2i+5}}\frac{d^2}{d\rho^2}w_{i+4} \text{ by (11),}$$

or

$$\frac{1}{r^{2i+1}}\frac{d^2 w_{i+2}}{d\rho^2} = \frac{d^2}{d\rho^2}\frac{w_{i-2}}{r^{2i-3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

But since $\rho = \frac{1}{4}(x^2+y^2)$, it follows that, in operating on a function involving x and y only in the form x^2+y^2,

$$\frac{d}{dx} = \tfrac{1}{2}x\frac{d}{d\rho}, \qquad \frac{d}{dy} = \tfrac{1}{2}y\frac{d}{d\rho}, \qquad \text{and} \qquad x\frac{d}{dx} - y\frac{d}{dy} = \tfrac{1}{2}(x^2-y^2)\frac{d}{d\rho}.$$

Also

$$\frac{d^2}{dx^2} = \tfrac{1}{2}\frac{d}{d\rho} + \tfrac{1}{4}x^2\frac{d^2}{d\rho^2}, \qquad \frac{d^2}{dy^2} = \tfrac{1}{2}\frac{d}{d\rho} + \tfrac{1}{4}y^2\frac{d^2}{d\rho^2},$$

so that

$$\frac{d^2}{dx^2} - \frac{d^2}{dy^2} = \tfrac{1}{4}(x^2-y^2)\frac{d^2}{d\rho^2}.$$

Now let us put

$$\delta^2 = \frac{d^2}{dx^2} - \frac{d^2}{dy^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Then

$$\delta^2 = \tfrac{1}{4}(x^2-y^2)\frac{d^2}{d\rho^2},$$

and therefore (12) may be written

$$\frac{1}{r^{2i+1}}\delta^2 w_{i+2} = \delta^2\frac{w_{i-2}}{r^{2i-3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

These expressions in (14) are obviously solid tesseral harmonics.

The transference formula required is for $\dfrac{\delta^2 W_{i+2}}{R^{2i+1}}$.

By the formula (4) we have

$$\frac{c^{i-2} W_{i-2}}{R^{2i-3}} = \frac{1}{c} \sum_{k=0}^{k=\infty} \frac{k+i-2!}{i-2!\,k!} \frac{w_k}{c^k},$$

operating on both sides by δ^2, and applying (14), we have

$$\frac{c^{i-2}}{R^{2i+1}} \delta^2 W_{i+2} = \frac{1}{c} \sum_{k=0}^{k=\infty} \frac{k+i-2!}{i-2!\,k!} \frac{\delta^2 w_k}{c^k}. \qquad (15)$$

Now the general formula (6) for the zonal harmonic shows us that $d^2 w_k/d\rho^2$ is zero when $k = 0, 1, 2, 3$, and hence $\delta^2 w_k$ vanishes for the same values of k. Thus the summation in (15) is from $k = \infty$ to $k = 4$, or, if we write $k + 2$ for k, from ∞ to 2. Hence (15) gives

$$\frac{c^i}{R^{2i+1}} \delta^2 W_{i+2} = \frac{1}{c} \sum_{k=2}^{k=\infty} \frac{k+i!}{i-2!\,k+2!} \left(\frac{a}{c}\right)^k \frac{\delta^2 w_{k+2}}{a^k}. \qquad (16)$$

This is the second transference formula required.

We observe that the transference of a negative zonal harmonic gives us positive zonals, and that tesseral harmonics of the type $\delta^2 W_{i+2}/R^{2i+1}$ give us harmonics of the type $\delta^2 w_{k+2}$.

§ 2 *The Mutual Influence of two Spheres of Fluid without Rotation.*

Imagine two approximately spherical masses of fluid of unit density, with their centres at the origins o and O respectively, and with mean radii a and A respectively.

We shall find that each exercises on the other certain forces, one part of which has a solid zonal harmonic of the first degree as potential. This part of the force must remain essentially unbalanced in the supposed system, but we shall see hereafter that it is balanced by the rotation to be afterwards imposed on the system.

Meanwhile it will be supposed that it is annulled in some way, and we shall content ourselves with finding the mutual influence of the spheroids, and the outstanding term of the first degree of harmonics.

Let us assume that the equations, referred to our two origins, of the surfaces of the two spheroids, when they mutually perturb one another, are

$$\left.\begin{aligned} \frac{r}{a} &= 1 + \left(\frac{A}{a}\right)^3 \sum_{i=2}^{i=\infty} \frac{2i+1}{2i-2} \left(\frac{a}{c}\right)^{i+1} h_i r^{-i} w_i \\ \frac{R}{A} &= 1 + \left(\frac{a}{A}\right)^3 \sum_{i=2}^{i=\infty} \frac{2i+1}{2i-2} \left(\frac{A}{c}\right)^{i+1} H_i R^{-i} W_{-i} \end{aligned}\right\} \qquad (17)$$

The h's and H's are unknown coefficients, to be determined.

We have now to find the potential at any point in space

The mass of the spheroid o is $\frac{4}{3}\pi a^3$, and its potential is $\frac{4}{3}\pi a^3/r$.

The potential due to the departure from sphericity, represented by the term in h_i in the first of (17), is

$$\frac{4\pi A^3}{3c}\frac{3h_i}{2i-2}\left(\frac{a}{c}\right)^i\left(\frac{a}{r}\right)^{i+1}\frac{w_i}{r^i}. \qquad (18)$$

This is written in a form convenient for passing to the case of $r = a$ It may also be written in the form

$$\tfrac{4}{3}\pi A^3\left(\frac{a}{c}\right)^{2i+1}\frac{3h_i}{2i-2}\frac{c^i w_i}{r^{2i+1}}, \qquad (19)$$

when it is in a suitable form for application of the transference formula (4).

We shall now introduce two new symbols, namely,

$$\gamma = \left(\frac{a}{c}\right)^2, \qquad \Gamma = \left(\frac{A}{c}\right)^2. \qquad (20)$$

Then (19) may be written

$$\tfrac{4}{3}\pi A^3\left(\frac{a}{c}\right)^3\frac{3h_i\gamma^{i-1}}{2i-2}\frac{c^i w_i}{r^{2i+1}},$$

and, of course, the similar potential with the other origin is

$$\tfrac{4}{3}\pi a^3\left(\frac{A}{c}\right)^3\frac{3H_i\Gamma^{i-1}}{2i-2}\frac{c^i W_i}{R^{2i+1}} \qquad (21)$$

The whole potential at any point of space consists of the potentials of the two spheres and of the inequalities on each. The potential of the inequalities of the sphere o may be written in the form (18), and of sphere O in the form (21).

Thus the whole potential is

$$\tfrac{4}{3}\pi a^3\cdot\frac{a}{r}+\frac{4\pi A^3}{3c}\sum_{k=2}^{k=\infty}\frac{3h_k}{2k-2}\left(\frac{a}{c}\right)^k\left(\frac{a}{r}\right)^{k+1}\frac{w_k}{r^k} \qquad (22\text{–i.})$$

$$+\frac{4\pi A^3}{3c}\cdot\frac{c}{R}+\frac{4\pi a^3}{3}\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{3H_i\Gamma^{i-1}}{2i-2}\frac{c^i W_i}{R^{2i+1}} \qquad (22\text{–ii.})$$

The first line of (22) refers to origin o, the second to origin O, and to this latter half the transference formula (4) must be applied.

Now apply (2) to the first term of the second line, and (4) to one term of the series in the second term, and we have

$$\frac{4\pi A^3}{3c}\frac{c}{R}=\frac{4\pi A^3}{3c}\sum_{k=0}^{k=\infty}\left(\frac{a}{c}\right)^k\frac{w_k}{a^k},$$

and

$$\left[\frac{4\pi a^3}{3}\left(\frac{A}{c}\right)^3\frac{3H_i\Gamma^{i-1}}{2i-2}\right]\frac{c^i W_i}{R^{2i+1}}=\frac{4\pi A^3}{3c}\left(\frac{a}{c}\right)^3\frac{3H_i\Gamma^{i-1}}{2i-2}\sum_{k=0}^{k=\infty}\frac{k+i\,!}{k!\,i!}\left(\frac{a}{c}\right)^k\frac{w_k}{a^k}.$$

Thus the second line of (22) when transferred is

$$\frac{4\pi A^3}{3c}\left[\sum_{k=0}^{k=\infty}\left(\frac{a}{c}\right)^k\frac{w_k}{a^k}+\tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{i=2}^{i=\infty}\sum_{k=0}^{k=\infty}\frac{k+i\,!}{k\,!\,i\,!}\frac{\Gamma^{i-1}}{i-1}H_i\left(\frac{a}{c}\right)^k\frac{w_k}{a^k}\right]. \quad . \quad (22\text{–ii.})$$

Then (22–i.) and (22–ii.) together constitute the potential now entirely referred to origin o

We want to choose the h's and H's so that each spheroid may be a level surface, save as to the outstanding term of the first degree

In order that (17) may be a level surface, when we substitute for r its value (17) in (22), the whole potential must be constant. In effecting this substitution, we may put $r = a$ in the small terms, but in $\frac{4}{3}\pi a^3/r$ we must give it the full value (17).

The constancy of the potential is secured by making the coefficient of each harmonic term vanish separately—excepting the first harmonic, which remains outstanding by supposition.

We may consider each harmonic term by itself.

As far as concerns the term involving w_k, we have, from (22–i.) and (22–ii.), as the value of the potential,

$$\tfrac{4}{3}\pi a^3\frac{a}{r}+\frac{4\pi A^3}{3c}\left[\frac{3h_k}{2k-2}\left(\frac{a}{c}\right)^k\left(\frac{a}{r}\right)^{k+1}\frac{w_k}{r^k}+\left(\frac{a}{c}\right)^k\frac{w_k}{a^k}+\tfrac{3}{2}\left(\frac{a}{c}\right)^3\left(\frac{a}{c}\right)^k\frac{w_k}{c^k}\sum_{i=2}^{i=\infty}\frac{k+i\,!}{k\,!\,i\,!}\frac{\Gamma^{i-1}}{i-1}H_i\right],$$

and the value of r which must make this constant is

$$\frac{r}{a}=1+\left(\frac{A}{a}\right)^3\frac{2k+1}{2k-2}\left(\frac{a}{c}\right)^{k+1}h_k\frac{w_k}{r^k},$$

but in the small terms inside [ ] we may put $r = a$ simply.

Make, therefore, the substitution, and equate the coefficient of w_k to zero. On dividing that coefficient by $\frac{4\pi A^3}{3c}\cdot\left(\frac{a}{c}\right)^k$, we find

$$-\frac{2k+1}{2k-2}h_k+\frac{3h_k}{2k-2}+1+\tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{k+i\,!}{k\,!\,i\,!}\frac{\Gamma^{i-1}}{i-1}H_i=0.$$

Therefore

$$h_k=1+\tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{k+i\,!}{k\,!\,i\,!}\frac{\Gamma^{i-1}}{i-1}H_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and, by symmetry,

$$H_r=1+\tfrac{3}{2}\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{r+i\,!}{r\,!\,i\,!}\frac{\gamma^{i-1}}{i-1}h_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Multiplying both sides of (24) by the coefficient of H_r in (23), we have

$$\tfrac{3}{2}\left(\frac{a}{c}\right)^3\frac{k+r\,!}{k\,!\,r\,!}\frac{\Gamma^{r-1}}{r-1}H_r=\tfrac{3}{2}\left(\frac{a}{c}\right)^3\frac{k+r\,!}{k\,!\,r\,!}\frac{\Gamma^{r-1}}{r-1}+\left(\tfrac{3}{2}\right)^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{r+i\,!\,r+k\,!}{r\,!\,i\,!\,k\,!\,r\,!}\frac{\gamma^{i-1}}{i-1}\frac{\Gamma^{r-1}}{r-1}h_i.$$

Performing $\sum\limits_{r=2}^{r=\infty}$ on both sides, and substituting from (23),

$$h_k - 1 = \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{r=2}^{r=\infty} \frac{k+r!}{k!\,r!}\frac{\Gamma^{r-1}}{r-1} + (\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3 \sum_{r=2}^{r=\infty}\sum_{i=2}^{i=\infty} \frac{r+i!\,r+k!}{r!\,i!\,k!\,r!}\frac{\gamma^{i-1}}{i-1}\frac{\Gamma^{r-1}}{r-1}h_i. \quad (25)$$

Now let

$$\left.\begin{aligned}(k, \Gamma) &= \sum_{r=2}^{r=\infty}\frac{k+r!}{k!\,r!}\frac{\Gamma^{r-1}}{r-1}\\ [k, i, \Gamma] &= \sum_{r=2}^{r=\infty}\frac{r+i!\,r+k!}{r!\,i!\,k!\,r!}\frac{\Gamma^{r-1}}{r-1}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

And (25) may be written

$$h_k = 1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 (k, \Gamma) + (\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}[k, i, \Gamma]\frac{\gamma^{i-1}}{i-1}h_i. \quad . \quad . \quad (27)$$

By imparting to k all integral values from 2 upwards, we get a system of linear equations for the determination of the h's, and it will appear below that as many of them may be found numerically as may be desired.

We now have to consider the series (26).

Let

$$\beta = \frac{\gamma}{1-\gamma}, \quad B = \frac{\Gamma}{1-\Gamma},$$

and denote the operations

$$\frac{1}{\lambda!}\frac{d^\lambda}{d\gamma^\lambda}\gamma^\lambda \quad \text{or} \quad \frac{1}{\lambda!}\frac{d^\lambda}{d\Gamma^\lambda}\Gamma^\lambda \quad \text{by} \quad E^\lambda.$$

Consider the function $\gamma^{-1}E^\lambda.\,\gamma\log(1+\beta)$.

Now

$$\log(1+\beta) = -\log(1-\gamma) = \sum_{r=2}^{r=\infty}\frac{\gamma^{r-1}}{r-1}.$$

Therefore

$$E^k.\,\gamma\log(1+\beta) = \frac{1}{k!}\frac{d^k}{d\gamma^k}\sum_{r=2}^{r=\infty}\frac{\gamma^{k+r}}{r-1} = \sum_{r=2}^{r=\infty}\frac{k+r!}{k!\,r!}\frac{\gamma^r}{r-1}.$$

Thus

$$(k, \gamma) = \frac{1}{\gamma}E^k.\,\gamma\log(1+\beta), \quad (k, \Gamma) = \frac{1}{\Gamma}E^k.\,\Gamma\log(1+B) \quad . \quad . \quad . \quad (28)$$

Next consider the function $\gamma^{-1}E^kE^i.\,\gamma\log(1+\beta)$.

As before,

$$E^i.\,\gamma\log(1+\beta) = \sum_{r=2}^{r=\infty}\frac{i+r!}{i!\,r!}\frac{\gamma^r}{r-1},$$

and

$$\begin{aligned}E^kE^i.\,\gamma\log(1+\beta) &= \frac{1}{k!}\frac{d^k}{d\gamma^k}\sum_{r=2}^{r=\infty}\frac{i+r!}{i!\,r!}\frac{\gamma^{k+r}}{r-1},\\ &= \sum_{r=2}^{r=\infty}\frac{i+r!\,k+r!}{i!\,r!\,r!\,k!}\frac{\gamma^r}{r-1}.\end{aligned}$$

Hence

$$[k, i, \gamma] = \frac{1}{\gamma} E^k E^i . \gamma \log(1+\beta), \qquad [k, i, \Gamma] = \frac{1}{\Gamma} E^k E^i . \Gamma \log(1+B). \quad (29)$$

We must now develop the symbolical sums of the series in (28) and (29). The following theorems are obvious :—

$$\frac{d^n}{d\gamma^n}\gamma^p = \frac{p\,!}{p-n\,!}\gamma^{p-n}, \qquad \frac{d^n}{d\gamma^n}\log(1+\beta) = \frac{n-1\,!}{(1-\gamma)^n},$$

$$\frac{d^n}{d\gamma^n}(1-\gamma)^{-p} = \frac{p+n-1\,!}{p-1\,!}(1-\gamma)^{-p-n}.$$

Then, by their aid, we have from LEIBNITZ's theorem—

$$\frac{d^k}{d\gamma^k}\gamma^{k+1}\log(1+\beta) = \sum_{t=0}^{t=k}\frac{k\,!}{t\,!\,k-t\,!}\frac{d^t}{d\gamma^t}\gamma^{k+1}\frac{d^{k-t}}{d\gamma^{k-t}}\log(1+\beta),$$

$$= \sum_{t=0}^{t=k}\frac{k\,!}{t\,!\,k-t\,!}\,\frac{k+1\,!\,k-t-1\,!}{k-t+1\,!}\,\frac{\gamma^{k-t+1}}{(1-\gamma)^{k-t}},$$

in which we interpret $(-1)\,!/(1-\gamma)^0$ as $\log(1+\beta)$

Thus

$$(k, \gamma) = \sum_{t=0}^{t=k}\frac{1}{(k-t+1)(k-t)}\,\frac{k+1\,!}{t\,!\,k-t\,!}\beta^{k-t}, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

with $\beta^0/0 = \log(1+\beta)$.

Again

$$\frac{1}{i\,!\,k\,!}\frac{d^i}{d\gamma^i}\gamma^i\frac{d^k}{d\gamma^k}(\gamma^{k+1}\log(1+\beta))$$

$$= \frac{1}{i\,!}\sum_{t=0}^{t=k}\frac{k+1\,!\,k-t-1\,!}{t\,!\,k-t\,!\,k-t+1\,!}\frac{d^i}{d\gamma^i}\frac{\gamma^{i+k-t+1}}{(1-\gamma)^{k-t}},$$

$$= \gamma\sum_{t=0}^{t=k}\sum_{r=0}^{r=i}\frac{k+1\,!\,k-t-1\,!\,i+k-t+1\,!\,i+k-r-t-1\,!}{t\,!\,k-t\,!\,r\,!\,i-r\,!\,k-t+1\,!\,k-t-1\,!\,i+k-r-t+1\,!}\beta^{i+k-r-t}.$$

Hence

$$[k, i, \gamma] = \sum_{t=0}^{t=k}\sum_{r=0}^{r=i}\frac{1}{(i+k-r-t+1)(i+k-r-t)}\,\frac{k+1\,!\,i+k-t+1\,!}{k-t+1\,!}\cdot\frac{\beta^{i+k-r-t}}{t\,!\,k-t\,!\,r\,!\,i-r\,!}. \quad (31)$$

In (30) and (31) the infinite series are replaced by finite series.

From the form of the series it is obvious that the result must be symmetrical with respect to k and i, so that $[k, i, \gamma] = [i, k, \gamma]$, but this is not obvious on the face of the formula (31).

We shall find, therefore, the symmetrical form of (31) for the first few terms.

If $t = k$, $r = i$, we obviously have

$$\textit{First term} = (k+1)(i+1)\log(1+\beta).$$

The second term arises from $t = k$, $r = i-1$, and $t = k-1$, $r = i$. The two corresponding values of (31) will be found to add together, and we get

$$\textit{Second term} = \tfrac{1}{4}(k+1)(i+1)[2(i+k)+ik]\beta.$$

The third term arises from $t = k, r = i - 2$; $t = k - 1, r = i - 1$; $t = k - 2$, $r = i$, and we find—

$$\textit{Third term} = \frac{1}{3.2}(k+1)(i+1)\left\{\frac{i(i-1)+k(k-1)}{2!} + \frac{ik(i+2)(k+2)}{2!\,3!} + \frac{ik(i+k+1)}{2!^2}\right\}\beta^2.$$

A symmetrical form for further terms may be obtained by writing (31) first with i before k and then with k before i, and taking half the sum of the two results. In computing these coefficients it is a useful check to compute from both unsymmetrical forms, when the identity of results verifies the computation.

The following Tables have been computed from (30) and (31). The numbers are the coefficients of the quantities at the heads of the columns for the values of k and i written in the first column. The series (k, γ) is terminable with β^k, and the series $[k, i, \gamma]$ is terminable with β^{k+i}.

In $[k, i, \gamma]$ the coefficients have only been computed as far as β^6, so that the last which is given completely is $[2, 4, \gamma]$; however, with such values of β as we require, the series are carried far enough to give numerical results with sufficient accuracy.

TABLE of (k, γ).

	$\text{Log}(1+\beta)$	$+\beta$	$+\beta^2$	$+\beta^3$	$+\beta^4$	$+\beta^5$
$k=2$	3	3	$\frac{1}{2}$			..
$k=3$	4	6	2	$\frac{1}{3}$	.	..
$k=4$	5	10	5	$1\frac{2}{3}$	$\frac{1}{4}$	..
$k=5$	6	15	10	5	$1\frac{1}{2}$	$\frac{1}{5}$

TABLE of $[k, i, \gamma]$.

	$\text{Log}(1+\beta)$	$+\beta$	$+\beta^2$	$+\beta^3$	$+\beta^4$	$+\beta^5$	$+\beta^6$	$+\beta^7$
$k=2, i=2$	9	27	$18\frac{1}{2}$	8	$1\frac{1}{2}$		..	..
$k=2, i=3$	12	48	46	31	12	2	..	..
$k=2, i=4$	15	75	$92\frac{1}{2}$	85	$50\frac{1}{4}$	17	$2\frac{1}{2}$	.
$k=2, i=5$	18	108	163	190	$151\frac{1}{2}$	$77\frac{2}{3}$	23	&c.
$k=3, i=3$	16	84	108	103	63	22	$3\frac{1}{3}$	..
$k=3, i=4$	20	130	210	260	219	118	$36\frac{2}{3}$	&c.
$k=3, i=5$	24	186	362	552	594	$434\frac{4}{5}$	206	&c.
$k=4, i=4$	25	200	400	625	$687\frac{3}{4}$	514	$248\frac{1}{3}$	&c.
$k=4, i=5$	30	285	680	1285	$1750\frac{1}{2}$	1681	1110	&c.
$k=5, i=5$	36	405	1145	2585	4272	$5098\frac{2}{3}$	4345	&c.

We must now go back and determine the value of the outstanding potential of the first degree of harmonics, which will be annulled when rotation is imposed on the system. The potential is given in (22–i.) and (22–ii.); (22–i.) contributes nothing, and (22–ii.) gives us, for $k=1$,

$$\frac{4\pi A^3}{3c}\left[\frac{a}{c}+\tfrac{3}{2}\left(\frac{a}{c}\right)^4\sum_{i=2}^{i=\infty}\frac{i+1\,!}{1\,!\,i\,!}\,\frac{\Gamma^{i-1}}{i-1}H_i\right]\frac{w_1}{a}.$$

Thus, if we call u_1, U_1 the outstanding potential of the first degree, when referred to the two origins respectively, we have

$$\left.\begin{aligned} u_1 &= \frac{4\pi A^3}{3c}\left[1+\tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{i+1}{i-1}\Gamma^{i-1}H_i\right]\frac{a}{c}\cdot\frac{w_1}{a} \\ U_1 &= \frac{4\pi a^3}{3c}\left[1+\tfrac{3}{2}\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{i+1}{i-1}\gamma^{i-1}h_i\right]\frac{A}{c}\cdot\frac{W_1}{A} \end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

§ 3. *The Potential due to Rotation.*

Intermediate between the two origins o and O take a third Q, and take the axes of ξ and η parallel to those of x and y, and that of ζ identical with that of z. Let $Qo=d$, $QO=D$.

Then suppose that the system of the two spheroids is in uniform rotation about the axes of ξ with an angular velocity ω.

The potential Ω of the centrifugal forces is given by

$$\Omega=\tfrac{1}{2}\omega^2(\eta^2+\zeta^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

But

$$\left.\begin{aligned} &z=\zeta+d, \quad Z=D-\zeta, \quad d+D=c \\ &y=\eta \qquad\quad Y=-\eta \\ &\xi=x \qquad\quad X=-\xi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Hence

$$\begin{aligned}\Omega &= \tfrac{1}{2}\omega^2(y^2+z^2-2zd+d^2) \\ &= \tfrac{1}{2}\omega^2\left[-\tfrac{1}{2}(x^2-y^2)+\tfrac{1}{3}(z^2-\tfrac{1}{2}x^2-\tfrac{1}{2}y^2)+\tfrac{2}{3}(x^2+y^2+z^2)-2zd+d^2\right].\end{aligned}$$

Then, remembering that

$$w_2=z^2-\tfrac{1}{2}x^2-\tfrac{1}{2}y^2, \quad w_1=z,$$

and if we put

$$q_2=x^2-y^2, \quad Q_2=X^2-Y^2,$$

we have

$$\Omega=-\tfrac{1}{4}\omega^2q_2+\tfrac{1}{6}\omega^2w_2-\omega^2dw_1+\tfrac{1}{3}\omega^2r^2+\tfrac{1}{2}\omega^2d^2. \quad . \quad . \quad . \quad . \quad (35)$$

Similarly the rotation potential, when developed with reference to the other origin O, is

$$\Omega=-\tfrac{1}{4}\omega^2Q_2+\tfrac{1}{6}\omega^2W_2-\omega^2DW_1+\tfrac{1}{3}\omega^2R^2+\tfrac{1}{2}\omega^2D^2. \quad . \quad . \quad (36)$$

The last terms of (35) and (36) are constants, and the term in r^2, and that in R^2 are symmetrical about each origin, and so the corresponding forces can produce no departure from sphericity in either mass; thus these terms may be dropped. Next we have in (35) and (36) the outstanding potentials $-\omega^2 dw_1$ and $-\omega^2 DW_1$, which will be annulled by other similar terms, and so need not be considered now. We are left, therefore, with the terms in q_2 and w_2, or in Q_2 and W_2. The q_2 is a sectorial harmonic, the w_2 a zonal, and it will be convenient to treat them separately. We shall begin with the zonal term.

§ 4. *Disturbance due to the Zonal Harmonic Rotational Term.*

The potential whose effects we are to consider is $\frac{1}{6}\omega^2 w_2$ or $\frac{1}{6}\omega^2 W_2$, according to the origin which we are considering.

If an isolated spheroid of fluid of unit density be rotating with angular velocity ω, the ellipticity of the spheroid is $15\omega^2/16\pi$; therefore we put

$$\epsilon = \frac{15\omega^2}{16\pi}. \qquad (37)$$

Let us assume, for the equations to the two spheroids,

$$\left.\begin{aligned}\frac{r}{a} &= 1 + \tfrac{1}{3}\epsilon\frac{w_2}{r^2} + \left(\frac{A}{a}\right)^3 \sum_{i=2}^{i=\infty} \frac{2i+1}{2i-2}\left(\frac{a}{c}\right)^{i+1} l_i \frac{w_i}{r^i} \\ \frac{R}{A} &= 1 + \tfrac{1}{3}\epsilon\frac{W_2}{R^2} + \left(\frac{a}{A}\right)^3 \sum_{i=2}^{i=\infty} \frac{2i+1}{2i-2}\left(\frac{A}{c}\right)^{i+1} L_i \frac{W_i}{R^i}\end{aligned}\right\}, \qquad (38)$$

where l_i, L_i, are unknown coefficients which are to be determined. We now have to determine the potentials at any point of the inequalities (38) on the two spheroids.

The potential of the inequality $\frac{1}{3}\epsilon\, w_2/r^2$ in the first of (38) is

$$\tfrac{4}{5}\pi\, a^3 . a^2 . \tfrac{1}{3}\epsilon\frac{w_2}{r^5} = \tfrac{1}{4}\omega^2 a^2 \left(\frac{a}{r}\right)^3 \frac{w_2}{r^2}. \qquad (39)$$

The similar inequality in the second of (38) gives us

$$\tfrac{4}{5}\pi\, A^3 . A^2 . \tfrac{1}{3}\epsilon\frac{W_2}{R^5} = \frac{4\pi A^3}{3c}\tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2 . \frac{c^3 W_2}{R^5}. \qquad (40)$$

The term in l_k in the first of (38) gives us, as in § 2,

$$\frac{4\pi A^3}{3c}\,\frac{3l_k}{2k-2}\left(\frac{a}{c}\right)^k \left(\frac{a}{r}\right)^{k+1} \frac{w_k}{r^k}. \qquad (41)$$

The term in L_i in the second of (38) gives us, as in § 2,

$$\tfrac{4}{3}\pi a^3 \left(\frac{A}{c}\right)^3 \frac{3L_i\Gamma^{i-1}}{2i-2}\,\frac{c^i W_i}{R^{2i+1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

The potential due to rotation is $\frac{1}{6}\omega^2 w_2$ or $\frac{1}{6}\omega^2 W_2$, being the second term of (35) or (36); this term we find it convenient to write

$$\tfrac{1}{6}\omega^2 a^2 \left(\frac{r}{a}\right)^2 \frac{w_2}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

The sums of the several terms (39), (40), (41), (42), and (43) are to be regarded as the potential of perturbing forces by which the spheroid a or the spheroid A is disturbed, and the arbitrary constants l and L are to be so chosen that each may be a figure of equilibrium.

We may consider the spheroid a by itself, and the solution for it will afford us the solution for the spheroid A by symmetry. In order to find the disturbance, the formulæ (40) and (42) must be transferred.

Now by (4), with $i = 2$,

$$\frac{4\pi A^3}{3c}\,\tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2 \frac{c^3 W_2}{R^5} = \frac{4\pi A^3}{3c}\,\tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2 \sum_{k=0}^{k=\infty} \frac{k+2!}{2!\,k!}\left(\frac{a}{c}\right)^k \frac{w_k}{a^k}. \quad . \quad . \quad . \qquad (40')$$

And again, by (4),

$$\frac{4\pi a^3}{3c}\left(\frac{A}{c}\right)^3 \frac{3L_i\Gamma^{i-1}}{2i-2}\,\frac{c^i W_i}{R^{2i+1}} = \frac{4\pi A^3}{3c}\,\tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{k=0}^{k=\infty} \frac{k+i!}{i!\,k!}\,\frac{\Gamma^{i-1}}{i-1}\,L_i\left(\frac{a}{c}\right)^k \frac{w_k}{a^k}. \quad . \qquad (42')$$

Then (39), (40′), the sum of (41) from $k = \infty$ to $k = 2$, the sum of (42′) from $i = \infty$ to $i = 2$, and (43) together constitute the disturbing potential, all now referred to the origin o.

In order to find the disturbance of the spheroid a, we add the perturbing potential to $\frac{4}{3}\pi a^3/r$, give r its value (38) in this term, put $r = a$ in the perturbing potential, and make the whole potential constant by equating to zero the coefficient of each harmonic term.

We will begin by putting $r/a = 1 + \frac{1}{3}\epsilon\, w_2/r^2$, and considering only the perturbing potentials (39) and (43). We have then, for the coefficient of w_2/r^2,

$$-\tfrac{4}{3}\pi a^2.\tfrac{1}{3}\epsilon + \tfrac{1}{4}\omega^2 a^2 + \tfrac{1}{6}\omega^2 a^2.$$

Now, with the value of ϵ in (37),

$$-\tfrac{4}{3}\pi a^2.\,\tfrac{1}{3}\epsilon = -\tfrac{5}{12}\omega^2 a^2 \quad \text{and} \quad -\tfrac{5}{12} + \tfrac{1}{4} + \tfrac{1}{6} = 0.$$

Hence the coefficient of w_2/r^2 vanishes, and the term in ϵ in (38) has been properly chosen to satisfy the perturbing potentials (39) and (43).

Following the similar process with the remaining terms of (38), and equating to zero the coefficient of w_k, we have from (40′), (41), and (42′),

$$-\frac{2k+1}{2k-2}l_k + \frac{3l_k}{2k-2} + \tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2\frac{k+2!}{k!\,2!} + \tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{k+i!}{k!\,i!}\,\frac{\Gamma^{i-1}}{i-1}L_i = 0\,;$$

whence

$$l_k = \tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2\frac{k+2!}{k!\,2!} + \tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{k+i!}{k!\,i!}\,\frac{\Gamma^{i-1}}{i-1}L_i. \quad . \quad . \quad . \quad . \qquad (44)$$

By symmetry, the condition that the spheroid A may be a level surface is

$$L_r = \tfrac{1}{5}\epsilon\left(\frac{a}{c}\right)^2\frac{r+2!}{r!\,2!} + \tfrac{3}{2}\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{r+i!}{r!\,i!}\,\frac{\gamma^{i-1}}{i-1}l_i. \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

Multiply both sides of (45) by $\tfrac{3}{2}\left(\frac{a}{c}\right)^3\frac{r+k!}{r!\,k!}\,\frac{\Gamma^{r-1}}{r-1}$, and perform $\sum_{r=2}^{r=\infty}$ on the whole, and substitute from (44); and we have

$$\begin{aligned} l_k - \tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2\frac{k+2!}{k!\,2!} &= \tfrac{1}{5}\epsilon\left(\frac{a}{c}\right)^2\tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{r=2}^{r=\infty}\frac{r+2!\,r+k!}{2!\,r!\,r!\,k!}\,\frac{\Gamma^{r-1}}{r-1} \\ &+ (\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\sum_{r=2}^{r=\infty}\frac{r+i!\,r+k!}{r!\,i!\,r!\,k!}\,\frac{\gamma^{i-1}}{i-1}\,\frac{\Gamma^{r-1}}{r-1}l_i. \quad . \quad . \quad . \quad . \qquad (46)\end{aligned}$$

Introducing the notation (26) for the series involved in (46), we have

$$\begin{aligned} l_k = \tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2&\left\{\tfrac{1}{2}(k+1)(k+2) + \tfrac{3}{2}\left(\frac{a}{A}\right)^2\left(\frac{a}{c}\right)^3[k,\,2,\,\Gamma]\right\} \\ &+ (\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}[k,\,i,\,\Gamma]\frac{\gamma^{i-1}}{i-1}l_i. \quad . \quad . \quad . \quad . \qquad (47)\end{aligned}$$

Each value of k gives a similar equation, and there is a similar series of equations with small and large letters interchanged.

Now put

$$\left.\begin{aligned} l_k &= \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2(k+1)(k+2)\,\lambda_k \\ L_i &= \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2(k+1)(k+2)\,\Lambda_k \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

and (47) becomes

$$\begin{aligned} \lambda_k = 1 &+ \frac{3}{(k+1)(k+2)}\left(\frac{a}{A}\right)^2\left(\frac{a}{c}\right)^3[k,\,2,\,\Gamma] \\ &+ (\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{(i+1)(i+2)}{(k+1)(k+2)}[k,\,i,\,\Gamma]\frac{\gamma^{i-1}}{i-1}\lambda_i. \quad . \quad . \quad . \qquad (49)\end{aligned}$$

We attribute to k in (49) all values from ∞ to 2, and thus find a series of equations for the λ's. A similar series of equations holds for the Λ's.

We must now find the outstanding potential of the first degree of harmonics. No such term exists in (39), (41), (43), but it arises entirely out of (40′) and (42′). If we write v_1 for the outstanding potential, we have clearly

$$v_1 = \frac{4\pi A^3}{3c}\left\{\tfrac{1}{5}\epsilon\left(\frac{A}{c}\right)^2 \frac{3\,!}{2\,!\,1\,!}\frac{a}{c}\cdot\frac{w_1}{a} + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{i+1\,!}{i\,!\,1\,!}\frac{\Gamma^{i-1}}{i-1} L_i \frac{a}{c}\frac{w_1}{a}\right\};$$

whence

$$v_1 = \frac{4\pi A^3}{3c}\,\tfrac{1}{10}\,\epsilon\left(\frac{a}{c}\right)^2\left\{6\left(\frac{A}{a}\right)^2 + \sum_{i=2}^{i=\infty}\frac{(i+1)^2\,(i+2)}{i-1}\,\Gamma^{i-1}\,\Lambda_i\right\}\frac{a}{c}\cdot\frac{w_1}{a}, \quad . \quad . \quad (50)$$

and, by symmetry,

$$V_1 = \frac{4\pi a^3}{3c}\,\tfrac{1}{10}\,\epsilon\left(\frac{A}{c}\right)^2\left\{6\left(\frac{a}{A}\right)^2 + \sum_{i=2}^{i=\infty}\frac{(i+1)^2\,(i+2)}{i-1}\,\gamma^{i-1}\,\lambda_i\right\}\frac{A}{c}\cdot\frac{W_1}{A}. \quad . \quad . \quad (51)$$

§ 5. *Disturbance due to the Sectorial Harmonic Rotational Term.*

In (35) and (36) we have found this term to be $-\tfrac{1}{4}\omega^2 q_2$ or $-\tfrac{1}{4}\omega^2 Q_2$.

We have already observed that, if the operation $\dfrac{d^2}{dx^2} - \dfrac{d^2}{dy^2}$ or δ^2 be performed on w_i, the result vanishes when $i = 1, 2, 3$.

Now, by (6),

$$\begin{aligned} w_4 &= \sum_{k=0}^{k=2} (-)^k \frac{4\,!}{k\,!^2\, 4-2k\,!}\, z^{4-2k}\rho^k \\ &= z^4 - \frac{4\,!}{1\,!^2\, 2\,!} z^2\rho + \frac{4\,!}{2\,!^2\, 0\,!}\rho^2. \end{aligned}$$

Hence $\tfrac{1}{4}d^2 w_4/d\rho^2 = 3$, and, since $\delta^2 w_4 = \tfrac{1}{4}(x^2 - y^2)\, d^2 w_4/d\rho^2$, it follows that

$$q_2 = x^2 - y^2 = \tfrac{1}{3}\delta^2 w_4, \qquad \text{and} \qquad Q_2 = \tfrac{1}{3}\delta^2 W_4. \quad . \quad . \quad . \quad . \quad (52)$$

Hence the sectorial rotational term is $-\tfrac{1}{12}\,\omega^2\,\delta^2 w_4$ or $-\tfrac{1}{12}\,\omega^2\,\delta^2 W_4$; this potential is of the second order of sectorial harmonics.

Now, with ϵ as defined in (37), let us assume as the equations to the two surfaces,

$$\left.\begin{aligned} \frac{r}{a} &= 1 - \tfrac{1}{6}\epsilon\frac{\delta^2 w_4}{r^2} - \left(\frac{A}{a}\right)^3 \sum_{i=2}^{i=\infty}\frac{2i+1}{2i-2}\left(\frac{a}{c}\right)^{i+1} m_i \frac{\delta^2 w_{i+2}}{r^i} \\ \frac{R}{A} &= 1 - \tfrac{1}{6}\epsilon\frac{\delta^2 W_4}{R^2} - \left(\frac{a}{A}\right)^3 \sum_{i=2}^{i=\infty}\frac{2i+1}{2i-2}\left(\frac{A}{c}\right)^{i+1} M_i \frac{\delta^2 W_{i+2}}{R^i} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (53)$$

We have now to determine the potentials of the inequalities on the two spheroids expressed by (53).

The potential of the inequality $-\frac{1}{10}\epsilon\,\delta^2 w_4/r^2$ in the first of (53) is

$$-\frac{4\pi a^3}{5}\cdot a^2\cdot\tfrac{1}{6}\epsilon\frac{\delta^2 w_4}{r^5} = -\tfrac{1}{8}\omega^2 a^2\left(\frac{a}{r}\right)^3\frac{\delta^2 w_4}{r^2}. \qquad (54)$$

The potential of the similar inequality in the second of (53) is

$$-\frac{4\pi A^3}{3c}\cdot\left(\frac{A}{c}\right)^2\tfrac{1}{10}\epsilon\frac{c^3\delta^2 W_4}{R^5}. \qquad (55)$$

The term in m_k in the first of (53) gives us

$$-\frac{4\pi A^3}{3c}\,\frac{3m_k}{2k-2}\left(\frac{a}{c}\right)^k\left(\frac{a}{r}\right)^{k+1}\frac{\delta^2 w_{k+2}}{r^2}. \qquad (56)$$

The term in M_i in the second of (53) gives us

$$-\frac{4\pi A^3}{3c}\left(\frac{a}{c}\right)^3\frac{3M_i}{2i-2}\Gamma^{i-1}\frac{c^i\,\delta^2 W_{i+2}}{R^{2i+1}}. \qquad (57)$$

Lastly, the sectorial term itself is

$$-\tfrac{1}{12}\omega^2 a^2\left(\frac{r}{a}\right)^2\frac{\delta^2 w_4}{r^2}. \qquad (58)$$

The sums of the several terms (54), (55), (56), (57), and (58) are to be regarded as the potential of perturbing forces by which the spheroid a, or the spheroid A, is disturbed, and the arbitrary constants m, M, are to be so chosen that they may each be figures of equilibrium. We may consider the spheroid a by itself, and the solution for it will afford the solution for the spheroid A by symmetry. In order to find the disturbance, the formulæ (55) and (57) must be transferred. For this purpose we require the second transference formulæ.

By (16), with $i = 2$, we have for (55)

$$-\frac{4\pi A^3}{3c}\left(\frac{A}{c}\right)^2\tfrac{1}{10}\epsilon\frac{c^3\delta^2 W_4}{R^5} = -\frac{4\pi A^3}{3c}\tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2\sum_{k=2}^{k=\infty}\frac{k+2\,!}{0\,!\,k+2\,!}\left(\frac{a}{c}\right)^k\frac{\delta^2 w_{k+2}}{a^k}. \qquad (55')$$

And by (16) we have for (57)

$$-\frac{4\pi A^3}{3c}\left(\frac{a}{c}\right)^3\frac{3M_i}{2i-2}\Gamma^{i-1}\frac{c^i\delta^2 W_{i+2}}{R^{2i+1}}$$

$$= -\frac{4\pi A^3}{3c}\,\tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{k=2}^{k=\infty}\frac{k+i\,!}{i-2\,!\,k+2\,!}\,\frac{\Gamma^{i-1}}{i-1}M_i\left(\frac{a}{c}\right)^k\frac{\delta^2 w_{k+2}}{a^k}. \qquad (57')$$

Then (54), (55'), the sum of (56) from $k=\infty$ to $k=2$, the sum of (57') from $i=\infty$ to $i=2$, and (58) together constitute the disturbing potential, all now referred to the origin a.

In order to find the disturbance of the spheroid a, we add the perturbing potential to $\frac{4}{3}\pi a^3/r$, give r its value (53) in this term, put $r = a$ in the perturbing potential, and make the whole potential constant by equating to zero the coefficients of each harmonic term.

We will begin by putting $r/a = 1 - \frac{1}{6}\epsilon\, \delta^2 w_4/r^2$, and considering only the perturbing potentials (54) and (58). We have then, for the coefficient of $\delta^2 w_4/r^2$,

$$\tfrac{4}{3}\pi a^3 \,.\, \tfrac{1}{6}\epsilon - \tfrac{1}{8}\omega^2 a^2 - \tfrac{1}{12}\omega^2 a^2.$$

Now, with the value of ϵ in (37),

$$\tfrac{4}{3}\pi a^3 \,.\, \tfrac{1}{6}\epsilon = \tfrac{5}{24}\omega^2 a^2, \quad \text{and} \quad \tfrac{5}{24} - \tfrac{1}{8} - \tfrac{1}{12} = 0.$$

Hence the coefficient of $\delta^2 w_4/r^2$ vanishes, and the term ϵ in (53) has been properly chosen to satisfy the perturbing potentials (54) and (58). Following the similar process with the remaining terms of (53), and equating to zero the coefficient of $\delta^2 w_{k+2}$, we have, from (55′), (56), (57′),

$$\frac{2k+1}{2k-2} m_k - \frac{3m_k}{2k-2} - \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2 - \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{k+i\,!}{i-2\,!\;k+2\,!}\,\frac{\Gamma^{i-1}}{i-1} M_i = 0,$$

or

$$m_k = \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{k+i\,!}{i-2\,!\;k+2\,!}\,\frac{\Gamma^{i-1}}{i-1} M_i \,. \quad . \quad . \quad . \quad . \quad (59)$$

By symmetry the condition that the spheroid A may be a level surface is

$$M_r = \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2 + \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{r+i\,!}{i-2\,!\;r+2\,!}\,\frac{\gamma^{i-1}}{i-1} m_i \,. \quad . \quad . \quad . \quad . \quad (60)$$

Multiply both sides of (60) by $\frac{3}{2}\left(\frac{a}{c}\right)^3 \frac{k+r\,!}{r-2\,!\;k+2\,!}\,\frac{\Gamma^{r-1}}{r-1}$, and perform $\sum_{r=2}^{r=\infty}$ on the whole, and substitute from (59), and we have

$$m_k - \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2 = \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2 \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{r=2}^{r=\infty} \frac{k+r\,!}{r-2\,!\;k+2\,!}\,\frac{\Gamma^{r-1}}{r-1}$$

$$+ \left(\tfrac{3}{2}\right)^2 \left(\frac{a}{c}\right)^3 \left(\frac{A}{c}\right)^3 \sum_{r=2}^{r=\infty} \sum_{i=2}^{i=\infty} \frac{r+i\,!\;k+r\,!}{i-2\,!\;r+2\,!\;r-2\,!\;k+2\,!}\,\frac{\Gamma^{r-1}}{r-1}\,\frac{\gamma^{i-1}}{i-1} m_i. \quad (61)$$

Now let us write

$$\left.\begin{aligned} \{k,\ \Gamma\} &= \sum_{r=2}^{r=\infty} \frac{k+r\,!}{r-1\,!\;k+2\,!}\Gamma^{r-1} \\ \boxed{k,\ i,\ \Gamma} &= \sum_{r=2}^{r=\infty} \frac{r+i\,!\;k+r\,!}{i-1\,!\;r+2\,!\;r-1\,!\;k+2\,!}\Gamma^{r-1} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (62)$$

so that (61) may be written

$$m_k = \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2 + \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2 \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \{k,\ \Gamma\} + \left(\tfrac{3}{2}\right)^2 \left(\frac{a}{c}\right)^3 \left(\frac{A}{c}\right)^3 \sum_{i=2}^{i=\infty} \boxed{k,\ i,\ \Gamma}\,\gamma^{i-1} m_i. \quad (63)$$

Next put

$$m_k = \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2 \mu_k, \qquad M_k = \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2 \mathrm{M}_k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

and (63) becomes

$$\mu_k = 1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3\left(\frac{a}{A}\right)^2\{k, \Gamma\} + (\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\boxed{k, i, \Gamma}\,\gamma^{i-1}\mu_i. \quad . \quad . \quad (65)$$

We attribute to k in (65) all values from ∞ to 2, and thus find a series of equations for the μ's. A similar series of equations holds for the M's.

We now have to sum the series (62).

Consider the function

$$\frac{1}{k+2}\left[(1+\beta)^{k+2} - 1\right].$$

$$\frac{1}{k+2}\left[(1+\beta)^{k+2} - 1\right] = \frac{1}{k+2}\left[(1-\gamma)^{-k-2} - 1\right] = \frac{1}{k+2}\left[\frac{k+2}{1!}\gamma + \frac{k+2\,.\,k+3}{2!}\gamma^2 + \ldots\right]$$

$$= \frac{1}{k+2}\sum_{r=2}^{r=\infty}\frac{k+r!}{k+1!\,r-1!}\gamma^{r-1} = \sum_{r=2}^{r=\infty}\frac{k+r!}{k+2!\,r-1!}\gamma^{r-1}.$$

Hence

$$\{k, \gamma\} = \frac{1}{k+2}\left[(1+\beta)^{k+2} - 1\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Next

$$\frac{1}{(k+2)\gamma^3}\cdot\frac{1}{i-1!}\frac{d^{i-2}}{d\gamma^{i-2}}\left\{\gamma^{i+1}\left[(1+\beta)^{k+2} - 1\right]\right\} = \frac{1}{\gamma^3.\,i-1!}\frac{d^{i-2}}{d\gamma^{i-2}}\sum_{r=2}^{r=\infty}\frac{k+r!}{k+2!\,r-1!}\gamma^{i+r}$$

$$= \frac{1}{\gamma^3}\sum_{r=2}^{r=\infty}\frac{k+r!\,i+r!}{i-1!\,k+2!\,r-1!\,r+2!}\gamma^{r+2}.$$

Hence

$$\boxed{k, i, \gamma} = \frac{1}{(k+2)\gamma^3}\cdot\frac{1}{i-1!}\frac{d^{i-2}}{d\gamma^{i-2}}\left\{\gamma^{i+1}\left[(1+\beta)^{k+2} - 1\right]\right\}. \quad . \quad (67)$$

The differential in (67) must now be evaluated. We have, by LEIBNITZ's theorem,

$$\frac{d^{i-2}}{d\gamma^{i-2}}\left\{\gamma^{i+1}\left[(1+\beta)^{k+2} - 1\right]\right\} = -\frac{d^{i-2}}{d\gamma^{i-2}}\gamma^{i+1} + \sum_{r=0}^{r=i-2}\frac{i-2!}{r!\,i-r-2!}\frac{d^r\gamma^{i+1}}{d\gamma^r}\frac{d^{i-r-2}}{d\gamma^{i-r-2}}(1-\gamma)^{-k-2}$$

$$= -\frac{i+1!}{3!}\gamma^3 + \sum_{r=0}^{r=i-2}\frac{i-2!}{r!\,i-r-2!}\,\frac{i+1!}{i-r+1!}\,\frac{i+k-r-1!}{k+1!}\,\frac{\gamma^{i-r+1}}{(1-\gamma)^{i+k-r}}$$

$$= -\frac{i+1!}{3!}\gamma^3 + \frac{\gamma^3}{(1-\gamma)^{k+2}}\sum_{r=0}^{r=i-2}\ldots\beta^{i-r-2}$$

Substituting in (67), we have

$$\boxed{k, i, \gamma} = \frac{i(i+1)}{6(k+2)}\left[-1 + (1+\beta)^{k+2}\sum_{r=0}^{r=i-2}3!\,\frac{i-2!\,i+k-r-1!}{r!\,i-r-2!\,i-r+1!\,k+1!}\beta^{i-r-2}\right]. (68)$$

The following Tables, computed from (66) and (68), give the values of $\{k, \gamma\}$ and $\boxed{k, i, \gamma}$ as far as $k = 5$, and $k = 5$, $i = 5$

Table of $\{k, \gamma\}$.

$$\{2, \gamma\} = \tfrac{1}{4}[(1+\beta)^4 - 1]$$
$$\{3, \gamma\} = \tfrac{1}{5}[(1+\beta)^5 - 1]$$
$$\{4, \gamma\} = \tfrac{1}{6}[(1+\beta)^6 - 1]$$
$$\{5, \gamma\} = \tfrac{1}{7}[(1+\beta)^7 - 1]$$

Table of $\boxed{k, i, \gamma}$.

$k=2, i=2$, $\frac{1}{4}[(1+\beta)^4-1]$
$k=2, i=3$, $\frac{1}{2}[(1+\beta)^5-1]$
$k=2, i=4$, $\frac{5}{6}[(1+\beta)^6-1]$
$k=2, i=5$; $\frac{5}{4}[(1+\beta)^7-1]$

$k=3, i=2$, $\frac{1}{5}[(1+\beta)^5-1]$
$k=3, i=3$, $\frac{2}{5}[(1+\beta)^5(1+\frac{5}{4}\beta)-1]$
$k=3, i=4$; $\frac{2}{3}[(1+\beta)^6(1+\frac{3}{2}\beta)-1]$.
$k=3, i=5$; $[(1+\beta)^7(1+\frac{7}{4}\beta)-1]$.

$k=4, i=2$; $\frac{1}{6}[(1+\beta)^6-1]$
$k=4, i=3$; $\frac{1}{3}[(1+\beta)^6(1+\frac{3}{2}\beta)-1]$
$k=4, i=4$, $\frac{5}{9}[(1+\beta)^6(1+3\beta+\frac{21}{10}\beta^2)-1]$
$k=4, i=5$, $\frac{5}{6}[(1+\beta)^7(1+\frac{7}{2}\beta+\frac{14}{5}\beta^2)-1]$

$k=5, i=2$; $\frac{1}{7}[(1+\beta)^7-1]$
$k=5, i=3$; $\frac{2}{7}[(1+\beta)^7(1+\frac{7}{4}\beta)-1]$
$k=5, i=4$, $\frac{10}{21}[(1+\beta)^7(1+\frac{7}{2}\beta+\frac{14}{5}\beta^2)-1]$
$k=5, i=5$, $\frac{5}{7}[(1+\beta)^7(1+\frac{21}{4}\beta+\frac{42}{5}\beta^2+\frac{21}{5}\beta^3)-1]$.

§ 6. *Determination of the Angular Velocity of the System.*

The angular velocity of the system must now be determined in such a way as to annul the outstanding potential of the first degree of harmonics.

Referring to origin o, we have from (35) $-\omega^2 dw_1$ directly from the rotation potential; the remaining terms are $u_1 + v_1$, since the sectorial harmonic term does not contribute anything.

Thus, taking u_1 from (32), and v_1 from (50), we get for the potential

$$-\omega^2 dw_1 + \frac{4\pi A^3}{3c}\frac{w_1}{c}\Bigg\{1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{i+1}{i-1}\Gamma^{i-1} H_i$$
$$+ \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2\left[6\left(\frac{A}{a}\right)^2 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{(i+1)^2(i+2)}{i-1}\Gamma^{i-1}\Lambda_i\right]\Bigg\}.$$

Equating this to zero,

$$\frac{3\omega^2 c^3}{4\pi} d = A^3\Bigg\{1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{i+1}{i-1}\Gamma^{i-1} H_i$$
$$+ \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2\left[6\left(\frac{A}{a}\right)^2 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{(i+1)^2(i+2)}{i-1}\Gamma^{i-1}\Lambda_i\right]\Bigg\}. \quad (69)$$

And, by symmetry,

$$\frac{3\omega^2 c^2}{4\pi} D = a^3 \Big\{ 1 + \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{i+1}{i-1} \gamma^{i-1} h_i$$
$$+ \tfrac{1}{10}\epsilon \left(\frac{A}{c}\right)^2 \left[6\left(\frac{a}{A}\right)^2 + \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{i=2}^{i=\infty} \frac{(i+1)^2(i+2)}{i-1} \gamma^{i-1} \lambda_i \right] \Big\}. \tag{70}$$

Add (70) to (69), note that $d + D = c$, and $\frac{1}{10}\epsilon = 3\omega^2/32\pi$, and solve for ω^2, and we have

$$\frac{3\omega^2}{4\pi} = \left[\left(\frac{A}{c}\right)^3 + \left(\frac{a}{c}\right)^3\right] \frac{1 + \tfrac{3}{2} \dfrac{A^3 a^3}{(A^3 + a^3) c^3} \displaystyle\sum_{i=2}^{i=\infty} \frac{i+1}{i-1} (\Gamma^{i-1} H_i + \gamma^{i-1} h_i)}{1 - \tfrac{3}{4}\left[\left(\dfrac{A}{c}\right)^5 + \left(\dfrac{a}{c}\right)^5\right] - \tfrac{3}{16}\left(\dfrac{A}{c}\right)^3 \left(\dfrac{a}{c}\right)^3 \displaystyle\sum_{i=2}^{i=\infty} \frac{(i+1)^2(i+2)}{i-1} (\gamma\Gamma^{i-1}\Lambda_i + \Gamma\gamma^{i-1}\lambda_i)}. \tag{71}$$

Now let $1 + K$ denote the factor by which $(A/c)^3 + (a/c)^3$ is multiplied in (71). Then, if the two masses were particles, K would be zero, and (71) would simply be the usual formula connecting masses, mean motion, and mean distance in a circular orbit. Hence $1 + K$ is an augmenting factor by which the value of the square of the angular velocity must be multiplied if it be derived from the law of the periodic time of two particles revolving about one another. K, in fact, gives the correction to KEPLER's law for the non-sphericity of the masses.

This completes the solution of the problem, for we have determined the angular velocity in such a way as to justify the neglect of the harmonic terms of the first degree in §§ 2 and 4.

§ 7. *Solution of the Problem.*

We may now collect from the preceding paragraphs the complete solution of the problem.

In (38) and (53) we have found that there are terms in r/a as follows:—

$$\tfrac{1}{3}\epsilon\frac{w_2}{r^2} - \tfrac{1}{6}\epsilon\frac{\delta^2 w_4}{r^2}.$$

Now

$$w_2 = z^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2, \quad \text{and} \quad \delta^2 w_4 = 3(x^2 - y^2);$$

hence

$$w_2 - \tfrac{1}{2}\delta^2 w_4 = z^2 - 2x^2 + y^2 = r^2 - 3x^2,$$

and these terms are therefore equal to $\epsilon(\frac{1}{3} - x^2/r^2)$.

We note that $\epsilon = 15\omega^2/16\pi = \frac{5}{4}\omega^2/\frac{4}{3}\pi$, and that $\omega^2/\frac{4}{3}\pi$ is the ratio generally written m in works on the figure of the Earth. Then, from (17), (38), (53), the equations to the two surfaces are

$$\left.\begin{aligned}\frac{r}{a} &= 1+\epsilon\left(\tfrac{1}{3}-\frac{x^2}{r^2}\right)+\left(\frac{A}{a}\right)^3\sum_{i=2}^{i=\infty}\frac{2i+1}{2i-2}\left(\frac{a}{c}\right)^{i+1}\left\{(h_i+l_i)\frac{w_i}{r^i}-m_i\frac{\delta^2 w_{i+2}}{r^i}\right\}\\ \frac{R}{A} &= 1+\epsilon\left(\tfrac{1}{3}-\frac{X^2}{R^2}\right)+\left(\frac{a}{A}\right)^3\sum_{i=2}^{i=\infty}\frac{2i+1}{2i-2}\left(\frac{A}{c}\right)^{i+1}\left\{(H_i+L_i)\frac{W_i}{R^i}-M_i\frac{\delta^2 W_{i+2}}{R^i}\right\}\end{aligned}\right\}. \tag{72}$$

From (27), (49), (65), we see that $h_2, h_3 \ldots h_i \ldots, \lambda_2, \lambda_3 \ldots \lambda_i \ldots, \mu_2, \mu_3 \ldots \mu_i \ldots,$ are to be found by solving the equations resulting from all values of k from 2 to infinity in the following:—

$$\left.\begin{aligned}h_k-1 &= \tfrac{3}{2}\left(\frac{a}{c}\right)^3(k,\Gamma)+(\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}[k,i,\Gamma]\frac{\gamma^{i-1}}{i-1}h_i\\ \lambda_k-1 &= \frac{3}{(k+1)(k+2)}\left(\frac{a}{c}\right)^3\left(\frac{a}{A}\right)^3[k,2,\Gamma]\\ &\quad+\frac{1}{(k+1)(k+2)}(\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}[k,i,\Gamma]\frac{(i+1)(i+2)}{i-1}\gamma^{i-1}\lambda_i\\ \mu_k-1 &= \tfrac{3}{2}\left(\frac{a}{c}\right)^3\left(\frac{a}{A}\right)^2\{k,\Gamma\}+(\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{i=2}^{i=\infty}\boxed{k,i,\Gamma}\,\gamma^{i-1}\mu_i\end{aligned}\right\}, \tag{73}$$

and symmetrical systems of equations for obtaining the H's, Λ's, and M's.

With the values found by the solution of these equations we then evaluate K by formula (71); and we have

$$\tfrac{4}{5}\epsilon=\frac{3\omega^2}{4\pi}=\left[\left(\frac{A}{c}\right)^3+\left(\frac{a}{c}\right)^3\right](1+K). \quad \ldots\ldots \tag{74}$$

We are now enabled to find the l's and m's by the formulæ (48) and (64), viz.,

$$\left.\begin{aligned}l_k &= \tfrac{1}{10}\epsilon(k+1)(k+2)\left(\frac{A}{c}\right)^2\lambda_k\\ m_k &= \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2\mu_k\end{aligned}\right\}, \quad \ldots\ldots \tag{75}$$

and the symmetrical forms give us the L's and M's.

Having thus evaluated all the auxiliary constants, (72) gives the solution of the problem.

It is well known that $\frac{5}{4}\times 3\omega^2/4\pi$ is the ellipticity of a single homogeneous mass of fluid rotating with angular velocity ω. Hence the first terms of (72) simply denote the ellipticity due to rotation in each of the masses, as if the other did not exist. Now the rigorous solution for the form of equilibrium of a rotating mass of fluid is an ellipsoid of revolution with eccentricity sin g, the value of g being given by the solution of

$$\frac{\omega^2}{2\pi}=\cot^3 g\,[(3+\tan^2 g)\,g-3\tan g].^* \quad \ldots\ldots \tag{76}$$

* See, for example, THOMSON and TAIT's 'Natural Philosophy' (3), § 771, with $f=\tan g$.

Hence it will undoubtedly be more correct to construct the surface, of which the equation is (72), by regarding the part of r under the symbol Σ as the correction to the radius-vector of an ellipsoid of revolution with eccentricity determined by (76), where $\omega^2/2\pi$ is found from (74).

§ 8. *Examples of the Solution.*

The principal object of the preceding investigation is to trace the forms of the two masses when they approach to close proximity, we shall thus be able to determine the forms when they are on the point of coalescing into a single mass, and shall finally obtain at least an approximate figure of the single mass. For this purpose we require to push the approximation by spherical harmonic analysis as far as it will bear. We shall below endeavour to estimate the degree of departure from correctness involved by the use of this analysis. The results will, therefore, be worked out numerically for such values of c/a as bring the two masses close together, and it will appear that the largest value of c/a assumed for numerical solution is such that the surfaces cross; in this case the reality will be a single mass of a shape which it will be possible to draw with tolerable accuracy.

The computations are facilitated if, instead of assuming c to be an exact multiple of a, we take c^2 a multiple of a^2; that is to say, we shall take $1/\gamma$ as an integer, and therefore $1/\beta$ also an integer.

We shall in the first instance suppose the two masses to be equal. In the following examples, then, we have $A = a$, $\Gamma = \gamma$, $B = \beta$, and the two masses assume the same shape.

The computations will be carried through in detail in two cases, viz., when $\beta = \frac{1}{7}$, and when $\beta = \frac{1}{5}$. The results will also be given for $\beta = \frac{1}{6}$.

When $\beta = \frac{1}{7}$, $\gamma = \frac{1}{8}$, $c/a = 2{\cdot}8284$, and when $\beta = \frac{1}{5}$, $\gamma = \frac{1}{6}$, $c/a = 2{\cdot}449$. Thus the distances of the centres apart are $2\frac{4}{5}$ and $2\frac{2}{5}$ of the mean radius respectively. The numerical details of the two computations may be stated *pari passû*, and the numbers applying to $\beta = \frac{1}{5}$ will be distinguished by being printed in small type.

In the case of $\beta = \frac{1}{6}$, we have $\gamma = \frac{1}{7}$, $c/a = 2{\cdot}6458$; but only the final result is given, without intermediate details.

The first step is to compute the values of the several series by means of the Tables in §§ 2 and 5.

The numerical results are as follows.

TABLE of (k, γ).

	$\beta = \frac{1}{7}$	$\beta = \frac{1}{8}$
$k = 2$	839	1 167
$k = 3$	1 433	2 012
$k = 4$	2 204	3 125
$k = 5$	3 163	4 536

TABLE of $\{k, \gamma\}$.

	$\beta = \frac{1}{7}$	$\beta = \frac{1}{8}$
$k = 2$	177	·268
$k = 3$	190	298
$k = 4$	205	331
$k = 5$	·221	369

TABLE of $[k, i, \gamma]$

	$\beta = \frac{1}{7}$	$\beta = \frac{1}{8}$
$k = 2,\quad i = 2$	5 460	7 847
$k = 2,\quad i = 3$	9 494	13 895
$k = 2,\quad i = 4$	14 875	22 201
$k = 2,\quad i = 5$	21 780	33 190
$k = 3,\quad i = 2$	9 494	13 895
$k = 3,\quad i = 3$	16 667	24 969
$k = 3,\quad i - 4$	26 384·	40 517
$k = 3,\quad i = 5$	39 047	61 574
$k = 4,\quad i = 2$	14 875	22 201
$k = 4,\quad i = 3$	26 384	40 517
$k = 4,\quad i = 4$	42 214	66 840
$k = 4,\quad i = 5$	63 183	103 372
$k = 5,\quad i = 2$	21 780	33 190
$k = 5,\quad i = 3$	39 047	61 574
$k = 5,\quad i = 4$	63 183	103 372
$k = 5,\quad i = 5$	95 690	162 831

TABLE of $\boxed{k, i, \gamma}$

	$\beta = \frac{1}{7}$	$\beta = \frac{1}{8}$
$k = 2,\quad i = 2$	177	268
$k = 2,\quad i = 3$	475	744
$k = 2,\quad i = 4$	1 024	1 655
$k = 2,\quad i = 5$	1 933	3 229
$k = 3,\quad i = 2$	190	298
$k = 3,\quad i = 3$	519	844
$k = 3,\quad i = 4$	1 137	1 921
$k = 3,\quad i = 5$	2 183	3 837
$k = 4,\quad i = 2$	205	331
$k = 4,\quad i = 3$	569	·961
$k = 4,\quad i = 4$	1 266	2 238
$k = 4,\quad i = 5$	2 471	4 578
$k = 5,\quad i = 2$	221	369
$k = 5,\quad i = 3$	624	1 096
$k = 5,\quad i = 4$	1·412	2 616
$k = 5,\quad i = 5$	2 803	5 479

With these values for the series, we have to compute the coefficients of the systems of simultaneous equations (73). The equations lend themselves more readily to solution if we consider $h_i - 1$, $\lambda_i - 1$, $\mu_i - 1$, as the unknowns instead of h_i, λ_i, μ_i. The results are given in the following equations.

The upper coefficients correspond to the case of $\beta = \frac{1}{7}$; the lower ones, printed in small type, to the case of $\beta = \frac{1}{5}$.

$h_2 - 1 =$	05902 +	00300 $(h_2 - 1)$ +	00033 $(h_3 - 1)$ +	00004 $(h_4 - 1)$ +	00001 $(h_5 - 1)$	+ ..
	13516	01362	00201	·00036	·00007	
$h_3 - 1 =$	·10086 +	00522	+ 00057	+ 00008	+ ·00001	+ . .
	·23385	02412	00361	·00065	·00012	
$h_4 - 1 =$	15529 +	·00817	+ 00091	+ 00012	+ ·00002	+ . .
	36467	03854	00586	·00108	·00021	
$h_5 - 1 =$	·22321 +	01196	+ 00134	+ ·00018	+ ·00003	+ . .
	53150	05762	·00891	00165	00033	

. . . .

$\lambda_2 - 1 =$	06400 +	00300 $(\lambda_2 - 1)$ +	00054 $(\lambda_3 - 1)$ +	00011 $(\lambda_4 - 1)$ +	00002 $(\lambda_5 - 1)$	+ ...
	·15158	01362	00335	00089	00023	
$\lambda_3 - 1 =$	·06677 +	·00313	+ 00057	+ 00011	+ 00002	+ ...
	16114	01447	00361	·00098	00026	
$\lambda_4 - 1 =$	06976 +	00327	+ 00060	+ 00012	+ ·00002	+ .
	17174	01542	00391	00108	00029	
$\lambda_5 - 1 =$	·07297 +	00342	+ 00064	+ ·00013	+ 00003	+ ..
	18352	·01646	·00424	00118	00033	

. . . .

$\mu_2 - 1 =$	01184 +	·00010 $(\mu_2 - 1)$ +	00003 $(\mu_3 - 1)$ +	00001 $(\mu_4 - 1)$ +	00000 $(\mu_5 - 1)$	+ .
	·02818	00047	·00022	·00008	00003	
$\mu_3 - 1 =$	·01274 +	00010	+ 00004	+ 00001	+ 00000	+ ..
	03127	00052	00024	00009	00003	
$\mu_4 - 1 =$	·01374 +	·00011	+ 00004	+ ·00001	+ ·00000	+ ..
	03478	00057	00028	·00011	00004	
$\mu_5 - 1 =$	01482 +	·00012	+ 00004	+ ·00001	+ 00000	+ ..
	03879	00064	00032	00013	00004	

. . . .

The solutions of these equations are obviously found by an easy approximation; they are

$h_2 = 1·0593$	$\lambda_2 = 1·0642$	$\mu_2 = 1·0118$
1·1377	1·1544	1·0282
$h_3 = 1·1012$	$\lambda_3 = 1·0670$	$\mu_3 = 1·0127$
1·2382	1·1642	1·0313
$h_4 = 1·1559$	$\lambda_4 = 1·0700$	$\mu_4 = 1·0137$
1·3719	1·1750	1·0348
$h_5 = 1·2241$	$\lambda_5 = 1·0732$	$\mu_5 = 1·0148$
1·5424	1·1870	1·0388

the small figures corresponding, as before, to the case of $\beta = \frac{1}{5}$.

With these values of the h's and λ's, I find

$$2\Sigma \frac{i+1}{i-1}\gamma^{i-1}h_i = {\cdot 8718 \atop 1\,3005}; \quad 2\Sigma \frac{(i+1)^2(i-2)}{i-1}\gamma^i\lambda_i = {1\,3949 \atop 2\,8542}; \quad \tfrac{3}{2}\left(\frac{a}{c}\right)^5 = {\cdot 00829, \atop 01701}$$

the summations, of course, stopping with $i = 5$.

Applying these in (71), we have, when

$$\beta = \tfrac{1}{7},\ 1 + K = \frac{1 + 02891}{1 - 00880} = 1\cdot 0380; \text{ or, when } \beta = \tfrac{1}{5},\ 1 + K = \frac{1 + 0664}{1 - 0195} = 1\cdot 0877,$$

whence

$$\tfrac{4}{5}\epsilon = \frac{3\omega^2}{4\pi} = {\cdot 08839 \atop \cdot 13608} \times {1\cdot 0380 \atop 1\,0877} = {\cdot 09175. \atop 1481}$$

Thus the angular velocity of the system has been found.

Next we have

$$\tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2 = {\cdot 001434. \atop 00309}$$

Introducing this into (48) and (64) with the previously found values of the λ's and μ's,

$$\left.\begin{array}{ll} l_2 = {0183, \atop 0428} & m_2 = {00145, \atop 0032} \\ l_3 = {\cdot 0306, \atop 0720} & m_3 = {\cdot 00145, \atop 0032} \\ l_4 = {\cdot 0460, \atop 1089} & m_4 = {00145, \atop 0032} \\ l_5 = {\cdot 0646, \atop \cdot 1541} & m_5 = {\cdot 00145, \atop 0032} \end{array}\right\}; \text{ and hence } \left\{\begin{array}{l} h_2 + l_2 = {1\cdot 0776, \atop 1\,1806} \\ h_3 + l_3 = {1\,1318, \atop \cdot 1\,3100} \\ h_4 + l_4 = {1\,2019, \atop 1\,4808} \\ h_5 + l_5 = {1\,2887. \atop 1\,6965} \end{array}\right.$$

By taking the differences of $h + l$, we may conclude that

$$h_6 + l_6 = {1\,39, \atop 1\,96}$$

and this sixth harmonic term will now be included.

It appears from the values of the m's that the harmonics of the type $\delta^2 w_{i+2}$ are practically negligible, excepting the term $\delta^2 w_4$, and that in that we may neglect the part depending on m_2.

Now, if r denotes the radius-vector due to the rotation, and δr the increase of radius-vector due to the mutual influence of the two masses, we have

$$\frac{\delta r}{a} = {\cdot 1191 \atop \cdot 2008}\frac{w_2}{r^2} + {\cdot 0309 \atop \cdot 0637}\frac{w_3}{r^3} + {\cdot 0100 \atop 0252}\frac{w_4}{r^4} + {\cdot 0035 \atop 0108}\frac{w_5}{r^5} + {\cdot 0013 \atop 0048}\frac{w_6}{r^6} + \dots \qquad (77)$$

We next have to consider the values of r, the radius-vector of the ellipsoid, due to rotation

We might compute from the spherical harmonic formula

$$\frac{r}{a} = 1 + \epsilon\left(\frac{1}{3} - \frac{x^2}{r^2}\right).$$

The results so computed will be compared with the others computed as shown below.

The following Table of the angular velocity and corresponding eccentricity e of the equilibrium ellipsoid of revolution is extracted from THOMSON and TAIT'S 'Natural Philosophy,' § 772 :—

e	$\frac{\omega^2}{2\pi}$
3	·0243
4	·0436
·5	·0690
·6	·1007
·7	·1387
·8	·1816

From this we find by interpolation that, when $3\omega^2/4\pi =$ 09175, $e =$ ·472 ; and, when $3\omega^2/4\pi =$ ·1481, $e =$ ·594.

These, then, are the eccentricities of the ellipsoids whose radius-vector is r in the two cases $\beta = \frac{1}{7}$, $\beta = \frac{1}{5}$

The equations to the generating ellipses are

$$\frac{r}{a} = \frac{1 - \cdot 0806}{1 - \cdot 2228 \cos^2\theta} \text{ for } \beta = \tfrac{1}{7}, \qquad \text{and} \qquad \frac{r}{a} = \frac{1 - \cdot 1353}{1 - \ 3535 \cos^2\theta} \text{ for } \beta = \tfrac{1}{5}.$$

The following are the computed values of r/a for each 15° of θ, the latitude, the small figures written below appertaining to the case of $\beta = \frac{1}{5}$.

	$\theta =$ 0°	15°	30°	45°	60°	75°	90°
$\beta = \frac{1}{7}: \frac{r}{a} =$	1 0429,	1 0330,	1·0074,	·9753,	·9461,	·9264,	·9194
$\beta = \frac{1}{5}$:	1·075,	1 056,	1·009,	·953,	906,	875,	·865

Computing from the spherical harmonic formula, I find

$\beta = \frac{1}{7}: \frac{r}{a} =$	1·0382,	1·0305,	1·0096,	·9809,	9522,	·9312,	·9235
$\beta = \frac{1}{5}$:	1·0616,	1·0490,	1·0154,	·9692	·9230,	·8892,	·8768

The greatest discrepancy occurs when $\beta = \frac{1}{5}$ and $\theta = 90°$, and the difference between the two results is $\frac{1}{70}$ of either. It follows, therefore, that in drawing the figures it is not of much importance which results we take But, as above remarked, the radius-vectors computed from the true ellipsoidal figure are the more correct

The formula (77) for δr consists of a series of zonal harmonics. The pole of symmetry of these harmonics lies in the equator of the ellipsoid of revolution defined by r, and is that point of each mass which lies nearest to the other Then, denoting by θ co-latitude estimated from this pole, I find that the numerical values of δr for each 15° of θ are as follows :—

$\theta =$	0°	15°	30°	45°	60°	75°	90°
$\beta = \frac{1}{7}$: $\frac{\delta r}{a} =$	$+ \cdot 165$,	$+ \cdot 141$,	$+ 084$,	$+ \cdot 019$,	$- \cdot 031$,	$- \cdot 055$,	$- \cdot 056$,
$\beta = \frac{1}{5}$ ·	$+ 280$,	$+ 257$,	$+ 142$,	$+ 024$,	$- \cdot 060$,	$- 094$,	$- 092$,

	105°	120°	135°	150°	165°	180°
$\beta = \frac{1}{7}$	$- 037$,	$- \cdot 004$,	$+ \cdot 032$,	$+ \cdot 065$,	$+ \cdot 088$,	$+ \cdot 096$
$\beta = \frac{1}{5}$	$- 059$,	$- 002$,	$+ 055$,	$+ 106$,	$+ 143$,	$+ 155$

These have to be combined with r, so as to give the radius-vectors of the mass of fluid along two sections, one perpendicular to the axis of rotation (which may be called the equatorial section), the other through the axis and the two centres (which may be called the section through the prime meridian). Taking the case of $\beta = \frac{1}{7}$, we add the successive values of δr to the equatorial value of r, viz., 1 043, and thus find the equally-spaced radius-vectors along the equatorial section. Next we add the successive values of δr to the corresponding values of r, and thus find the equally-spaced radius-vectors along the prime meridian. The results are as follows .—

$\theta =$	0°	15°	30°	45°	60°	75°	90°
$\beta = \frac{1}{7}$: Equator, $\frac{r + \delta r}{a} =$	1·208,	1·184,	1·126,	1 062,	1·012,	·988,	·987,
Pr. Merid. $=$	1·208,	1 174,	1·091,	·994,	·915,	·871,	·863,

	105°	120°	135°	150°	165°	180°
Equator	1 006,	1·039,	1 075,	1 108,	1·131,	1 139
Pr. Merid.	·889,	·942,	1·008,	1·072,	1·121,	1·139

These results apply to the case of $\beta = \frac{1}{7}$; those for $\beta = \frac{1}{5}$ are found in the same way, and are given in the figures referred to below.

When $\beta = \frac{1}{7}$ the distance between the centres is given by $c/a = 2{\cdot}828$. I have also worked out the case of $\beta = \frac{1}{6}$, although none of the numerical details are given here.

In figs. 2, 3, 4, and 5 (Plates 22 and 23) are exhibited the figures which result from some of these computations.

Figs. 2 and 3 refer to $\beta = \frac{1}{5}$, 4 and 5 to that of $\beta = \frac{1}{6}$, and the numerical values for $\beta = \frac{1}{7}$, given above, make it easy to draw a figure for $\beta = \frac{1}{7}$.

Since in these cases the masses are equal, the two halves of the figure are the images of one another. The numerical value of each radius-vector is entered on the plates; and other numerical data and explanations are given.

Figs. 2 and 3 correspond to $\beta = \frac{1}{5}$, and here the figures as computed cross one another. The reality must, therefore, be two bulbs joined by a stalk, like a dumb-bell. The dotted lines have been filled in conjecturally, and must show pretty closely what that single figure, formed by the coalescence of the two masses, must be

Figs. 4 and 5 show in a similar manner the case of $\beta = \frac{1}{6}$, and here the two masses are separate, although nearly in contact. When $\beta = \frac{1}{7}$ the shapes present similar characters, but are wider apart.

§ 9. *On the Use of Spherical Harmonic Analysis as a Method of Approximation.*

Spherical harmonic analysis gives less accuracy as the bodies considered depart more and more from spheres. How far, then, do our results present an approach to accuracy? To answer this question, we have to find how nearly the potentials at the surfaces of these figures may be computed from the spherical harmonic formulæ.

It would be laborious to make an accurate computation of the potential, and it fortunately appears to be unnecessary to do so, since a sufficient answer may be obtained in another way.

The potential of an ellipsoid of revolution may be computed either rigorously or by harmonic analysis. With a certain degree of eccentricity the approximate result will agree badly with the rigorous one.

If the ellipsoid consists of a fluid of unit density, there is a certain angular velocity which makes it a level surface. If ω be that angular velocity, then we know that the spherical harmonic solution would give $1 - 15\omega^2/16\pi$ as the ratio of the minor to the major axis. If then c, a, are the rigorous values of the minor and major semi-axes, the harmonic approximation is good if c/a does not differ much from $1 - 15\omega^2/16\pi$.

If we denote by $1 - \mu$ the factor by which the approximate value of the ratio of the axes is to be multiplied in order to obtain the rigorous value, we have

$$\mu = 1 - \frac{c/a}{1 - 15\omega^2/16\pi},$$

and μ may be regarded as a measure of inaccuracy.

A table of the values of $\omega^2/2\pi$, corresponding to various eccentricities $e = \sqrt{(1 - (c/a)^2)}$, is computed from the transcendental equation in THOMSON and TAIT's 'Natural Philosophy,' § 772. From these I compute as follows:—

$e=\sqrt{\left(1-\frac{c^2}{a^2}\right)}$	$\frac{c}{a}$	$1-\frac{15\omega^2}{16\pi}$	Difference	$\frac{1}{\mu}$
1	9949	9949	0000	Large
2	9798	9799	0001	9799
3	9539	9544	·0005	1909
4	·9165	9182	0017	540
5	8660	·8705	0045	193
6	·8000	8111	0111	73
7	7141	·7399	0258	29
8	·6000	6595	0595	11 1
9	4359	5869	1510	3 9

The measures of inaccuracy corresponding to the values of e in the first column, or the values of c/a in the second, are the reciprocals of the numbers in the last column. We thus see that there is still a considerable degree of approximation when $e=·8$, or when the ratio of the axes is 3 to 5, for the measure of inaccuracy is $\frac{1}{11}$; but for $e=·9$ the approximation is bad.

Now the shapes of certain egg-like bodies have been computed by the spherical harmonic method, and it seems safe to assume that the approximation has given about the same degree of accuracy as would hold in the case of an ellipsoid of revolution whose minor axis bears to its major axis the same ratio as the shorter axis of the egg to the longer.

Turning now to our computation, and considering only the more elongated or meridional sections, we see that, when $\beta=\frac{1}{5}$, the longer axis is $1·355+1\,230=2·585$, and the shorter $2(1-·227)=1·546$; and the ratio $1·546:2·585$ is ·6, which corresponds to the measure of inaccuracy $1/11·1$. It might, however, be more legitimate to adopt two different measures, and at the pointed end of the egg to take the ratio $773:1·355=·57$, which will correspond to a measure of inaccuracy about $\frac{1}{10}$; and at the blunt end to take the ratio $·773:1·230=63$, which would correspond to a measure of inaccuracy $\frac{1}{12}$ or $\frac{1}{13}$.

In the case of $\beta=\frac{1}{5}$ the two masses cross one another, and the result has been used to give an approximate picture of the dumb-bell figure of equilibrium. We now see that even in this case there is a sufficient degree of approximation to give a very good idea of the accurate result.

In the case of the meridional section, where $\beta=\frac{1}{6}$, we have for the ratio of axes at the pointed end of the egg $\frac{1-1717}{1\,2712}=\frac{·8283}{1·2712}=·65$, and measure of inaccuracy about $\frac{1}{25}$; and at the blunt end $\frac{1-1717}{1\,1746}=\frac{8283}{1\,1746}=·71$, and measure of inaccuracy $\frac{1}{29}$.

In the case of $\beta=\frac{1}{7}$ the similar figures are, for the pointed end, $\frac{1-137}{1\,208}=\frac{·863}{1\,208}=·72$, and measure of inaccuracy about $\frac{1}{30}$; and for the blunt end $\frac{1-137}{1·139}=\frac{863}{1·139}=·76$, and measure of inaccuracy perhaps about $\frac{1}{50}$.

It thus appears that as the bodies recede the accuracy increases with great rapidity, and in the two cases considered last it is hardly necessary, from a physical point of view, to consider greater accuracy than that attained.

It must be remarked, however, that this way of estimating the degree of inaccuracy must necessarily give much too unfavourable a view.

If we have a single mass of fluid departing considerably from the spherical form, it is clear that the potential computed on the hypothesis of a layer of surface density on the true sphere will come to depart largely from the potential at the surface of the fluid. If, however, we compute the potential of such a mass at points a little remote from the surface, the approximation will be much closer. Now, where there are two masses, as in our problem, the potential at the surface of either mass consists of two parts, one due to the mass itself, the other due to the other mass. As regards the first of these two parts, the above criterion is applicable, but as regards the second part it gives too unfavourable a view.

Now in the case of the single mass the deformative forces due to centrifugal force are considerably vitiated by computation at the spherical surface instead of the true surface, whilst in the case of the two masses the tide-generating forces are computed with greater accuracy than is shown by the criterion

Under these circumstances it has appeared worth while to give another figure below, which, judged by the criterion, would be no approximation at all.

The reasons for giving this figure will be stated when we come to it.

§ 10. *To find the Moment of Momentum of the System.*

Rotating figures of equilibrium are classified according to the amount of moment of momentum with which they are endued. It is, therefore, interesting to determine the moment of momentum of the systems now under consideration.

We must begin by finding the moments of inertia of the two masses Let δI, δi, denote the moments of inertia of the shells of zero mass lying on the mean spheres of radii A, a.

Then

$$\delta i = \iint (y^2 + z^2)(r - a)\, a^2\, d\varpi,$$

where $d\varpi = \sin\theta\, d\theta\, d\phi$, and where the integral is taken throughout angular space.

Now

$$y^2 + z^2 = \tfrac{2}{3}a^2 + \tfrac{1}{3}a^2\left(\frac{w_2}{r^2} - \tfrac{1}{2}\frac{\delta^2 w_4}{r^2}\right),$$

and $r - a$ is the sum of a series of harmonics. Then, in consequence of the properties of harmonic functions, we need only consider the harmonics of the second degree in $r - a$, and

$$\delta i = \tfrac{1}{3}a^5 \iint \left\{\tfrac{1}{3}e\left[\left(\frac{w_2}{r^2}\right)^2 + \left(\tfrac{1}{2}\frac{\delta^2 w_4}{r_2}\right)^2\right] + \tfrac{5}{2}\left(\frac{A}{c}\right)^3\left[(h_2 + l_2)\left(\frac{w_2}{r^2}\right)^2 + 2m_2\left(\tfrac{1}{2}\frac{\delta^2 w_4}{r^2}\right)^2\right]\right\} d\varpi.$$

But

$$\iint \left(\frac{w_2}{r^2}\right)^2 d\varpi = \tfrac{4}{5}\pi, \qquad \iint \left(\tfrac{1}{2}\frac{\delta^2 w_4}{r^2}\right)^2 d\varpi = \tfrac{4}{5}\pi \,.\, 3, \qquad \tfrac{1}{3}\epsilon = \frac{5\omega^2}{16\pi},$$

and the moment of inertia of the mean sphere is $\frac{8}{15}\pi a^5$; hence, if we write

$$f = \frac{5\omega^2}{8\pi} + \tfrac{5}{2}\left(\frac{A}{c}\right)^3 [\tfrac{1}{2}(h_2 + l_2) + 3m_2]$$

$$F = \frac{5\omega^2}{8\pi} + \tfrac{5}{2}\left(\frac{a}{c}\right)^3 [\tfrac{1}{2}(H_2 + L_2) + 3M_2],$$

the moments of inertia, i and I, are given by

$$i = \tfrac{8}{15}\pi a^5 (1 + f),$$

$$I = \tfrac{8}{15}\pi A^5 (1 + F)$$

We already have in (71)

$$\frac{3\omega^2}{4\pi} = \left[\left(\frac{A}{c}\right)^3 + \left(\frac{a}{c}\right)^3\right](1 + K).$$

Hence the sum of the rotational momenta of the two masses is

$$(i + I)\,\omega = \tfrac{2}{5}\left(\frac{4\pi}{3}\right)^{\frac{3}{2}} [a^5 (1 + f) + A^5 (1 + F)]\left[\left(\frac{a}{c}\right)^3 + \left(\frac{A}{c}\right)^3\right]^{\frac{1}{2}} (1 + K)^{\frac{1}{2}}.$$

The whole system revolves orbitally about the centre of inertia with an angular velocity ω hence the orbital momentum is

$$\tfrac{4}{3}\pi\,[a^3\omega\, d^2 + A^3\omega\, D^2].$$

But

$$d = \frac{A^3 c}{a^3 + A^3}, \qquad D = \frac{a^3 c}{a^3 + A^3}.$$

Hence the orbital momentum is

$$\frac{\frac{4}{3}\pi\, A^3 a^3}{A^3 + a^3}\,\omega c^2,$$

and this is equal to

$$(\tfrac{4}{3}\pi)^{\frac{3}{2}} \frac{a^3 A^3 c^{\frac{1}{2}}}{(A^3 + a^3)^{\frac{1}{2}}} (1 + K)^{\frac{1}{2}}.$$

It will be convenient to refer the mass to the radius of a sphere of the same mass as the sum of the two.

Let b be the radius of such a sphere; then

$$b^3 = A^3 + a^3,$$

Thus the whole moment of momentum is

$$(\tfrac{4}{3}\pi)^{\frac{3}{2}}\, b^5 \left\{\tfrac{2}{5}\left[\left(\frac{a}{b}\right)^5 (1 + f) + \left(\frac{A}{b}\right)^5 (1 + F)\right]\left(\frac{b}{c}\right)^{\frac{1}{2}} + \left(\frac{a}{b}\right)^3 \left(\frac{A}{b}\right)^3 \left(\frac{c}{b}\right)^{\frac{1}{2}}\right\} (1 + K)^{\frac{1}{2}}$$

We shall therefore compute the coefficient of $(\frac{4}{3}\pi)^{\frac{3}{2}}\, b^5$.

Computing from this formula, I find the following values of the moment of momentum in the case where the masses are equal, when

$$\beta = \tfrac{1}{7}, \qquad \left(\frac{4\pi}{3}\right)^{\frac{1}{2}} b^5 \times \cdot 468$$
$$\beta = \tfrac{1}{6}, \qquad \times \cdot 472$$
$$\beta = \tfrac{1}{5}, \qquad \times \cdot 482$$

Now I find by a numerical investigation* that, if we imagine a mass of fluid equal to $\frac{4}{3}\pi b^3$ rotating in the form of a Jacobian ellipsoid of three unequal axes, then, when the momentum is $(\frac{4}{3}\pi)^{\frac{1}{2}} b^5 \times \cdot 392$, the axes of the ellipsoid are $1 \cdot 898b$, $0 \cdot 8113b$, $0 \cdot 649b$, and, when the momentum is $(\frac{4}{3}\pi)^{\frac{1}{2}} b^5 \times \cdot 644$, the axes are $3 \cdot 136b$, $0 \cdot 586b$, $0 \cdot 545b$.

It seems probable, then, that the Jacobian ellipsoid of mass $\frac{4}{3}\pi b^3$ becomes unstable, at least as soon as when the moment of momentum is somewhere about $(\frac{4}{3}\pi)^{\frac{1}{2}} b^5 \times \cdot 5$.

It may be worth mentioning that the greatest moment of momentum for which the ellipsoid (of mass $\frac{4}{3}\pi b^3$) is stable, when it is a figure of revolution, is $(\frac{4}{3}\pi)^{\frac{1}{2}} b^5 \times \cdot 3038$.

§ 11. *On the Conditions under which the two Masses may be close to one another.*

If at any point on the surface of either mass the sum of the tide-generating and centrifugal forces is greater than gravity, it is obvious that equilibrium cannot subsist It is also clear that, if this condition is to be found anywhere, it will be at that point of the smaller mass which lies nearest to the larger mass. Hence, in order that the system may be a possible one, we must satisfy ourselves that at that point gravity of the body itself exceeds the sum of the tide-generating and centrifugal forces.

To determine the limitations of size and proximity of the smaller of the two masses to a high degree of approximation would be very laborious, and we shall, therefore, content ourselves with a rough investigation, to be explained below.

We shall now find approximations for the shapes of the two masses and for their potentials

The radius-vector of either mass and the potential may be expanded in powers of a/c and A/c, and a term involving c^n in the denominator will be referred to as being of the n^{th} order.

Now the term of the highest order which can be included without the introduction of great complication is the 7th, and we shall content ourselves with that term.

The expressions for the various parts of the potential have been developed above, but it may be observed that the terms involving the first order of harmonics may be

* 'Roy. Soc Proc.,' vol 41, 1887, p. 319

omitted, since they are subsequently annulled by a proper choice of the angular velocity.

From (22–i.) we have

$$\frac{4\pi a^2}{3}\frac{a}{r}+\frac{4\pi A^3}{3c}\tfrac{3}{2}\left[h_2\left(\frac{a}{c}\right)^2\left(\frac{a}{r}\right)^3\frac{w_2}{r^2}+\tfrac{1}{2}h_3\left(\frac{a}{c}\right)^3\left(\frac{a}{r}\right)^4\frac{w_3}{r^3}+\ldots\right]$$

The last term in the development to the 7th order is that involving w_6. Then it is clear that we require h_2 correct to the 4th order, h_3 to the 3rd, and so on. But (25) shows us that the h's are equal to unity to the 4th order inclusive. Hence, in the above, all the h's may be treated as unity.

Again (22–ii.) when written in reference to the origin o affords other terms, in which all those included under $\Sigma\Sigma$ are of the 8th and higher orders, and negligible; and the rest (with omission of the first harmonic term) gives

$$\frac{4\pi A^3}{3c}\left[\left(\frac{a}{c}\right)^2\frac{w_2}{a^2}+\left(\frac{a}{c}\right)^3\frac{w_3}{a^3}+\ldots\right]$$

Thus this first part of the potential is, to the 7th order inclusive,

$$\frac{4\pi a^2}{3}\cdot\frac{a}{r}+\frac{4\pi A^3}{3c}\sum_{k=2}^{k=6}\left(\frac{a}{c}\right)^k\left\{\frac{3}{2(k-1)}\left(\frac{a}{r}\right)^{k+1}+\left(\frac{r}{a}\right)^k\right\}\frac{w_k}{r^k}.$$

Next, from the expression for Ω in § 3, we have a term in the potential due to rotation $+\frac{1}{3}\omega^2r^2$. The remaining terms due to rotation will be taken up later.

From (71) we see that, to the 7th order inclusive,

$$\frac{3\omega^2}{4\pi}=\left(\frac{A}{c}\right)^3+\left(\frac{a}{c}\right)^3.$$

Hence ω^2 and ϵ are of the 3rd order; and from (48) and (64) it follows that the factors by which the l's and m's are derived from the λ's and μ's are of the 5th order And, since the λ's and μ's only differ from unity in terms of the 5th order, it follows that the l's and m's are of the 5th order. Then (41) and (56) show us that all the terms in l and m are negligible

The first set of terms due to rotation and to the corresponding deformation are given in (39) and (43), and together contribute

$$\tfrac{1}{6}\omega^2a^2\left[\tfrac{3}{2}\left(\frac{a}{r}\right)^3+\left(\frac{r}{a}\right)^2\right]\frac{w_2}{r^2}.$$

The second set of terms due to rotation, and to the corresponding deformation, are given in (54) and (58), and together contribute

$$-\tfrac{1}{6}\omega^2a^2\left[\tfrac{3}{2}\left(\frac{a}{r}\right)^3+\left(\frac{r}{a}\right)^2\right]\tfrac{1}{2}\frac{\delta^2w_1}{r^2}.$$

Hence, to the 7th order inclusive, we have

$$V = \frac{4\pi a^3}{3}\frac{a}{r} + \frac{4\pi A^3}{3c}\sum_{k=2}^{k=6}\left(\frac{a}{c}\right)^k\left[\frac{3}{2(k-1)}\left(\frac{a}{r}\right)^{k+1} + \left(\frac{r}{a}\right)^k\right]\frac{w_k}{r^k}$$
$$+ \tfrac{1}{3}\omega^2 a^2\left(\frac{r}{a}\right)^2 + \tfrac{1}{6}\omega^2 a^2\left[\tfrac{3}{2}\left(\frac{a}{r}\right)^3 + \left(\frac{r}{a}\right)^2\right]\left(\frac{w_2 - \frac{1}{2}\delta^2 w_4}{r^2}\right) \quad . \quad . \quad . \quad . \quad (78)$$

Now the expression (72) for the radius-vector of the mass a to the same order of approximation gives us

$$\frac{r}{a} = 1 + \frac{5\omega^2}{16\pi}\left(\frac{w_2 - \frac{1}{2}\delta^2 w_4}{r^2}\right) + \left(\frac{A}{c}\right)^3\sum_{k=2}^{k=6}\frac{2k+1}{2k-2}\left(\frac{a}{c}\right)^{k-2}\frac{w_k}{r^k},$$

and a similar expression for R/A

To determine the inward force at the pole of the mass a, where it is nearest to the mass A, we must evaluate $-dV/dr$, and in the first term substitute the above expression for r, and in the remaining terms put $r = a$; also at this pole $w_2/r^2 = 1$, and $\delta^2 w_4 = 0$.

Then, differentiating (78),

$$-\frac{dV}{dr} = \frac{4\pi a}{3}\frac{a^2}{r^2} - \frac{4\pi a}{3}\left(\frac{A}{c}\right)^3\sum_{k=2}^{k=6}\left(\frac{a}{c}\right)^{k-2}\left\{-\frac{3(k+1)}{2(k-1)} + k\right\}$$
$$- \tfrac{2}{3}\omega^2 a - \tfrac{1}{6}\omega^2 a^2\left\{-\frac{3\ 3}{2} + 2\right\}. \quad . \quad . \quad . \quad . \quad (79)$$

But at the pole

$$\frac{4\pi a}{3}\frac{a^2}{r^2} = \frac{4\pi a}{3} - \tfrac{5}{6}\omega^2 a - \frac{4\pi a}{3}\left(\frac{A}{c}\right)^3\sum_{k=2}^{k=6}\frac{2k+1}{k-1}\left(\frac{a}{c}\right)^{k-2}.$$

Substituting this for the first term of (79), we have

$$-\frac{dV}{dr} = \frac{4\pi a}{3} - \tfrac{13}{12}\,\omega^2 a - \frac{4\pi a}{3}\left(\frac{A}{c}\right)^3\sum_{k=2}^{k=6}(k+\tfrac{1}{2})\left(\frac{a}{c}\right)^{k-2}.$$

But

$$\omega^2 a = \frac{4\pi a}{3}\left[\left(\frac{A}{c}\right)^3 + \left(\frac{a}{c}\right)^3\right];$$

hence

$$-\frac{dV}{dr} = \frac{4\pi a}{3}\left[1 - \tfrac{13}{12}\left[\left(\frac{A}{c}\right)^3 + \left(\frac{a}{c}\right)^3\right] - \left(\frac{A}{c}\right)^3\left\{\tfrac{5}{2} + \tfrac{7}{2}\frac{a}{c} + \tfrac{9}{2}\left(\frac{a}{c}\right)^2 + \tfrac{11}{2}\left(\frac{a}{c}\right)^3 + \tfrac{13}{2}\left(\frac{a}{c}\right)^4\right\}\right]$$
$$= \frac{4\pi a}{3}\left[1 - \frac{43A^3 + 13a^3}{12c^3} - \left(\frac{A}{c}\right)^3\left\{\tfrac{7}{2}\frac{a}{c} + \tfrac{9}{2}\left(\frac{a}{c}\right)^2 + \tfrac{11}{2}\left(\frac{a}{c}\right)^3 + \tfrac{13}{2}\left(\frac{a}{c}\right)^4\right\}\right].$$

Thus the criterion of the possibility of equilibrium is that

$$C = 1 - \frac{43A^3 + 13a^3}{12c^3} - \left(\frac{A}{c}\right)^3 \left\{ \frac{7}{2}\frac{a}{c} + \frac{9}{2}\left(\frac{a}{c}\right)^2 + \frac{11}{2}\left(\frac{a}{c}\right)^3 + \frac{13}{2}\left(\frac{a}{c}\right)^4 \right\} \quad . \quad . \quad (80)$$

should be positive.

But the radius-vectors of the poles are

$$\frac{r}{a} = 1 + \frac{5}{12}\left[\left(\frac{A}{c}\right)^3 + \left(\frac{a}{c}\right)^3\right] + \left(\frac{A}{c}\right)^3 \left[\frac{5}{2} + \frac{7}{4}\left(\frac{a}{c}\right) + \frac{9}{6}\left(\frac{a}{c}\right)^2 + \frac{11}{8}\left(\frac{a}{c}\right)^3 + \frac{13}{10}\left(\frac{a}{c}\right)^4\right],$$

and, similarly,

$$\frac{R}{A} = 1 + \frac{5}{12}\left[\left(\frac{A}{c}\right)^3 + \left(\frac{a}{c}\right)^3\right] + \left(\frac{a}{c}\right)^3 \left[\frac{5}{2} + \frac{7}{4}\left(\frac{A}{c}\right) + \frac{9}{6}\left(\frac{A}{c}\right)^2 + \frac{11}{8}\left(\frac{A}{c}\right)^3 + \frac{13}{10}\left(\frac{A}{c}\right)^4\right]$$

Therefore

$$r + R = a + A + \frac{5}{12c^3}\left[(7a + A)A^3 + (a + 7A)a^3\right] + \frac{7}{4c^4}A^2a^2(A + a)$$
$$+ \frac{3}{c^5}A^3a^3 + \frac{11}{8c^6}A^3a^3(a + A) + \frac{13}{10c^7}A^3a^3(a^2 + A^2).$$

Now the interval between the two masses is $c - (r + R)$; hence, if the two masses are just in contact,

$$c = a + A + \frac{5}{12c^3}\left[(7a + A)A^3 + (a + 7A)a^3\right] + \frac{7}{4c^4}A^2a^2(A + a)$$
$$+ \frac{3}{c^5}A^3a^3 + \frac{11}{8c^6}A^3a^3(a + A) + \frac{13}{10c^7}A^3a^3(a^2 + A^2). \quad . \quad (81)$$

In order, then, to test whether equilibrium is still possible when the two masses are just in contact, it is necessary to determine c from (81); and then, substituting in (80), find whether C is positive or not.

The solution of an equation

$$c = \alpha + \frac{\beta}{c^3} + \frac{\gamma}{c^4} + \frac{\delta}{c^5} + \frac{\epsilon}{c^6} + \frac{\zeta}{c^7},$$

and the determination of

$$C = 1 - \frac{\mathrm{B}}{c^3} - \frac{\Gamma}{c^4} - \frac{\Delta}{c^5} - \frac{\mathrm{E}}{c^6} - \frac{\mathrm{Z}}{c^7}$$

can only be performed by trial and error.

Now suppose that the solution is $c_0 + \delta c$, where δc is small; and that

$$c_1 = \alpha + \frac{\beta}{c_0^3} + \frac{\delta}{c_0^4} + \ldots, \qquad C_1 = 1 - \frac{\mathrm{B}}{c_0^3} - \frac{\Gamma}{c_0^4} - \ldots$$

Then it is obvious that

$$\frac{\delta c}{c_0} = \frac{c_1 - c_0}{c_0 + \frac{3\beta}{c_0^3} + \frac{4\gamma}{c_0^4} + \ \ .}$$

and

$$\delta C = \left(\frac{3\mathrm{B}}{c_0^3} + \frac{4\Gamma}{c_0^4} + \ldots\ \right)\frac{\delta c}{c_0}, \qquad \text{and} \qquad C = C_1 + \delta C$$

It is not hard to find an approximate solution c_0 by trial and error, and the correct results may then be found thus.

Consider the case where the two bodies are equal to one another, and put $a = A = 1$ The equations then become

$$c = 2 + \frac{20}{3c^3} + \frac{7}{2c^4} + \frac{3}{c^5} + \frac{11}{4c^6} + \frac{13}{5c^7},$$

$$C = 1 - \frac{14}{3c^3} - \frac{7}{2c^4} - \frac{9}{2c^5} - \frac{11}{2c^6} - \frac{13}{2c^7}.$$

By trial and error we find $c = 2{\cdot}535$, $C = +{\cdot}557$.

From this we conclude that equilibrium still subsists when the two masses are in contact.

When $a = A = 1$, $c = 2{\cdot}535$, we have $\gamma = (a/c)^2 = \frac{1}{6 \cdot 43}$, and $\beta = \gamma/(1-\gamma) = \frac{1}{5 \cdot 43}$. Our figures showed that when $\beta = \frac{1}{6}$ the two masses were nearly in contact, and when $\beta = \frac{1}{5}$ they crossed.

This result is, therefore, in accordance with the figures.

Next pass to the case of an infinitesimal satellite, and suppose a infinitely small compared with A and c, and that $A = 1$. The equations are

$$c = 1 + \frac{5}{12c^3},$$

$$C = 1 - \frac{43}{12c^3}.$$

The solution of the first equation is $c = 1{\cdot}226$, and this value of c makes $C = -{\cdot}94$. Hence we conclude that an infinitesimal fluid satellite cannot revolve with its surface in contact with its planet.

C vanishes when $c = (\frac{43}{12})^{\frac{1}{3}} = 1{\cdot}89$. Hence it appears that the nearest approach of the infinitesimal satellite to the planet is 1·89 mean radii of the planet. The nature of the approximation adopted is, however, such that in reality the satellite must lie further from the planet than this, perhaps at two radii distance.* The satellite and planet of which we here speak are, of course, supposed to revolve as parts of a rigid

* [See ROCHE, 'Montpellier, Acad. Sci. Mém.,' vol. 1, 1847–1850, p. 243 (added Oct. 5, 1887)]

body. Now, if for equal masses equilibrium still subsists when the two masses are in contact, whilst for infinitesimal mass of one equilibrium is impossible with the masses in contact, it follows that for some ratio of masses equilibrium can just subsist when they are in contact.

The question, therefore, remains to determine this limiting ratio of masses.

I find, then, that when $a = 1$, $A = 3\cdot4$, we have

$$c = 4\cdot4 + [2\cdot25684]c^{-3} + [1\cdot94945]c^{-4} + [2\cdot07156]c^{-5} + [2\cdot37619]c^{-6} + [2\cdot80737]c^{-7},$$
$$C = 1 - [2\cdot15205]c^{-3} - [2\cdot13850]c^{-4} - [2\cdot24765]c^{-5} - [2\cdot33480]c^{-6} - [2\cdot40735]c^{-7},$$

the numbers in [ ] being the logarithms of the coefficients

The solution of this is $c = 5\cdot57$, which makes $C = -\cdot006$

Again, when $a = 1$, $A = 3\cdot3$, we have

$$c = 4\cdot3 + [2\cdot21556]c^{-3} + [1\cdot91353]c^{-4} + [2\cdot03266]c^{-5} + [2\cdot32731]c^{-6} + [2\cdot74467]c^{-7},$$
$$C = 1 - [2\cdot11347]c^{-3} - [2\cdot09961]c^{-4} - [2\cdot20876]c^{-5} - [2\cdot29591]c^{-6} - [2\cdot36846]c^{-7},$$

the solution of which is $c = 5\cdot45$, which makes $C = +\cdot010$. Since $(3\cdot4)^3 = 39\cdot3$, and $(3\cdot3)^3 = 35\cdot9$, it follows that the ratio of the masses in the first case is $1 : 39\cdot3$, and in the second $1 : 35\cdot9$.

From this it appears that when the ratio of the masses is about 1 to 38 equilibrium is still just possible when the two masses touch.

It must be borne in mind, however, that the nature of the approximations adopted in this investigation is such that the results in this limiting case are only given very roughly, and it is certain that actually the limiting size of the smaller of the two masses must be greater than as thus computed.

We can only conclude that the limiting case occurs when the ratio of the masses is about 1 to 30, or the ratio of the radii about 1 to 3.

There is one other case which it is interesting to consider, namely, to find the limiting proximity of the Moon to the Earth, both bodies being treated as homogeneous fluids of the same density, revolving as a rigid body.

The case of Moon and Earth is well represented by $a = 1$, $A = 4\cdot333$; for this gives 1 to 81·35 as the ratio of the masses With these values I find

$$C = 1 - [2\cdot46626]c^{-3} - [2\cdot45443]c^{-4} - [2\cdot56358]c^{-5} - [2\cdot65073]c^{-6} - [2\cdot72328]c^{-7},$$

and

$$r + R = 5\cdot333 + [2\cdot59898]c^{-3} + [2\cdot24358]c^{-4} + [2\cdot38748]c^{-5} + [2\cdot77563]c^{-6} + [3\cdot32042]c^{-7}.$$

Now $c = 7\cdot0$ will be found to make C vanish, and, with this value of c, $c - (r + R) = \cdot414$

If A be 4000 miles, $c = 6500$ miles, and $c - (r + R) = 380$ miles

Thus, as far as this investigation goes, it appears that when the fluid Moon is on the point of breaking up from stress of tidal and centrifugal forces the distance between the centres of Moon and Earth is 6500 miles, and the shortest distance between the two surfaces is 380 miles.

This result must, however, from the nature of the approximation, be an under-estimate of the distances.

The whole of the present section has been suggested by a pamphlet by Mr. JAMES NOLAN* in which he criticises some of my previous papers. I have commented elsewhere on his criticisms.†

§ 12. *On the Case where the two Masses are unequal.*

The results of the previous section point to a very remarkable limitation to the possibility of approach of two masses of unequal size. It has, therefore, seemed worth while to consider this case numerically, and a case is therefore chosen which shall approach near to that which we know is the limit of possibility. I choose, therefore, $a = 1$, $A = 3$, which makes the ratio of the masses 1 to 27, and $c = 5 \cdot 3$, which brings the protuberances into close proximity.

The numerical details are omitted, but figs. 5 and 6 (Plate 23) give the results, the numerical values of the radius-vectors being, as before, entered on the figure.

The elongation of the smaller mass is so extreme that it is obvious that, rigorously speaking, the spherical harmonic approximation must be considered to break down. Nevertheless, I conceive that these curious figures may be held to indicate the general nature of the true result.

It is remarkable that the smaller mass exhibits a marked furrowing round the middle. This seems to indicate that such a system tends to break up by the separation of the smaller mass into two parts.

§ 13. *Summary.*

The intention of this paper is, first, to investigate the forms which two masses of fluid assume when they revolve in close proximity about one another, without relative motion of their parts; and, secondly, to obtain a representation of the single form of equilibrium which must exist when the two masses approach so near to one another as just to coalesce into a single mass.

When the two masses are far apart the solution of the problem is simply that of the equilibrium theory of the tides. Each mass may, as far as its action on the other is

* 'DARWIN's Theory of the Genesis of the Moon.' ROBERTSON, Melbourne, Sydney, Adelaide, and Brisbane, 1885.

† 'Nature,' February 18 and July 29, 1886.

concerned, be treated as spherical, and the tide-generating potential is given with sufficient accuracy by a single term of the second order of harmonics As the masses are brought nearer to one another, this approximation ceases to be sufficient, terms of higher orders of harmonics become necessary to represent the potential adequately, and the departure from sphericity of each mass begins to exercise a sensible deforming influence on the other

When the departure from sphericity of one body produces a sensible deformation in the other, that deformation in its turn reacts on the first, and thus the actual figure assumed by either mass may be regarded as a deformation due to the primitive influence of the other mass, on which is superposed the sum of an infinite series of reflected deformations.

But each mass is deformed, not only by the tidal action of the other, but also by its own rotation about an axis perpendicular to its orbit. The departure from sphericity of either body due to rotation also exercises an influence on the other, and thus there arises another infinite series of reflected deformations It is shown in this paper how the summations of these two kinds of reflections are to be made by means of the solution of three sets of linear equations for the determination of three sets of coefficients

The first set of coefficients are augmenting factors. by which the tides of each order of harmonics are to be raised above the value which they would have if the perturbing mass were spherical. It appears that, the higher the order of harmonics, the more do these factors exceed unity.

The second set of coefficients correspond to one part of the rotational effects. They appertain to terms of exactly the same form as the tidal terms, and in the final result the terms to which they apply become fused with the tidal terms. These terms are the zonal harmonics of the several orders with respect to the axis joining the centres of the two masses.

The third set of coefficients correspond to the remainder of the rotational effect, and they appertain to a different kind of deformation These deformations are represented by sectorial harmonics involving $\cos 2\phi$, where ϕ is azimuth measured from the plane passing through the axis of rotation of the system and the centres of the two masses. That term of this set which is of the second order of harmonics, and which represents the ellipticity of either mass augmented by mutual influence, is the only term which is considerable, even when the two masses are very close together; but the existence of the other harmonic deformations of this class is interesting. We may say, then, that all the tides of either mass are augmented above the values which they would have if the other mass were spherical; that the ellipticity corresponding to rotation is augmented; and that the deformation due to rotation is no longer exactly elliptic-spheroidal.

The angular velocity of the system is found by the consideration that the repulsion due to centrifugal force between the two masses shall exactly balance the resultant

attraction between them. If the masses were spherical, the square of the orbital angular velocity, multiplied by the cube of the distance between the centres, would be equal to the sum of the masses. When the masses are deformed, however, this law is no longer true, and the angular velocity has to be augmented by a factor a little greater than unity, which depends on the amounts of the deformations.

The theory here sketched is applied above numerically to several cases, and the results will be found in the preceding paragraphs. We shall first consider the cases where the two masses are equal to one another.

In the first example ($\beta = \frac{1}{7}$) solved numerically, the distance between the centres of the two masses is 2 83 times the mean radius of either of them. The two bodies are found to be elongated until they approach near to one another; but, as the character of the distortion is better illustrated in a subsequent case, the result is not given graphically. All the data, however, are found which will enable the reader to draw the figure if he should wish to do so.

In the next example ($\beta = \frac{1}{6}$), with the masses still equal, the distance between the centres is reduced to 2 646 of the mean radius of either. The result of the solution is illustrated by two figures. In fig. 4, Plate 22, the section of the masses by a plane perpendicular to the axis of rotation is shown, and in fig. 5, Plate 23, we have the section by a plane passing through the axis and the centres of the two masses. On both figures are inscribed the values of the radii for each 15° of latitude in terms of the mean radius as unity, and the mean sphere, from which the distortion is computed, is marked by a short line on each radius. The elongation of the masses is, of course, considerably greater in the section through the axis than in the other section. Each mass is shaped somewhat like an egg, and the small ends face one another and come very nearly into contact.

In the headings to the figures, amongst other numerical data, are given the square of the angular velocity and the angular momentum of the system. The density of the fluid being unity, the angular velocity ω is given by the value of $\omega^2/4\pi$; this is the function of angular velocity which is usually given when reference is made to figures of equilibrium of rotating fluid, such as the revolutional or Jacobian ellipsoids of equilibrium. The moment of momentum of the system is given by reference to the angular velocity of a sphere, of the same mass as the sum of our two masses, rotating so as to have the same momentum. If, in fact, b be such a length that a sphere of fluid of that radius has the same mass as our system (so that $b^3 = a^3 + A^3$), then the moment of momentum is given by a number μ in the expression $(\frac{4}{3}\pi)^{\frac{3}{2}} b^5 \times \mu$. By this notation the angular velocity and moment of momentum are made comparable with the results given in a previous paper* on the Jacobian ellipsoid of equilibrium. From that paper the following table of the axes, angular velocity, and moment of momentum of several solutions of JACOBI'S problem is extracted.

* 'Roy. Soc. Proc.', vol. 41, 1887, p 319.

JACOBI'S ELLIPSOIDS.

	Axes			Ang vel	Mom
	Greatest $\frac{a}{b}$	Mean $\frac{b}{b}$	Least $\frac{c}{b}$	$\frac{\omega^2}{4\pi}$	μ
1	1 1972	1 1972	6977	09356	30375
2	1 279	1 123	696	093	306
3	1 383	1 045	692	0906	3134
4	1 601	924	677	0830	3407
5	1 899	811	649	0705	·3920
6	2 346	702	607	0536	4809
7	3 136	586	545	·0334	644
8	5 04	45	44	013	1 016
9	∞	00	00	000	∞

In figs. 4 and 5, Plates 22, 23, $\omega^2/4\pi$ is 038, and the momentum μ is 472. On comparison with the Table of JACOBI's ellipsoids, we see that this corresponds with a considerably slower rotation than the 6th solution, and nearly the same moment of momentum.

In the next case the two masses are still closer ($\beta = \frac{1}{5}$), the distance between the centres being only 2 449 times either mean radius. The result is illustrated in figs. 2 and 3, the explanation of figs. 4 and 5 serves, *mutatis mutandis*, for these two figures also.

This case is interesting because the masses have approached so near to one another that they partially overlap. Two portions of matter cannot, of course, occupy the same space, and the continuity of figures of equilibrium leads us to believe that the reality must consist of a single mass of fluid. In figs 2 and 3 conjectural dotted lines are drawn to show how it is probable that the overlapping of the two masses is replaced by a neck of fluid joining them. The figures as thus amended serve to give a good representation of the single dumb bell shaped figure of equilibrium

The angular velocity is here given by $\omega^2/4\pi = \cdot 049$, and the moment of momentum by ·482. In the sixth entry of the Table of Solutions of JACOBI's problem we find $\omega^2/4\pi = \cdot 0536$, and the moment of momentum $\mu = \cdot 481$. This ellipsoid has, then, the same moment of momentum, and only about 4 per cent. more angular velocity, than our dumb-bell. It has seemed, therefore, worth while to mark (in chain-dot) on figs 2 and 3 the outline of this Jacobian ellipsoid of the same mass as the dumb-bell. The actual vertex of the ellipsoid just falls outside the limits to which it was possible to extend the figure.

In the paper above referred to it is shown how the energy of the Jacobian ellipsoid is to be computed. If we denote the kinetic energy by $(\frac{4}{3}\pi)^2 \mathrm{b}^5 \times \epsilon$, and the intrinsic energy by $(\frac{4}{3}\pi)^2 \mathrm{b}^5 \times (i - 1)$,* then it appears that in the case of the ellipsoid drawn in these figures $\epsilon = 0964$, $i = \cdot 4808$, and the total energy $E = \epsilon + i = 5772$.

* The intrinsic energy being negative, it is more convenient to tabulate i a positive quantity.

Now in the case of our dumb-bell figure it appears, from calculations referred to in the Appendix, that $\epsilon = \cdot 0925$, $\iota = \cdot 4873$, and $E = \cdot 5798$. Hence in the dumb-bell figure the kinetic energy is less, but the intrinsic energy is so much greater that the total energy is about a half per cent greater. These numbers are, of course, computed from the approximate formulæ, and must not be taken as rigorously correct for the dumb-bell figure of equilibrium.

With reference to a figure of transition from the Jacobian ellipsoid, Sir WILLIAM THOMSON has remarked :—*

"We have a most interesting gap between the unstable Jacobian ellipsoid, when too slender for stability, and the case of smallest moment of momentum consistent with stability in two equal detached portions The consideration of how to fill up this gap with intermediate figures is a most attractive question, towards answering which we at present offer no contribution." †

Figs. 2 and 3 are intended to form such a contribution, but it is certain that the matter is far from being probed to the bottom.

M. POINCARÉ has made an admirable investigation of the forms of equilibrium of a single rotating mass of fluid, and has especially considered the stability of JACOBI'S ellipsoid. ‡ He has shown, by a difficult analytical process, that when the ellipsoid is moderately elongated (he has not arrived at a numerical result) instability sets in by a furrowing of the ellipsoid along a line which lies in a plane perpendicular to the longest axis It is, however, extremely remarkable that the furrow is not symmetrical with respect to the two ends, and thus there appears to be a tendency to form a dumb-bell with unequal bulbs

If M. POINCARÉ'S result shall appear to be not only true, but to contain the whole truth concerning the mode in which instability of the ellipsoid supervenes, then there must be some other transitional form between the unsymmetrically furrowed Jacobian

* THOMSON and TAIT'S 'Natural Philosophy,' 1883, § 778″ (*i*)

† In 778″ (*g*) he remarks that "a deviation from the ellipsoidal figure in the way of thinning it in the middle and thickening it towards the end would, with the same moment of momentum, give less energy." I conceive that the energy referred to throughout this paragraph is kinetic only, and we have seen that the kinetic energy is less for the dumb-bell than for the ellipsoid

[If we write U for a quantity proportional to the excess of kinetic above intrinsic energy, so that $U = \epsilon + (1 - \iota)$, then figures of equilibrium are to be determined by making U stationary for variations of the parameters involved in it This course is actually pursued in the Appendix below, the function (viii.) being, in fact, this U; and the variations of it, being made stationary, afford a controlling solution of the problem of this paper The similar method may easily be applied to the case of JACOBI'S ellipsoids. From this point of view the interesting function to tabulate is $\epsilon + (1 - \iota)$, and we observe that in the case of the Jacobian ellipsoid referred to on the last page it is ·6052, and for the dumb-bell it is ·6156. Is not the energy referred to by Sir W. THOMSON this function U? (*Addition to foot-note, dated October 10, 1887.*)]

‡ 'Acta Mathematica,' 7, 3 and 4, 1885.

ellipsoid and the dumb-bell, except, perhaps, in the case where the two bulbs pass on to two masses of a definite ratio.

M. POINCARÉ's work seemed so important that this paper was kept back for a year, whilst I endeavoured to apply the principles, which he has pointed out, to the discussion of the stability of the two masses. The attempt, which is given in the Appendix, is apparently abortive, on account of the imperfections of spherical harmonic analysis when applied to bodies which depart considerably from the spherical shape.

We must, therefore, leave this complex question in abeyance, and merely point to the Appendix as an example of the method which must almost certainly be pursued if this problem is to yield its answer to analysis

Allusion has just been made to the imperfection of spherical harmonic analysis, and this brings us naturally to face the question whether that analysis may not have been pushed altogether too far in the computation of the figures of equilibrium under discussion. This question is considered in § 9, and a rough criterion of the limits of applicability of this analysis is there found. From this it appears that even in the cases of figs. 2 and 3 the result must present a fair approximation to correctness. The criterion, indeed, appears to be such as necessarily to give too unfavourable a view of the correctness of the result.

The rigorous method of discussing the stability of the system having failed, certain considerations are adduced in § 11 which bear on the conditions under which there is a form of equilibrium consisting of two fluid masses in close promixity. It appears that there cannot be such a form with the two masses just in contact, unless the smaller of the two masses exceeds in mass about one-thirtieth of the larger.

If we take into consideration the fact that the criterion of the applicability of harmonic analysis is too severe, it appears to be worth while to find to what results the analysis leads when two masses, one 27 times as great as the other, are brought close together. The numerical work of the calculation is omitted, since the numbers can only represent the true conclusion very roughly, but the result is illustrated graphically in figs. 6 and 7, Plate 23. These figures can only serve to give a general idea of the truth, but the form into which the smaller mass is thrown is so remarkable as to be worthy of attention. The deep furrow round the smaller mass, lying in a plane parallel to the axis of rotation, cannot be due merely to the imperfection of the solution; and it appears to point to the conclusion that there is a tendency for the smaller body to separate into two, just as we have seen the Jacobian ellipsoid become dumb-bell shaped and separate into two parts.

In this paper, indeed, we have sought to trace the process in the opposite direction, and to observe the coalescence of two masses into one. The investigation is complementary to, but far less perfect than, that of M. POINCARÉ, who describes the series of changes which he has been tracing in the following words:—

"Considérons une masse fluide homogène animée originairement d'un mouvement de rotation; imaginons que cette masse se contracte en se refroidissant lentement,

mais de façon à rester toujours homogène. Supposons que le refroidissement soit assez lent et le frottement intérieur du fluide assez fort pour que le mouvement de rotation reste le même dans les diverses portions du fluide. Dans ces conditions le fluide tendra toujours à prendre une figure d'équilibre séculairement stable. Le moment de la quantité de mouvement restera d'ailleurs constant.

"Au début, la densité étant très faible, la figure de la masse est un ellipsoïde de révolution très peu différent d'une sphère. Le refroidissement aura d'abord pour effet d'augmenter l'aplatissement de l'ellipsoïde, qui restera cependant de révolution. Quand l'aplatissement sera devenu à peu près égal à $\frac{2}{5}$, l'ellipsoïde cessera d'être de révolution et deviendra un ellipsoïde de JACOBI. Le refroidissement continuant, la masse cessera d'être ellipsoïdale; elle deviendra dissymétrique par rapport au plan des yz, et elle affectera la forme représentée dans la figure, p. 347.*

"Comme nous l'avons fait observer à propos de cette figure, l'ellipsoïde semble se creuser légèrement dans sa partie moyenne, mais plus près de l'un des deux sommets du grand axe, la plus grande partie de la matière tend à se rapprocher de la forme sphérique, pendant que la plus petite partie sort de l'ellipsoïde par un des sommets du grand axe, comme si elle cherchait à se détacher de la masse principale.

"Il est difficile d'annoncer avec certitude ce qui arrivera ensuite si le refroidissement continue, mais il est permis de supposer que la masse ira en se creusant de plus en plus, puis en s'étranglant dans la partie moyenne, et finira par se partager en deux corps isolés.

"On pourrait être tenté de chercher dans ces considérations une confirmation ou une réfutation de l'hypothèse de LAPLACE, mais on ne doit pas oublier que les conditions sont ici très différentes, car notre masse est homogène, tandis que la nébuleuse de LAPLACE devait être très fortement condensée vers le centre." †

It was in the hope that the investigation might throw some light on the nebular hypothesis of LAPLACE and KANT that I first undertook the work. It must be admitted, however, that we do not obtain much help from the results. It is justly remarked by M. POINCARÉ that the conditions for the separation of a satellite from a nebula differ from those of his problem in the great concentration of density in the central body. But both his investigation and the considerations adduced here seem to show that, when a portion of the central body becomes detached through increasing angular velocity, the portion should bear a far larger ratio to the remainder than is observed in the satellites of our system as compared with their planets; and it is hardly probable that the heterogeneity of the central body can make so great a difference in the result as would be necessary if we are to make an application of these ideas. It appears then at present necessary to suppose that after the birth of a satellite, if it takes place at all in this way, a series of changes occur which are quite unknown.

* The furrowed ellipsoid of JACOBI.

† POINCARÉ, 'Acta Mathemat.,' 7, 1885, p. 379.

APPENDIX.

On the Energy and Stability of the System.

M POINCARÉ has shown in his admirable memoir, referred to in the Summary, how the dynamical stability of a rotating fluid system in relative equilibrium depends on the energy Certain factors in the expression for the energy, which he calls coefficients of stability, are there proved to afford the required criterion

It will now be shown how in this case these coefficients of stability are determinable, at least as far as spherical harmonic analysis permits The results will also cast an interesting light on the methods by which the equations to the two masses have been obtained.

The task before us is to determine the "exhaustion of potential energy" of the two masses in presence of one another as due to the deformation of each from the spherical figure by yielding to gravitation and to centrifugal force.

The work will be rendered simpler by the introduction of a new notation Let us write, then, as the equations to two shapes, which are not necessarily together a figure of equilibrium —

$$\left.\begin{aligned}\frac{r}{a} &= 1 + \sum_{k=2}^{k=\infty} \frac{2k+1}{2k-2}\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^{k-2}\left\{n_k\frac{w_k}{r^k} - p_k\frac{\delta^2 w_{k+2}}{r^k}\right\}\\ \frac{R}{A} &= 1 + \sum_{k=2}^{k=\infty} \frac{2k+1}{2k-2}\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^{k-2}\left\{N_k\frac{W_k}{R^k} - P_k\frac{\delta^2 W_{k+2}}{R^k}\right\}\end{aligned}\right\} \qquad (\text{i})$$

It will be observed that these equations have the same form as (72), but that the constants introduced are different from the h, l, m, ϵ, which were determined, so that the figures might be figures of equilibrium At present we do not assume that (i) do represent figures of equilibrium

The energy lost may be divided into several parts —

e_1, the energy lost by the mass a yielding from its spherical figure to the gravitation of the mean sphere a

e_2, the exhaustion of mutual energy of that layer of matter on the mass a which constitutes its departure from sphericity

e_3, the loss of energy due to the deformation of the mass a in presence of the mean sphere A.

E_1, E_2, E_3, the similar quantities for the mass A.

$(Ee)_4$, the loss of mutual energy of the two layers in presence of one another.

e_5, the loss of energy due to the deformation of the mass a in the presence of centrifugal force due to rotation ω

E_5, the similar loss for A

1st. e_1 is equal and opposite to the work required to raise each element of the layer on a through half its own height against the gravity due to the mean sphere a This gravity is $\frac{4}{3}\pi a$ Co-latitude and longitude being denoted by θ, ϕ, let $d\varpi = \sin\theta\, d\theta\, d\phi$, an element of solid angle. In effecting the integrations, the properties of spherical harmonic functions are used without comment, viz .—

$$\iint\left(\frac{w_k}{r^k}\right)^2 d\varpi = \frac{4\pi}{2k+1}, \qquad \iint\left(\frac{\delta^2 w_{k+2}}{r^k}\right)^2 d\varpi = \frac{4\pi}{2k+1}\,\frac{k+2!}{2\;k-2!}, \qquad \iint w_k\,\delta^2 w_{k+2}\, d\varpi = 0,$$

$$\iint w_i w_k\, d\varpi = 0, \qquad \iint w_i\,\delta^2 w_{k+2}\, d\varpi = 0.$$

Then, taking only a typical term of the first of (i),

$$e_1 = -\iint \tfrac{4}{3}\pi a^5\cdot\tfrac{1}{2}\cdot\left[\frac{2k+1}{2k-2}\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^{k-2}\left\{n_k\frac{w_k}{r^k} - p_k\frac{\delta^2 w_{k+2}}{r^k}\right\}\right]^2 d\varpi$$

$$= -\left(\frac{4\pi}{3}\right)^2\tfrac{1}{2}a^5\,\frac{2k+1}{(2k-2)^2}\,3\left(\frac{A}{c}\right)^6\left(\frac{a}{c}\right)^{2k-4}\left[n_k^2 + \frac{k+2!}{2\,.\,k-2!}\,p_k^2\right],$$

whence, with all the terms, and remembering that $(a/c)^2 = \gamma$, $(A/c)^2 = \Gamma$,

$$e_1 = -\tfrac{1}{2}\left[\left(\frac{4\pi}{3}\right)^2 \frac{A^3a^3}{c}\right] \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{k=2}^{k=\infty} \frac{2k+1}{2k-2}\, \frac{\gamma^{k-1}}{k-1}\left[n_k^2 + \frac{k+2!}{2\;k-2!}\,p_k^2\right] \qquad \text{(ii.)}$$

The formula for E_1 may be written down by symmetry

2nd e_2, the exhaustion of mutual energy of the layer on itself, is half the potential of the layer at any element, multiplied by the mass of the element, and integrated over the whole sphere.

The potential of the layer is

$$\frac{4\pi A^3}{3c} \sum_{k=0}^{k=\infty} \frac{3}{2k-2}\left(\frac{a}{c}\right)^k \left(\frac{a}{r}\right)^{k+1}\left(n_k\frac{w_k}{r^k} - p_k\frac{\delta^2 w_{k+2}}{r^k}\right)$$

Then, at an element of the layer $r = a$, and taking a typical term only, we have

$$e_2 = \tfrac{1}{2}\;\frac{4\pi A^3}{3c}\;\frac{3(2k+1)}{(2k-2)^2}\left(\frac{a}{c}\right)^{2k-2}\left(\frac{A}{c}\right)^3 a^3 \iint\left[n_k\left(\frac{w_k}{r^k}\right)^2 + p_k\left(\frac{\delta^2 w_{k+2}}{r^k}\right)^2\right] d\varpi,$$

whence

$$e_2 = \tfrac{1}{2}\left[\left(\frac{4\pi}{3}\right)^2 \frac{A^3a^3}{c}\right] \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{k=2}^{k=\infty} \frac{3}{2k-2}\, \frac{\gamma^{k-1}}{k-1}\left[n_k^2 + \frac{k+2!}{2\;k-2!}\,p_k^2\right]. \qquad \text{(iii.)}$$

The formula for E_2 may be written down by symmetry

The addition of e_1 to e_2, and of E_1 to E_2, simplifies these expressions by cutting out the factor immediately following the Σ in either, and replacing it by unity

3rd. e_3 is the loss of energy due to raising the layer on a in presence of the mean sphere A. We multiply the potential of the sphere A by the mass of the element on a, and integrate throughout angular space

The potential of the sphere A, when transferred to the origin o, is

$$\frac{4\pi A^3}{3c}\sum_{k=0}^{k=\infty}\left(\frac{a}{c}\right)^k\left(\frac{r}{a}\right)^k \frac{w_k}{r^k}.$$

Then, at an element of the layer $r = a$ and taking a typical term,

$$e_3 = \frac{4\pi A^3}{3c}\;\frac{2k+1}{2k-2}\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^{2k-2} a^3 \iint n_k\left(\frac{w_k}{r^k}\right)^2 d\varpi,$$

whence

$$e_3 = \left(\frac{4\pi}{3}\right)^2 \frac{A^3a^3}{c}\;\tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{k=2}^{k=\infty} \frac{\gamma^{k-1}}{k-1}\, n_k \qquad \text{(iv.)}$$

The expression for E_3 may be written down by symmetry On collecting results from (ii.), (iii.), and (iv.), we have

$$e_1 + e_2 + e_3 = \left(\frac{4\pi}{3}\right)^2 \frac{A^3a^3}{c}\;\tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{k=2}^{k=\infty} \frac{\gamma^{k-1}}{k-1}\left\{n_k - \tfrac{1}{2}n_k^2 - \frac{k+2!}{4\,.\,k-2!}\,p_k^2\right\}, \qquad \text{(v.)}$$

and a similar expression for $E_1 + E_2 + E_3$.

4th $(Ee)_4$ is the loss of energy of one layer in the presence of the other. We take the potential of the layer on A, multiply it by the mass of an element on a, and integrate

The potential of the layer on A when transferred to the origin o, as in (22–ii.), is

$$\frac{4\pi A^3}{3c}\,\tfrac{3}{2}\left(\frac{a}{c}\right)^3\left\{\sum_{k=2}^{k=\infty}\sum_{i=2}^{i=\infty}\left[\frac{k+i!}{k!\,i!}\,\frac{\Gamma^{i-1}}{i-1}\,N_i\left(\frac{a}{c}\right)^k\left(\frac{r}{a}\right)^k\frac{w_k}{r^k} - \frac{k+i!}{i-2!\;k+2!}\,\frac{\Gamma^{i-1}}{i-1}\,P_i\left(\frac{a}{c}\right)^k\left(\frac{r}{a}\right)^k\frac{\delta^2 w_{k+2}}{r^k}\right]\right\}.$$

Introducing this into the integral, only taking a typical term, and neglecting those terms in the integral which must vanish, we get

$$(Ee)_4 = \frac{4\pi A^3}{3c}\tfrac{3}{2}\left(\frac{a}{c}\right)^3 \frac{2k+1}{2k-2}\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^{2k-2} a^3 \frac{\Gamma^{i-1}}{i-1}\left\{\iint \frac{k+i!}{k!\,i!} N_i n_k \left(\frac{w_k}{r^k}\right)^2 d\varpi + \iint \frac{k+i!}{i-2!\,k+2!} P_i p_k \left(\frac{\delta^2 w_{k+2}}{r^k}\right)^2 d\varpi\right\}.$$

Then, effecting the integrations, and putting in the $\Sigma\Sigma$, we get

$$(Ee)_4 = \left(\frac{4\pi}{3}\right)^2\frac{A^3a^3}{c}\cdot(\tfrac{3}{2})^2\left(\frac{a}{c}\right)^3\left(\frac{A}{c}\right)^3\sum_{k=2}^{k=\infty}\sum_{i=2}^{i=\infty}\frac{\Gamma^{i-1}}{i-1}\frac{\gamma^{k-1}}{k-1}\left\{\frac{k+i!}{k!\,i!}N_i n_k + \frac{k+2!}{2\;k-2!}\frac{k+i!}{i-2!\,k+2!}P_i p_k\right\}. \quad \text{(vi)}$$

This involves the two figures symmetrically.

5th e_5 is the loss of energy in the yielding of the figure a to centrifugal force To find it, we multiply the potential due to rotation by the mass of each element of the layer a, and integrate.

By (35) and (52) we know that the rotation potential is

$$\tfrac{1}{6}\omega^2a^2\left(\frac{r}{a}\right)^2\left\{\frac{w_2}{r^2} - \tfrac{1}{2}\frac{\delta^2 w_4}{r^2}\right\}.$$

As this only involves harmonics of the second order, we may neglect in the layer a all terms except those of the second order. Thus we get

$$\begin{aligned} e_5 &= \tfrac{1}{6}\omega^2a^2\tfrac{3}{2}\left(\frac{A}{c}\right)^3 a^3\iint\left\{n_2\left(\frac{w_2}{r^2}\right)^2 + \tfrac{1}{2}p_2\left(\frac{\delta^2 w_4}{r^2}\right)^2\right\}d\varpi \\ &= 4\pi\;\tfrac{1}{12}\omega^2a^5\left(\frac{A}{c}\right)^3\left\{n_2 + \frac{4!}{2\,.\,2\;\;0!}p_2\right\} \\ &= \left(\frac{4\pi}{3}\right)^2\frac{A^3a^3}{c}\left\{\frac{3\omega^2}{16\pi}\left(\frac{a}{c}\right)^2(n_2+6p_2)\right\}. \quad \text{(vii)}\end{aligned}$$

The expression for E_5 may be written down by symmetry. Collecting results from (v), (vi), and (vii.), we get, for the whole exhaustion of energy,

$$\begin{aligned} E - \left(\frac{4\pi}{3}\right)^2\frac{A^3a^3}{c} &= \sum_{k=2}^{k=\infty}\left[\tfrac{3}{2}\left(\frac{A}{c}\right)^3\frac{\gamma^{k-1}}{k-1}\left\{n_k - \tfrac{1}{2}n_k^2 - \frac{k+2!}{4\;k-2!}p_k^2\right\} + \tfrac{3}{2}\left(\frac{a}{c}\right)^3\frac{\Gamma^{k-1}}{k-1}\left\{N_k - \tfrac{1}{2}N_k^2 - \frac{k+2!}{4\,.\,k-2!}P_k^2\right\}\right] \\ &+ (\tfrac{3}{2})^2\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^3\sum_{k=2}^{k=\infty}\sum_{i=2}^{i=\infty}\frac{\Gamma^{i-1}}{i-1}\frac{\gamma^{k-1}}{k-1}\left\{\frac{k+i!}{k!\,i!}N_i n_k + \frac{k+2!}{2\,.\,k-2!}\frac{k+i!}{i-2!\,k+2!}P_i p_k\right\} \\ &+ \frac{3\omega^2}{16\pi}\left[\left(\frac{a}{c}\right)^2(n_2+6p_2) + \left(\frac{A}{c}\right)^2(N_2+6P_2)\right]. \quad \text{(viii.)}\end{aligned}$$

The expression is found without any assumption that the two masses are bounded by level surfaces, and therefore in equilibrium But the condition for equilibrium is that the differential coefficients of E with respect to any one and all of the parameters n, p, N, P, shall vanish. If we equate to zero dV/dn_k, we get

$$1 - n_k + \tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{i=2}^{i=\infty}\frac{\Gamma^{i-1}}{i-1}\frac{k+i!}{k!\,i!}N_i = 0.$$

If, however, $k = 2$, there is on the left-hand side an additional term

$$\frac{3\omega^2}{16\pi}\left(\frac{a}{c}\right)^2 \div \tfrac{3}{2}\left(\frac{A}{c}\right)^3\frac{\gamma}{1} = \frac{\omega^2}{8\pi}\left(\frac{c}{A}\right)^3 = \tfrac{3}{16}e\left(\frac{c}{A}\right)^3.$$

The equation of dV/dN_i to zero gives a similar equation.

If we equate to zero dV/dp_k, we get

$$-p_k + \tfrac{3}{2}\left(\frac{a}{c}\right)^3\Sigma\frac{\Gamma^{i-1}}{i-1}\frac{k+i!}{i-2!\,k+2!}P_i = 0.$$

If, however, $k = 2$, there is on the left-hand side an additional term

$$\frac{3\omega^2}{16\pi} 6 \left(\frac{a}{c}\right)^2 \div \tfrac{3}{2} \left(\frac{A}{c}\right)^3 \frac{\gamma}{1} \frac{4!}{2!\,0!} = \frac{\omega^2}{8\pi}\left(\frac{c}{A}\right)^3 \frac{6}{4\ 3} = \tfrac{1}{15}\epsilon \left(\frac{c}{A}\right)^3.$$

The equation of dV/dP_2 to zero gives a similar equation.

Now, if we put $h_k + l_k$ for n_k, except when $k = 2$, and then put $h_2 + l_2 + \frac{2}{15}\epsilon\ (c/A)^3 = n_2$, and similarly introduce the H's and L's, and if we put $p_k = m_k$, except when $k = 2$, and then put $p_2 = m_2 + \frac{1}{15}\epsilon\,(c/A)^3$, and similarly introduce the M's, it is easy to see that the equations (i.) to the two surfaces become the same as (72), and the equations of condition between n and N, and between p and P, become exactly those which we found by a different method above in (23), (44), and (59) The only difference is that the equations for h and l are fused together.

This, therefore, forms a valuable confirmation of the correctness of the long analysis employed for the determination of the forms of equilibrium.

The formula (viii.) also enables us to obtain the intrinsic energy of the system, that is to say, the exhaustion of energy of the concentration of the matter from a state of infinite dispersion to its actual shape, with its sign changed.

The last line of (viii.) depends on the yielding of the fluid to centrifugal force, and must be omitted from the exhaustion of energy.

Besides the rest of (viii.), we have in the exhaustion of energy of the system, the exhaustion of the two spheres and their mutual exhaustion.

It is clear, then, that the intrinsic energy is

$$-(\tfrac{4}{3}\pi)^2 \cdot \tfrac{3}{5}(a^5 + A^5) - (\tfrac{4}{3}\pi)^2 \frac{A^3a^3}{c}$$

$$-(\tfrac{4}{3}\pi)^2 \frac{A^3a^3}{c} \sum_{k=2}^{k=\infty} \left[\tfrac{3}{2}\left(\frac{A}{c}\right)^3 \frac{\gamma^{k-1}}{k-1} \left\{ n_k - \tfrac{1}{2}n_k^2 - \frac{k+2!}{4\,.\,k-2!}\,p_k^2 \right\} + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \frac{\Gamma^{k-1}}{k-1} \left\{ N_k - \tfrac{1}{2}N_k^2 - \frac{k+2!}{4\,.\,k-2!} P_k^2 \right\} \right]$$

$$-(\tfrac{4}{3}\pi)^2 \frac{A^3a^3}{c} \sum_{k=2}^{k=\infty} \sum_{i=2}^{i=\infty} (\tfrac{3}{2})^2 \left(\frac{A}{c}\right)^3 \left(\frac{a}{c}\right)^3 \frac{\Gamma^{i-1}}{i-1} \frac{\gamma^{k-1}}{k-1} \left\{ \frac{k+i!}{k!\,i!} N_i n_k + \frac{k+i!}{2\ k-2!\,i-2!}\,P_i p_k \right\}, \quad . \quad . \quad . \quad \text{(ix.)}$$

where the n, N, P, p, have their values determined in accordance with the condition that the surfaces are level surfaces.

In evaluating the intrinsic energy from this formula, it is convenient to refer the energy to that of a sphere of such radius, b, that its mass is equal to the whole mass of the system. Then $b^3 = a^3 + A^3$, and we may take the intrinsic energy as

$$(\tfrac{4}{3}\pi)^2 b^5 (i - 1).$$

Thus i will be a numerical quantity which is positive.

I find from (ix.), with $\beta = \frac{1}{5}$ and $a = A$, that $i = \cdot 4873$.

With regard to the kinetic energy of the system, we have seen in § 10 that the moment of momentum is $(\frac{4}{3}\pi)^{\frac{3}{2}} b^5 \times \mu$, where μ is a numerical quantity, and we find in the course of the determination the function $3\omega^2/4\pi$. Then, since the energy is half the moment of momentum multiplied by the angular velocity, it is clear that the kinetic energy is

$$(\tfrac{4}{3}\pi)^2 b^5 \times \tfrac{1}{2}\mu \left(\frac{3\omega^2}{4\pi}\right).$$

The kinetic energy, as represented by $\epsilon = \frac{1}{2}\mu\,(3\omega^2/4\pi)^{\frac{1}{2}}$, is comparable with the intrinsic energy as represented by i.

In the case of $\beta = \frac{1}{5}$, and $a = A$, I find the kinetic energy $\epsilon = \cdot 0925$.

Thus the total energy $\mathrm{E} = \epsilon + i = \cdot 5798$.*

[* See foot-note added to the Summary above. October 10, 1887.]

If the energy of a system be expressed as the sum of a number of coefficients, each multiplied by the square of a parameter, it has been shown by M POINCARÉ that the stability of the system depends on the signs of these coefficients, which he calls "the coefficients of stability" But, if the expression for the energy involves the products as well as the squares of the parameters, the coefficients of stability are the roots of a determinantal equation involving the second differentials of the energy with respect to the parameters

Let V, a linear quadratic function of x, y, z, &c., be the energy of a system in equilibrium, then the determinantal equation is

$$\begin{vmatrix} \frac{d^2V}{dx^2}-\lambda, & \frac{d^2V}{dx\,dy}, & \frac{d^2V}{dx\,dz}, & \&c \\ \frac{d^2V}{dy\,dx}, & \frac{d^2V}{dy^2}-\lambda, & \frac{d^2V}{dy\,dz}, & \&c. \\ \frac{d^2V}{dz\,dx}, & \frac{d^2V}{dz\,dy}, & \frac{d^2V}{dz^2}-\lambda, & \&c \\ \&c, & \&c, & \&c. & \end{vmatrix} = 0 \quad . \quad . \quad (x)$$

The solution of this equation in λ involves the determination of the several fundamental modes of vibration of the system, and the roots are the coefficients of stability

Now suppose that V involves a constant, then, in causing that constant to vary continuously, we have a series of systems of equilibrium of the same kind, and the coefficients of stability vary continuously at the same time If the system be initially in stable equilibrium, the stability ceases when a coefficient of equilibrium vanishes The system at the moment of instability is in a condition of "bifurcation," that is to say, there is another series of shapes of a different kind, of which this shape is a member In making the constant vary past the critical value, we find this second series of shapes stable, whilst the first is unstable

If the system be in uniform rotation, so that instead of absolute equilibrium there is equilibrium relatively to uniformly rotating axes, the same theorems hold true, provided that only one root of the determinantal equation vanishes at a time.

This last is the case which we are considering, and the constant, which we suppose to vary continuously, is c, the distance between the two centres of the mean spheres of radii a and A.

When the two masses are far apart the equilibrium is stable, but when they are brought closer a time may come when one of the coefficients of stability vanishes.

The condition for the vanishing of a coefficient of stability is determined by the determinant (x) with $\lambda = 0$.

To find the determinant, we have to evaluate the second differentials of E with respect to the parameters n, N, p, P

If we form the determinant corresponding to (x) with $\lambda = 0$, it is obvious that two infinite squares of entries which are diagonally opposite to one another, and which meet at a corner, are to be filled in with differentials involving $dn\,dp$, $dn\,dP$, $dN\,dP$, $dN\,dp$, in the denominators All these entries are zero, and hence the infinite determinant splits into two independent infinite determinants, one only involving the differentials with respect to N, n, and the other only those with respect to P, p The N, n determinant may be called "the tidal determinant," the P, p one "the rotational determinant"; for the origin of the terms in each is obvious

By considering only the tidal determinant, we see how the other may be treated very shortly.

For the sake of brevity, write

$$n_k\,N_i = -\frac{d^2E/dn_k\,dN_i}{(-d^2E/dn_k^2)^{\frac{1}{2}}\,(-d^2E/dN_i^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (xi.)$$

Then stability vanishes, as far as regards the tidal forces, when

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot\,, & 1, & 0, & 0, & n_4N_2, & n_4N_3, & n_4N_4, & \cdot \\ \cdot\,\cdot\,, & 0, & 1, & 0, & n_3N_2, & n_3N_3, & n_3N_4, & \cdot \\ , & 0, & 0, & 1, & n_2N_2, & n_2N_3, & n_2N_4, & \cdot \\ \cdot\,, & N_2n_4, & N_2n_3, & N_2n_2, & 1, & 0, & 0, & \cdot\,\cdot \\ , & N_3n_4, & N_3n_3, & N_3n_2, & 0, & 1, & 0, & \cdot \\ \cdot\,, & N_4n_4, & N_4n_3, & N_4n_2, & 0, & 0, & 1, & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0. \quad . \quad . \qquad \text{(xii)}$$

It will be observed that, by the notation (xi), and the appropriate division of the columns and rows of the tidal determinant, it has been converted into a determinant in which the diagonal consists of ones.

If we drop certain factors common to the whole, we have, by differentiating (viii),

$$\frac{d^2E}{dn_k^2} = -\tfrac{3}{2}\left(\frac{A}{c}\right)^3 \frac{\gamma^{k-1}}{k-1}, \quad \frac{d^2E}{dN_k^2} = -\tfrac{3}{2}\left(\frac{a}{c}\right)^3 \frac{\Gamma^{k-1}}{k-1}, \quad \frac{d^2E}{dn_k\,dn_i} = \frac{d^2E}{dN_k\,dN_i} = 0,$$

$$\frac{d^2E}{dn_k\,dN_i} = (\tfrac{3}{2})^2\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^3 \frac{\gamma^{k-1}}{k-1}\,\frac{\Gamma^{i-1}}{i-1}\,\frac{k+i\,!}{k\,!\,i\,!},$$

$$\frac{d^2E}{dp_k^2} = -\tfrac{3}{2}\left(\frac{A}{c}\right)^3 \frac{k+2\,!}{2\,.\,k-2\,!}\,\frac{\gamma^{k-1}}{k-1}, \quad \frac{d^2E}{dP_k^2} = -\tfrac{3}{2}\left(\frac{a}{c}\right)^3 \frac{k+2\,!}{2\,.\,k-2\,!}\,\frac{\Gamma^{k-1}}{k-1}, \quad \frac{d^2E}{dp_k\,dp_i} = \frac{d^2E}{dP_k\,dP} = 0$$

$$\frac{d^2E}{dp_k\,dP_i} = (\tfrac{3}{2})^2\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^3 \frac{k+2\,!}{2\,.\,k-2\,!}\,\frac{\gamma^{k-1}}{k-1}\cdot\frac{\Gamma^{i-1}}{i-1}\,\frac{k+i\,!}{i-2\,!\,k+2\,!},$$

$$= (\tfrac{3}{2})^2\left(\frac{A}{c}\right)^3\left(\frac{a}{c}\right)^3 \tfrac{1}{2}\,\frac{\gamma^{k-1}}{k-1}\,\frac{\Gamma^{i-1}}{i-1}\,\frac{k+i\,!}{i-2\,!\,k-2\,!}; \quad . \quad . \quad . \quad . \quad . \qquad \text{(xiii)}$$

$$\frac{d^2E}{dn_k\,dp_i} = \frac{d^2E}{dn_k\,dP_i} = \frac{d^2E}{dN_k\,dP_i} = \frac{d^2E}{dN_k\,dp_i} = 0.$$

With these values (xiii.) we easily find, by substitution in (xi), that

$$n_kN_i = -\tfrac{3}{2}\left(\frac{A}{c}\right)^{\frac{3}{2}}\left(\frac{a}{c}\right)^{\frac{3}{2}}\left[\frac{\gamma^{k-1}}{k-1}\,\frac{\Gamma^{i-1}}{i-1}\right]^{\frac{1}{2}}\frac{k+i\,!}{k\,!\,i\,!}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xiv)}$$

[This expression gives the value of each of the entries in the infinite determinant (xii).

Now it is possible, by a certain laborious investigation which I do not here reproduce, to develope this infinite "tidal" determinant in the form of a series, and then to show that, however close the two masses may be to one another, the series arising from the tidal determinant can never vanish. It may also be proved that the other infinite determinant, which results from "rotational" terms, is necessarily greater than the tidal determinant, and *à fortiori* can never vanish.]*

Thus it might be held that stability must subsist in the figures of equilibrium until the two masses come into contact. But, as appears from § 11, it is certain that, if one of the masses be smaller than the other, this cannot be the case. In fact, the investigation must break down on account of the imperfection arising from the use of spherical harmonic analysis.

We have seen that the infinite determinant, which gives the coefficients of stability, splits into two parts when we rely on spherical harmonic analysis; but when instability ensues it must be brought about by the joint action of the tidal and rotational forces. It appears certain then that, if a rigorous analysis were used, this separation would not take place.

Although the present investigation proves thus to be abortive, I have thought it worth while to sketch the process, because it almost certainly indicates the line that must be pursued whenever a more rigorous analysis shall be applied to this difficult problem.

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* This paragraph, replacing the investigation referred to, added July 12, 1887.

XIV. *On Thermal Radiation in Absolute Measure.*

By J. T. Bottomley, *M.A.*

Communicated by Sir W. Thomson, *Knt., LL.D., F.R.S.*

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[Plates 24–26.]

In June, 1884, I had the honour of laying before the Royal Society a communication "On the Permanent Temperature of Conductors through which an Electric Current is passing, and on Surface Emissivity."* In carrying out the experiments described in that communication, it became evident that the method then adopted would lend itself readily to the determination in absolute measure of the loss of heat, under various circumstances, from the surface of electrically conducting wires; from metallic wires, for example, and from carbon filaments, such as those used in incandescent electric lamps. Accordingly, at the conclusion of the paper just referred to, the results were given of some preliminary experiments on radiation from metallic wires in high vacuums; and I desire in the present communication to give an account of a more extended investigation in the same direction.

Although loss of heat by radiation and convection has been studied by various experimenters, few determinations in absolute measure of the loss under definite circumstances have been made; and, with the exception of those of Schleiermacher, to be mentioned immediately, no determinations, so far as I am aware, have been made through any considerable range of temperatures, or with a difference of temperatures between cooling body and surroundings of considerably more than 100° C.

Probably the best known results are those of D. Macfarlane,† who experimented on the cooling of polished and blackened copper globes in air, at ordinary pressure and saturated with moisture. Professor Tait has also published‡ the results of observations by J. P. Nicol on loss of heat from polished and blackened copper at three different pressures, viz., 760 mm. of mercury, 102 mm., and 10 mm. Recently

* 'Roy. Soc. Proc.,' vol. 37 (No. 232), 1884, p. 177.

† 'Roy. Soc. Proc.,' vol. 20, 1872.

‡ 'Edinburgh, Roy. Soc. Proc.,' 1869–70.

18.11.87

A. SCHLEIERMACHER,* using a method precisely similar to one of two which I described in the preliminary paper already referred to, has experimented in high vacuum through wide ranges of temperature. The object of these experiments was to test the supposed law of STEFAN, viz, that radiation of heat is in proportion to the fourth power of the absolute temperature of the radiating surface. SCHLEIERMACHER's results are not stated in absolute measure; but, from the data given in his paper, I have been able to make the calculations, and to compare his results with my own.

Of other recent experimenters it is only necessary to mention KUNDT and WARBURG, who in 1875 published an important research on friction and heat conduction in rarified gases;† and Mr. CROOKES,‡ who has given a very valuable comparative determination of the loss of heat from the same surface, the bulb of a mercury in glass thermometer, at different pressures from full atmospheric pressure down to a pressure of two millionths of an atmosphere.

In order to make this paper complete, it seems advisable that I should here explain fully the method of experimenting, and the apparatus used; even though this has already been partially done in my preliminary paper.

The general principle of my method of experimenting may be thus described. the wire for which it is desired to measure the surface loss of heat is stretched in the air or so-called vacuum, and an electric current is passed through it. The wire becomes heated; but very shortly the temperature becomes steady, a balance having been established between the energy supplied by the current and that lost by emission from the surface and by conduction to the fixed supports at the extremities. The latter part is of but small importance in the fine wires, one metre long, which I have been using, and it is allowed for in accordance with a calculation given by Sir WILLIAM THOMSON in a note appended to my first paper.§

When a permanent temperature has been attained by the wire, the current passing is determined in absolute measure. Simultaneously the resistance of the wire is measured; and from these measurements the energy is calculated which is expended in maintaining the temperature of the wire—that is, as has been said, the energy lost at the surface of the wire. The measurement of the electric resistance of the wire also enables me to calculate the temperature of the wire, by means of the results of a separate determination of the electric resistance at different temperatures of the particular wire in use. Lastly, the dimensions of the wire are measured, so that the loss of heat per square centimetre may be calculated; and the circumstances of the experiment and condition of the surroundings are noted.

Fig. 1, Plate 24, shows the electric connections of the apparatus chiefly used for

* 'WIEDEMANN's Annalen,' vol. 26, 1885, p. 287.

† 'POGGENDORFF's Annalen,' vols. 155 and 156, 1875.

‡ 'Roy. Soc. Proc.,' vol. 21, 1881, p. 239.

§ 'Roy. Soc. Proc.,' vol. 37, 1884, p. 187.

sending a known current through the radiation wire, and simultaneously measuring the current and the electric resistance of the wire. This arrangement is one of two described in my first paper on this subject. It has proved entirely satisfactory; and I have used the other method but little, except in some experiments on incandescent lamps, and in a special experiment described at the end of this paper.*

The arrangement shown in fig 1 is a WHEATSTONE's bridge adapted to suit the purpose in hand. The wire w under experiment is represented in this diagram as stretched between the points c and d. In reality it is surrounded with a tube which is immersed in a water-jacket, as will be described presently. ab is a frame in which there are stretched a large number of somewhat fine copper wires, sometimes 12 or 15, or more These wires are stretched between two very stout copper bars, shown cross-hatched at a and b. The multiple copper wires offer but little resistance to the battery currents, and present so large a surface to the air that the heating due to any current they are called upon to carry is absolutely insensible. At G is shown the THOMSON's current galvanometer (of extremely small resistance, $\frac{1}{500}$ ohm) which measures the current flowing in the branch $abcd$ of the WHEATSTONE's bridge—that is, the current passing though the wire w.

The other branch of the WHEATSTONE's bridge is a long fine platinum wire, specially drawn, and excellent as to uniformity It is so long and fine that the minute proportion of current passing through it is quite insufficient to warm it sensibly. It is stretched backwards and forwards on a board of polished pine; and a sliding contact piece e, guided by a V-groove, runs along one length of the wire, the position of the slider being read off on the scale ss. One electrode from the testing galvanometer T of the bridge is attached to the slider, while the other is attached at c in the other branch of the bridge. The battery consists of six large secondary cells arranged in series; and a THOMSON's rheostat wound with thick platinoid wire is used to control the current.

It will be seen that the arrangement here described gives a WHEATSTONE's bridge in which the only part sensibly heated is the radiation wire w, and that its resistance at any temperature can be measured in terms of the other three parts of the bridge. Every connection in the four branches of the bridge is soldered.

The current galvanometer used during my earlier experiments was one of Sir W. THOMSON's graded current-galvanometers with a magnet of low power to give sufficient sensitiveness. More recently I have substituted for the graded galvanometer one of a different pattern, called by Sir W. THOMSON a 'lamp-counter,' and designed for finding the number of lamps in use on an electric light circuit. To keep

* The alternative method referred to in the text consists in measuring the current passing through the radiation wire, and, simultaneously, the difference of potentials between two points at known distance apart (See figs 5A, 5B, Plate 24.) It possesses one great advantage, viz, the avoidance of the disturbance by conduction of heat from the ends, but it would be difficult or impossible to use this method with a vacuum, except in an envelope of glass

a check on the current galvanometers, the absolute values of their readings were frequently determined by electrolysis; and this was done specially after the chief series of experiments, and before calculation of the several portions of the results tabulated and shown in the curves given with this Paper In such determinations the galvanometer was not removed from its place, nor was any alteration made in the soldered connections, with the exception of opening the main branch of the bridge at the point c to admit the electrolytic cell. By this means local disturbance (if any) of the current galvanometer due to electrodes was taken into account; and this method of ascertaining the absolute value of the indications of the current galvanometer was satisfactory because of the complicated nature of the circuits. Influence from the electrodes was done away with as far as possible by laying side by side, insulated with paraffined paper, the electrodes coming to and going from any particular point; but a certain amount of complication was unavoidable, due regard being given to convenience of working, on account of the great number and variety of the pieces of apparatus used in these experiments, and on account of the necessity of placing the tube which contained the radiation wire, and which was attached to the SPRENGEL pump, at some little distance from the testing parts of the WHEATSTONE's bridge.

The radiation wire, in the present experiments of platinum, was stretched from end to end of a long straight tube, and this tube was kept permanently connected to a five-fall SPRENGEL pump. In my earlier experiments, including in particular one long and very complete series of measurements,* the platinum wire was contained in a glass tube, being sealed into the glass at the ends. I also tried platinum spirals in glass globes similar to the globes of incandescent electric lamps, although these spirals had the disadvantage of having the sides of the turns radiating towards each other. A glass envelope surrounding the radiation wire is, however, extremely unsatisfactory. It is impossible to tell with a material like glass, of low thermal conductivity and nearly opaque to heat radiations of low refrangibility, at what temperature the inner skin may be of a tube half an inch in internal diameter, with walls $\frac{1}{16}$-inch thick, and containing a red-hot wire stretched at its centre from end to end. And it is still more difficult to form an opinion as to the amount of heat that may be returned by reflection and radiation together to the radiation wire. I shall have occasion to refer to this matter somewhat later.

My reason for using glass envelopes in my earlier experiments was simply that I was unable to find a method of stretching a wire in a metal tube with one end at least of the wire insulated, and connecting such a tube in a satisfactory manner with the glass exhausting tube of the SPRENGEL pump. Very moderate experience in the production of high vacuums shows the futility of attempting to work with connections of india-rubber, vulcanite, or the ordinary cements and varnishes.

I have, however, been fortunate enough to find a cement which, when properly

* 'British Association Report,' 1885, and 'Nature,' 1885.

used, does act perfectly in keeping as good a vacuum, I believe, as can be made, and which does not give off at common temperatures any vapour which can be detected by the tests known at present. This invaluable material is "Siegelwachs,"* a small piece of which was given to me by Professor QUINCKE, with the remark that it would "hold a vacuum", and with it I construct my vacuum-tube with a wire stretched in it in the following way. *ab* (figs. 2A and 2B) is a copper tube one metre long. The bottom is closed with a metal plate, which is soldered on: but in the centre of the plate there is a small hole through which is brought out the extremity of the hard-drawn copper spiral shown in the diagram. At the top of the copper tube there is soldered in a short piece of re-entrant tube of a nearly conical shape, *e* is a glass tube which is prepared to fit the mouth of the tube *ab*. To do this, the glass tube, closed for the time at *o*, is heated to softening round about, and is blown into the tube *aa′* as into a mould. It is then quickly withdrawn and carefully annealed. Lastly, two or three stout platinum wires are sealed into the glass piece at *d*, and these, after the glass splint at *o* has been cut off, are brought out to a point to which one end of the radiation wire is silver-soldered. The other extremity of the radiation wire is attached to the spiral spring already inside the copper tube, but which for the purpose of silver-soldering is brought to the top *a* of that tube. When the soldering has been effected the tail-piece of the spiral is pulled down and out through the hole at *b*, and when all other arrangements are completed is soldered in its place. Previously to this, however, the glass piece *e* is put into its place with an extremely minute quantity of "siegelwachs", and then, with the help of a pointed glass rod, heated in a spirit lamp, the cement is put in drops round the junction of copper and glass, and carefully worked in so as to fill up every re-entrant corner, and make a coating everywhere convex outward except just where it joins the vertical part of the glass tube. Here the wax is thinned away very gradually, the greatest care being taken to make sure that all round the edges the cement is adhering perfectly to the glass. With a joint made as described I have maintained a high vacuum for weeks together; and, in fact, I have no reason for thinking that such a joint is not quite as good at atmospheric temperature as the usual joint made by fusion of glass on to the exhausting tube of the SPRENGEL pump.

It is unnecessary for me to enter into details as to the production of high vacuums by means of the SPRENGEL pump. In this matter the splendid researches of Mr. CROOKES have, in every detail, pointed the way for obtaining the best results; and the five-fall SPRENGEL pump of Mr. GIMINGHAM† is admirably convenient for exhausting vessels of moderate capacity. In measuring the vacuums produced, I use

* "Siegelwachs" is not similar to English sealing-wax ("Siegellac") It is a soft wax easy to manipulate and not liable to crack, and it shows no tendency to contract and draw away from the glass or metal to which it has been made to adhere I obtain it from the firm of J GAUTSCH, Königliche Hof-Wachswaaren-Fabrik, Munich, whose name was given me by Professor QUINCKE

† 'Roy. Soc. Proc,' vol. 25, 1877.

the McLeod gauge as modified by Mr. C. H. Gimingham,* and shown in figs. 3A and 3B. The modification consists in supplying at the top of the volume-tube, as commonly constructed, a short piece (3 centims.) of fine thermometer tubing carefully calibrated. For the lower exhaustions the air is compressed into the wider portion of the volume-tube; but for the higher exhaustions it is compressed into the divisions of the thermometer tube. To avoid errors from capillarity, there are two pressure tubes. that shown on the left in the diagram being a portion of the same tube as the wide part of the volume-tube, while the tube on the right in the diagram is made of a piece of the same thermometer-tube as is used for the finer part of the volume-tube. The mercury rises, of course, simultaneously in both pressure-tubes. It is usual, I think, to work the McLeod gauge by means of mercury from the main reservoir; but it will be found far more convenient to have an independent mercury reservoir as shown in the diagram, and, indeed, as it was originally described by the inventor.

In my experiments up to the present time I have not introduced the iodine-sulphur-silver tube proposed and used by Crookes† as a stopper against the mercury vapour of the pump; and, as the McLeod gauge measures only the pressure of that which is not collapsible in the vacuum space, its results are rendered imperfect for my purposes by the existence of mercury vapour. I must confess, however, that I cannot feel confidence in the power of finely divided silver or copper for stopping the vapour of sulphur out of the vacuum space, and unfortunately the testing spectroscopically, or otherwise, for vapour of sulphur is not so easy as for vapour of mercury. In experiments on radiation of heat the presence of any vapour whatever vitiates the results just to the extent to which it is present. In this connection I must also call attention to the difficulty of avoiding vapour from the material used in the drying tubes of the pump. It is now usual to employ drying tubes of phosphorus pentoxide; but it is extremely difficult to prepare this material perfectly free from small unburnt particles of phosphorus, and these will undoubtedly give rise to vapour of phosphorus. It may, I think, be doubted whether, after all, sulphuric acid is not in this respect a better drying material than phosphoric anhydride. I propose, however, shortly to make a series of experiments in which all vapours shall as far as possible be removed from the vacuum space, and afterwards stopped out of it by the interposition of a freezing chamber.

Another reason, however, prevented my using Mr. Crookes' mercury stopper, and, as it has also thrown great difficulty in the way of my measurements of the vacuum, I desire to mention it here. In determining the radiation, as will be seen presently, it is necessary to record *simultaneously* the energy lost and the pressure. But, unfortunately, I have been unable to find a platinum wire which does not incessantly give off small quantities of gas at high temperatures. This would be of no consequence at moderate pressures ($\frac{1}{100}$ mm., for example); but in extreme vacuums the effect is very

* 'Journal of the Society of Chemical Industry,' 1884.

† 'Phil. Trans.,' 1885, p. 693.

great indeed. Now it is, perhaps, scarcely realised that the time taken for a high vacuum ($\frac{1}{10}$ M, one-tenth of one-millionth of an atmosphere, for example) to equalise itself throughout the narrow glass-tubes used for connections with the SPRENGEL pump is very considerable, and may amount to 20 minutes, half-an-hour, an hour, or two, according to circumstances. It follows from this that the indications of the McLEOD gauge are certain to be behind time in the case of a high, but variable, vacuum; and the introduction of any considerable length of tubing that can possibly be avoided is very disadvantageous The difficulty here referred to has proved a more fruitful source of trouble, and seems to me less likely to be successfully dealt with, than any other connected with these experiments; though it is just possible that the use of platinum-iridium wire or platinum-silver may prove of considerable advantage

It has been already explained that the temperature at which the radiation is taking place is determined in these experiments by finding the resistance of the wire at the moment, and by knowing from separate experiments the resistance of the wire at different temperatures. The variation of electric resistance of platinum wire with temperature is very different in different specimens. It seems likely that small impurities in the platinum, in the way of admixture with it of minute quantities of the iridium class of metals, may be the cause of this variation. I, myself, have found that a very minute quantity of tungsten combined with German silver makes a vast difference as to the variation of resistance with temperature of that alloy* The great variableness as to this quality in platinum wires is abundantly attested by many experimenters,† and my own experiments fully confirm those of others I have, therefore, in every case taken a portion of the radiation wire itself, and determined its resistance through as wide a range of temperature as was attainable in the way to be described presently. The results of these experiments are laid down in the form of a curve having temperatures as abscissas and resistances as ordinates; and from this curve, which turns out to be almost a straight line with different inclinations in different specimens, the temperature of the wire in any experiment is taken off, its resistance being known.

The apparatus used in determining the resistance of platinum wires at high temperatures is shown in fig. 4, *a*, *b*, *c*. In my earlier experiments, and at lower temperatures, I employed the vapours of liquids with high boiling points as first proposed by ANDREWS,‡ some of the liquids suggested by Drs. RAMSAY and YOUNG§ proving convenient. The boiling points of liquids, however, would not give temperatures nearly high enough for the purpose in hand, and I was about to construct a

* "On the Electric Resistance of a new Alloy named Platinoid," 'Roy. Soc. Proc.,' vol. 38, 1885, p. 340.

† SIEMENS Bakerian Lecture for 1871, but first published in the 'Transactions of the Society of Telegraph Engineers' for 1874 SCHLEIERMACHER · 'WIEDEMANN, Annalen,' vol. 26, 1885, p. 295 H. L. CALLENDAR 'Roy Soc Proc,' vol. 41, 1887, p 231.

‡ 'Comptes Rendus,' vol 81, 1875, p 279

§ 'Chem. Soc Journ,' 1885.

massive copper-jacket for the purpose, when Sir WILLIAM THOMSON proposed to me to use a cylindrical jacket made up of a large number of thin co-axal copper-tubes, and pointed out to me the great advantages possessed by such a composite jacket in equalising the temperature Accordingly, the heating jacket shown in fig 4, *b*, is made up of eight co-axal copper tubes, each with a copper bottom, fitting closely one inside another, a few turns of the finest asbestos yarn being wrapped spirally round each of the tubes before it is pushed into the tube in which it fits. The internal diameter of the smallest tube is $3\frac{1}{2}$ centims., the outside diameter of the outermost tube is 6 centims., the length over all is 33 centims The sheet copper used for making the tubes is $\frac{3}{4}$ mm in thickness, and the weight of the whole is 4 kilos. I have used electrolytic copper, technically known as "conductivity copper," supposing it probable (though I have not had time to verify the supposition) that the pure copper, which shows such marked superiority to less pure copper as to electric conductivity, may also possess at least some superiority as to thermal conductivity.

The copper-jacket is closed with a stopper of woven asbestos about 5 centims. long. Holes are pierced in the stopper, through which pass two thick copper electrodes and the stem of an air-thermometer, to be described presently. The jacket is heated sometimes by two rows of six Bunsens in each row; the BUNSEN tubes in each row being screwed into holes tapped in a $\frac{1}{2}$-inch iron pipe. I find the most convenient heater, however, to be one of FLETCHER's large "solid flame" burners. This burner gives an enormous flame, enveloping the whole of the copper-jacket (which is placed inclined as shown in fig. 4, *a*), and easily raising it to a good red heat through and through. The electrical measuring apparatus which I use is amply sensitive to a change of two or three degrees of temperature; and, using the jacket and heater as just described, it is easy to maintain the internal cavity at a good red heat without any such variation of temperature for ten minutes or twenty minutes at a time, or longer if need be.

The air-thermometer used is shown separately in fig. 4, *c*. It is made from combustion tubing which remains perfectly hard and unyielding at a moderate red heat. The thermometer is formed from tubing, half-an-inch in internal diameter, by drawing off a portion *ab*, about $2\frac{1}{2}$ inches in length, and then drawing out the extremities so as to form capillary tubes *ad* and *bc*. One of these, *bc*, is for convenience bent up nearly close along the bulb of the thermometer. The other capillary tube is sufficiently long to pass through the asbestos stopper of the heating jacket, and to project down from it so far as to bring the bulb to the middle of the heating jacket.

The thermometer, as soon as it is drawn off, is attached, with both ends *c* and *d* open, to a water aspirator, and a current of air, purified and dried, is drawn through it. While the current of air is passing, the whole of the glass is raised to a good red heat two or three times by means of a powerful BUNSEN flame. In this way every trace of moisture and of condensed gas is driven off from the walls of the tube; and it appears

probable from a recent investigation by BUNSEN,* and from experiments which I have myself made, that when the moisture has been thoroughly removed there is no considerable condensation of air at the surface of the glass of the kind frequently supposed to disturb the indications of the air-thermometer. It is during the last of the heatings just described that the capillary tube *bc* is turned up as shown in the figure. When the glass has thoroughly cooled, with the stream of pure dry air still flowing through it, a tap leading to the aspirator is closed, and the ends *c* and *d* are drawn out and sealed with the help of a portable blowpipe, care being taken to draw off small portions of the stems in such a way as not to allow any air from the blowpipe flame to enter the thermometer.

I may here, for convenience, describe the subsequent use of the thermometer. The bulb is passed into the coil of platinum wire whose temperature is to be determined, and the point *d* is passed through a hole in the asbestos stopper and secured there by a small brass wedge. A wrapping of thin asbestos paper is then secured with asbestos yarn round the platinum wire and thermometer bulb, and the whole is passed into the copper jacket as shown figs. 4A, 4B. A screen of thick asbestos mill-board prevents the gases from the heating flame from playing round the point of the thermometer. As soon as the heating has commenced the point *d* is opened with a sharp file and when the temperature to be measured has been reached this point is closed up again by means of a portable blowpipe, care being taken, as before, to avoid the introduction into the thermometer of gases from the blowpipe flame. The barometric height at the time of sealing is also noted.

When the jacket and thermometer have cooled down, the latter is removed and weighed as follows. First the weight of the glass itself is taken, the air within being neglected. Next the point *c* of the thermometer is cut off under mercury or water, and the glass and its contents are weighed. Lastly, the point *d* is cut off; the glass is drawn quite full of mercury or water, and again weighed. The barometric height is noted at the time of opening the thermometer, and also the temperature of the mercury or water. The expansion of the glass envelope is approximately known; and, if water be used for filling the thermometer, the barometric height is corrected for the pressure of water-vapour. The temperature to which the thermometer was raised at sealing can be calculated from these data by well-known formulas

When water is used for filling the thermometer the precautions pointed out by BUNSEN and others in connection with gas analysis for avoiding absorption of the air by the water are, of course, attended to; and I find water preferable to mercury for the purpose in hand, owing to the largeness of the error introduced by any slight inaccuracy as to equality of mercury levels inside and outside the vessel at the first opening of the thermometers

Thermometers on the same principle as is described here were, I find, used by

* 'WIEDEMANN, Annalen,' vol 29, 1886, p 161

REGNAULT* in determining the boiling point of mercury. BUNSEN† has also used an air-thermometer which was closed with a stopper when the high temperature was reached. The quantity of air left in the thermometer at the high temperature was determined volumetrically instead of by weighing. The form of thermometer here described, with a capillary tube at each end, will, I think, be found an improvement in point of practical convenience.

Fig. 4, *a*, shows the electric connections used in determining the resistance of the platinum wire at various temperatures. The potential method has been chiefly employed, and has been found very convenient and satisfactory. For this purpose there are connected in series a single gravity DANIELL'S cell, a THOMSON rheostat, the platinum wire to be tested, and a standard coil of platinoid wire about double the resistance of the platinum wire At low temperatures the difference of potentials between the two ends of the standard coil is half, or considerably less than, that between the ends of the platinum wire, and when the temperature has been raised to 500° C. or 600° C. this relation is reversed. The standard coil is silk-covered wire wound in a single layer on a brass can which is kept filled with cold water; and, being of platinoid, its variation of resistance for two or three degrees of temperature is quite unimportant. A potential galvanometer having a resistance of 5000 ohms, and with an interposed resistance of 10,000 ohms or more, was used for determining the differences of potential on the standard coil and on the platinum wire; and the rheostat enabled me to obtain convenient deflections of the potential galvanometer.

In recent experiments on this subject I have also used a WHEATSTONE'S bridge instead of the potential method; the four resistances of the bridge being two standard coils of platinoid, the platinum wire to be tested, and a calibrated rheostat. As the resistance of the platinum wire rises with temperature in one branch, increased resistance is thrown in in the rheostat branch so as to maintain the balance in the bridge.

I come now to give an account of the main results which have been obtained up to the present time.‡

Three modes of experimenting have been used.

1. A constant current was kept flowing through the wire, and the SPRENGEL pump was worked so as gradually to improve the vacuum. The resistance of the wire at different vacuums was determined, and thus the temperature which the given current would maintain in the wire at a measured vacuum was determined.

* 'Paris, Acad. Mém.,' vol. 21, p 230.

† 'WIEDEMANN, Annalen,' vol. 24, 1884

‡ In doing this, I may perhaps be permitted to remark that, owing to the very limited amount of time at my disposal, these results have been obtained in the course of experiments scattered over long intervals of time. They have also been obtained with apparatus which I have incessantly altered wherever I could see a possibility of improving. Moreover, it was impossible to foresee the bearing of any one result in the long series of experiments, and few of them could even be calculated till long after they were obtained and recorded. I have, therefore, reason to be satisfied with their agreement among themselves

This method has been but little used recently. It was convenient at first on account of the battery power at my disposal, and on account of the current galvanometer I was using. It also gave me useful preliminary information.

2. A second method, and that which I have mainly employed, is to take the wire at a definite air-pressure—a vacuum—and to maintain this with the help of the pump if need be * then to apply a measured current, increasing step by step from the lowest to the highest attainable with the battery at my disposal, or at high vacuums safe to use with the wire, and, as before, to find the temperature of the wire. A complete series of currents having been taken at one vacuum, a higher exhaustion was produced and a new series taken, and so on. The curves appended represent this kind of experimenting.

3. Keeping the temperature of the wire constant (that is, its resistance constant), the SPRENGEL pump was gradually worked, and the current required to maintain the given temperature was measured. This method of experimenting has only become practicable to me recently. It requires a specially sensitive and readable current galvanometer. I propose to make considerable use of it in future, and meantime have employed it in the asymptotic experiment described below, p. 446.

The curves, Plate 25, show emission of heat in gramme-water-degree-Centigrade units, per square centimetre of surface of the heated wire at different temperatures. The temperature of the outer case was always within 1° of 15° C. Each curve shows the loss at the air-pressure marked on it. On the axis of abscissas the temperature is given in degrees Centigrade; while on the axis of ordinates, which shows emission, the scale unit is $\frac{1}{100}$ gramme-water-degree unit per square centimetre per second. The pressures range from 50 mm. to about 0·0000675 mm. (about 0·09 M, where M stands for a pressure equal to the one-millionth of the atmospheric pressure). I cannot, however, say whether the whole or any part of the most outlying curve on the right corresponds precisely to this pressure. It is here that the difficulty enters to which I have already referred, viz., the giving off of gas by the wire. When gas is being given off by the wire, however small the quantity, it becomes impossible to interpret the observations of the McLEOD gauge on account of the long time that is required for the pressure to equalise itself through the connecting tubes. For the same reason I have not been able to push the method of experimenting, in which a series of radiations at the same pressure but at different temperatures is determined, higher than about ·09 M.

In order, if possible, to get rid of this exhalation of air from the platinum wire—for such I take it to be—I have made many experiments in keeping the wire at a high temperature and under extreme vacuum for lengthened periods, but all to no purpose. Previous to the series of experiments represented by the most outlying two curves shown in Plate 25, the wire was kept for five days and nights with a current flowing through

* This was necessary at high vacuums, when the wire became very hot, as will be explained further on, and required experience and caution.

it which maintained it at a good red heat; and each morning and evening the traces of gas that had come off were removed by working the pump. In spite of this, however, the supply was by no means exhausted, although an improvement had certainly been made, and in the course of ten minutes' heating of the wire the vacuum would become sensibly deteriorated. I was at first inclined to attribute these effects to some extremely minute leakage, such as one may get from electric perforation of a glass tube or from a *nearly* perfect sealing of a platinum wire or of a joint. The pump, however, and all the connections were absolutely faultless so long as the wire was maintained at lower temperatures, say below dull-red heat, but, on being raised perhaps to 700° C. or so, I might make a determination of the radiation, and, on making a second determination one or two minutes later, might find that a distinct cooling influence had set in. The rapidity with which the change supervenes proves at any rate that the minute quantity of gas is generated in the immediate proximity of the wire, and it is not till a considerable time later that the McLeod gauge begins to be affected.

On the other hand, with regard to pressures higher than about 50 mm., I was unable to obtain satisfactory series of results on account of the incessant disturbance of the temperature of the wire at high temperatures by the air-currents which its own temperature produces. Special experiments were, however, made at not very high temperatures (see p. 445) in order to bridge over to some extent the interval between normal air-pressures and the highest pressure to which these curves refer.

As the results of the experiments just described are shown on the curves, the actual points experimentally found being marked, it would serve no useful purpose for me to quote the long Tables of numbers from which the curves are plotted. It may, however, be of interest that I should put down a single specimen of a series observed and calculated. I therefore insert here an extract from my laboratory note-book of June 22, 1886, and show side by side with it the results obtained by calculation from the observations.

EXTRACT from Note-book, date June 22, 1886. (Pressure to be about $\frac{1}{40}$ mm., actual pressure $\frac{1}{41}$ mm)

Observed						Calculated from observations					
Hour	Current galvanometer.	Slider	Water-jacket.	McLEOD gauge P	McLEOD gauge V	(1) Current (absolute)	(2) Resistance of wire Ohms	(3) r/r_{15}	(4) Temperature of wire	(5) C^2R (absolute).	(6) C^2R/Js Emission.
A M 11	Im *	938 0	° 15·4	9 6	35				°		
	1	889 5	Temperature rising to 16°	.		0174	2 590	1 068	40	7847×10^6	$27\ 99 \times 10^{-4}$
	2	742 5		9 6	35	0348	3·118	1 284	118	3 773 „	134 6 „
	3	567 7				0522	4 105	1 691	282	11 16 „	398 2 „
	4	472 0		9 8	35	0696	4 952	2 041	430	24 52 „	874 6 „
	5	424 2			..	0870	5 521	2 275	530	41·79 „	1490 „
	6	397 9		10	..	1044	5 889	2 428	592	64 19 „	2290 „
	7	368 3		.		1218	6 371	2 629	677	94 3 „	3362 „
	8	344 0		Pressu	re† low	1392	6 826	2 817	752	105 2 „	3752 „
	9	325 0		10	10	1566	7 231	2 980	820	177·2 „	6319 „

(1) Current calculated by knowing galvanometer constant, which was found by electrolysis (p 432)
(2) Resistance of wire from observation of position of slider (See p 431 and fig. 1)
(3) r/r_{15}, ratio of resistance of wire to its resistance at 15° C
(4) Temperature found from (3) (See pp 435, 436)
(5) C^2R, electric energy (absolute).
(6) is (5) divided by J (JOULE's equivalent) and by s, the surface of the wire.

An examination of the emission curves shows very clearly the part taken by the air in carrying away heat from the wire; and, in the process of experimenting, the diminution in the amount of this phenomenon produced by diminution of pressure is very striking. It will be seen from the curves that but little diminution of carrying power is produced by reducing from 50 mm to 5 mm., or even to $2\frac{1}{2}$ mm., and it is very remarkable (p. 445) how small is the effect produced by reducing from 760 mm. down to 50 mm. On reducing to 1 mm. or to $\frac{1}{2}$ mm , however, the effect produced by the smallest alterations of pressure is very striking; so much so that it would be easy to show it even as a popular or lecture-room experiment. Suppose a steady current flowing through the radiation wire, fig 1, which is in a vacuum of $\frac{1}{4}$ mm.; and suppose that a balance has been obtained on the WHEATSTONE bridge, the testing galvanometer giving the zero or *nil* indication. Let the mercury of the SPRENGEL

* Im , unmeasurably small Taken at commencement of each series, partly to ascertain that electric connections are right.

† The pressure was increasing slightly; the pump was worked and pressure diminished too much The gas coming from wire puts this to rights after a little time, or else a minute quantity of air is admitted from one of the air-traps It has been impossible for me to describe all the minutiæ of working in this paper.

pump be started to flow. The dropping of the mercury is scarcely heard to commence before the balance is disturbed, and the index spot of the reflecting galvanometer is seen travelling along the scale in the direction which shows that the temperature of the wire has begun to rise.

The difference also as to speed of heating and cooling at common pressures and in high vacuums is often surprising during experimenting. At common pressures, from 760 mm. to 10 mm., the times which elapse on starting or stopping the current, before the wire assumes a permanent condition (that is, the times for heating and for cooling), are not strikingly different; but at very low pressures, $\frac{1}{1000}$ mm. or so, the heating on starting the current is seemingly instantaneous, while the cooling, when the current is stopped, of the fine platinum wire, weighing only a few grains, is so slow sometimes as to be almost a tedious process.

It has been pointed out by Mr. CROOKES, and it is not difficult to understand, how it is that a change of pressure between 760 mm. and 10 mm. has so small an influence on the carrying power of the air, while a slight change at a pressure of $\frac{1}{4}$ mm. or less has a great influence. In the first case, the number of molecules is great and the length of the free path small; and, although, on diminishing the density, the number of carrying molecules is reduced, yet at the lower pressure these molecules meet with correspondingly less obstruction in their movements, and have correspondingly greater facility for transferring the heat outwards which they receive from the wire. But at low pressures and in a small vessel the free paths of the molecules become comparable with the distance from the hot wire to the cool envelope, and molecules can move to and fro between the wire and the envelope, experiencing but few collisions and comparatively little obstruction from other molecules. In this case diminishing the number of molecules reduces their aggregate carrying power, but does not correspondingly increase the facility with which they can, as it were, carry their charges to the cold walls of the enclosure.

It is a matter of considerable interest to compare these experimental results with the well-known "Law" of STEFAN, in accordance with which the radiation from a given surface for any particular wave-length at different temperatures is supposed to be proportional to the fourth power of the absolute temperature; and the loss of energy, therefore, proportional to the difference between the fourth powers of the absolute temperatures of the cooling body and its surroundings. It was to test the exactness of this law that the experiments of SCHLEIERMACHER,* already referred to, were undertaken. To compare STEFAN's Law with the results of my experiments (though it is to be remarked that the law as stated refers to pure radiation), I have calculated the values of $T^4 - T_0^4$ for a sufficient number of temperatures, taking $T_0 = 273 + 15$. Choosing then an experimental point on one of the curves traced with high vacuum, I laid down the curve $y = \alpha(T^4 - T_0^4)$, making it pass through the point $y = 0$, $t = 15°$ C., and through the chosen experimental point. This curve is

* 'WIEDEMANN, Annalen,' vol. 26, 1885, p. 287.

indicated on the diagram by the dots surrounded by dotted circles. It will be seen that, so far as they have gone, my experiments give no confirmation to this supposed law; and the results of Herr SCHLEIERMACHER lead to a precisely similar conclusion.

Comparing the results of SCHLEIERMACHER with my own, I find a satisfactory agreement between them. His results, except for what is termed, in connection with STEFAN'S Law, "the radiation constant,"* are not stated in absolute measure;† but the particulars given in his paper enable me to calculate the loss, in heat-units lost per square centimetre per second. SCHLEIERMACHER used in his experiments three wires —two of bright polished platinum, and one of platinum coated with black oxide of copper. These are called respectively Pt_1, Pt_2, and Pt_3.

The following Tables, and the corresponding curves, Plate 26, show the results of SCHLEIERMACHER reduced to absolute measure. The column headed t gives the temperature Centigrade of the radiation wire; that headed C^2R/Js is the loss of heat in gramme-water-degree units per square centimetre per second, while the column headed Em is the emissivity, and is found by dividing the absolute loss by the difference of temperatures of radiation wire and surroundings. The temperature of the water surrounding the envelope, which was of glass, was in each case quoted 0° C. Along with these results I give for comparison a similarly calculated set of results from one of my own series

Pt_1, No. 1. (Enclosure at 0° C.)

t	C^2R/Js	Em
130°	28.08×10^{-4}	21.6×10^{-6}
200	60.0 "	30.0 "
337	181.1 "	53.8 "
581	793 "	137 "
826	2600 "	315 "

Pt_2, No. 1. (Enclosure at 0° C.)

t	C^2R/Js	Em
65°	9.45×10^{-4}	14.5×10^{-6}
110	20.53 "	18.7 "
232	74.7 "	32.2 "
383	236 "	61.6 "
740	1468 "	198 "
900	3218 "	358 "

* The radiation constant is $C^2R/Js\,(T^4 - T_0^4)$

† It is surprising to find the paucity of the experimental results as to emission of heat which are stated in absolute measure; so that, although we have many comparisons between surfaces of lampblack, black paper, white paper, oxidised metal, polished metal, &c., there is scarcely a trustworthy result to be

Pt_3 (Enclosure at 0° C.)

t	C^2R/Js	Em
° 16	9.75×10^{-4}	60.9×10^{-6}
38	25.7 ,,	67.6 ,,
94	78.7 ,,	83.7 ,,
228	335.5 ,,	147 ,,
403	1182 ,,	293 ,,
585	3157 ,,	540 ,,

B_5, Wire experimented on by myself. (Enclosure at 15°.)

t	C^2R/Js	Em
° 302	186.7×10^{-4}	65.05×10^{-6}
425	493.5 ,,	120·3 ,,
613	1689 ,,	282 ,,
744	3918 ,,	537 ,,
806	5163 ,,	653 ,,

From these Tables it will be seen that of the two polished platinums, Pt_1 and Pt_2, the latter has a somewhat greater emissivity; while in the case of the blackened platinum, Pt_3, the emissivity is very much greater, being at 585° C., the highest temperature to which the blackened platinum was carried in these experiments, about four times that of the polished platinums. The numbers which I have quoted from one of my own experiments for comparison give an emissivity for polished platinum, with the vacuum of July 1 and 2, 1886, much higher than that found by SCHLEIERMACHER. This may be due partly to difference in the polish of the surfaces, as I find that a very slight difference in this respect produces a very great effect. It may be, also, that the vacuum at which the determinations of SCHLEIERMACHER were made was of a higher order than mine of July 1 and 2, though I do not think this probable.* I think, however, it is most likely that the true explanation of the greater part, at least, of the difference lies in the fact that SCHLEIERMACHER surrounded the wire with a glass envelope. It is impossible to estimate with a non-diathermanous and badly conducting substance like glass how much heat may be returned to the wire from the polished walls of the tube; and it

found, with the exception of those given by MACFARLANE, a few given by PICTET, and one or two by KUNDT and WARBURG, expressed in units of heat or units of energy emitted, under definite circumstances, per square centimetre per second.

* SCHLEIERMACHER used the BESSEL-HAGEN pump. My experience of this pump does not lead me to [illegible] as being capable of producing a very high vacuum. My reasons for this statement were given [illegible] Association, September, 1886, in a paper "On an Improved Mercury Pump."

was to avoid these uncertainties that I was led to abandon the far more manageable (for these experiments) glass envelope, and to take in its place a copper envelope blackened on the inside.

The next series of experiments which I desire to describe was undertaken for the purpose of tracing the rates of emission of heat at one fixed difference of temperatures between cooling body and surroundings, but at different pressures; commencing with ordinary atmospheric pressure, and passing to the highest vacuum I could command. The experiments, of which the results are given in the Tables, pp. 447, 448, were made with the apparatus of fig. 1. The slider was set at a particular point of the scale; and, by adjusting with the rheostat, the current was found which was required to maintain the temperature in the wire corresponding with the fixed position of the slider, while the pump was worked and the vacuum gradually produced. A complete series of readings was obtained from a common atmospheric pressure downwards for the temperature 408° C., and a partial series for the temperature 505° C., commencing with the pressure ·094 mm This kind of experimenting, as I have already remarked, only recently became possible, or at all events practicable. It requires a current galvanometer of great sensitiveness, and easily readable in a continuous way, qualities which have only now been supplied by the invention and perfecting of Sir W. THOMSON's current galvanometers. It is, however, an extremely convenient and satisfactory method.

An inspection of the Table shows the smallness of the falling away of the rate of loss of heat between full atmospheric pressure and pressures of 10 mm., 5 mm., or even 1 mm.; and the very rapid fall at pressures slightly lower than 1 mm. This has already been shown for temperatures below 100° C. by the experiments of KUNDT and WARBURG and of CROOKES, and is in accordance with the theory and experiments of MAXWELL, which show that heat conduction in gases is independent of the pressure so long as the density is such that the average length of free path of the molecules remains small in comparison with the dimensions of the containing vessel. The numbers given in my Tables are, as in all other cases throughout this paper, in absolute measure. The curve (see Plate 26) shows graphically the relation between pressure and energy lost at the temperature 408° C. The line of abscissas represents the pressure, while the ordinates represent the energy lost.

Carrying on the experiment as described above, I was able, as the vacuum gradually improved, to reach an interesting condition of matters in which further rarefaction of the space produced no further diminution of the rate of loss of energy from the wire. This was reached in the following way. The radiation wire was kept from the morning of March 28 till the morning of March 30 at a moderately high vacuum, and with the current incessantly flowing through it. During parts of March 28 and 29 experiments on radiation were being carried on, and during the evenings and nights of the 28th and 29th the current was raised so as to maintain the wire at a temperature very much higher than that at which I proposed to experiment. On

March 30, at 11 A.M., I commenced running the five-fall SPRENGEL pump continuously, making frequent observations on radiation and with the McLEOD gauge. Starting with a pressure of ·0071 mm. (= 10 M), the pressure was reduced in an hour and a half to 0·1 M, as indicated by the McLEOD gauge, though the actual pressure surrounding the wire may have been considerably different from this. As this point was reached the radiation of heat seemed to become steady, so that further working of the pump did not make any diminution in the amount of current required to maintain the fixed temperature of the wire. The pumping was continued for another half-hour, with a vacuum gradually improving, but with no change as to rate of loss of heat from the wire. I am therefore, I think, entitled to conclude that the limiting value of the radiation reached in this asymptotic experiment is the absolute radiation of the platinum wire for the difference of temperatures stated in a space freed from everything that the mercurial pump will remove, and that is not collapsible in the McLEOD gauge. There was no doubt vapour of mercury present, and possibly, seeing that it is so difficult to remove, some minute trace of vapour of water. There seems, however, very little doubt, from the concurrent testimony of recent experimenters, that the numbers commonly set down from REGNAULT's observations as representing the pressure of mercury vapour at ordinary temperatures are very much in excess of the truth.

The numbers which I take, for the present at least, as representing the radiation from my particular polished platinum wire contained in a non-reflecting envelope of copper, at the temperatures 16° C. and 17° C. respectively, are as follow :—

At 408° C. . . . . . . $378·8 \times 10^{-4}$ gramme-water-degree Centigrade units per square centim, per sec.

At 505° C. . . . . $726·1 \times 10^{-4}$,, ,, ,,

Two sets of experiments, made at an interval of a fortnight, gave me precisely the same number for the radiation at the lower temperature. That at the higher temperature I have only determined on one occasion. I hope, with experiments to be shortly undertaken, and with perhaps improved arrangements, to confirm or possibly modify these numbers, and to obtain radiations at other temperatures.

ASYMPTOTIC Experiments, March 22 to March 30, 1887. Slider placed at 500 of scale (see fig. 1). Barometric height, 740 mm. Temperature of wire, 408° C.

Temperature of water-jacket	Pressure	Current	$C^2R/J\vartheta$	Total emission divided by lowest emission observed (378 8 × 10^{-4}), which is taken as unity
° C	mm.	Amperes		
16	740	2 1776	8137 × 10^{-4}	21 48
	560	2 1603	8004 ,,	21 13
	440	2 1560	7971 ,,	21 04
	340	2 1538	7956 ,,	21 00
	240	2 1517	7941 ,,	20 96
	140	2 1431	7875 ,,	20 79
	90	2 1345	7818 ,,	20 64
	64	2 1172	7686 ,,	20 29
	52	2 1129	7658 ,,	20 21
	49	2 1108	7643 ,,	20 18
	42	2 1042	7591 ,,	20 04
	34	2 0999	7563 ,,	19 96
17	24	2 0784	7408 ,,	19 56
	17 2	2 0558	7249 ,,	19 14
	13 2	2 0352	7104 ,,	18 75
	5 7	1 9880	6314 ,,	16 67
	4 0	1 8757	6036 ,,	15 93
	2 5	1 7291	5125 ,,	13 53
	1 7	1 5954	4364 ,,	11 52
	0 88	1 4264	3487 ,,	9 206
	444	1 2505	2683 ,,	7 096
15	141	9357	1502 ,,	3 965
	·094	·8300	1181 ,,	3 118
	·070	·7805	1045 ,,	2 759
	053	·7287	910 5 ,,	2 404
	·034	6511	727 3 ,,	1 920
	012	5606	539 2 ,,	1·423
	0071*	·5174	459 1 ,,	1 212
	0051*	5045	436 4 ,,	1 152
15	00015*	4786	392 7 ,,	1 037
16	·00007*	4700	378 8† ,,	1 0

* Pressures observed, but uncertain, owing to lagging of McLEOD gauge.

† Further diminution of pressure by continued working of SPRENGEL pump made no diminution in the amount of radiation, which remained unaltered during more than an hour of working.

ASYMPTOTIC Experiments, March 25 to March 30, 1887. Slider at 450 of scale.
Temperature of wire, 505° C

Temperature of water-jacket	Pressure	Current	C^2R/J_s.	Total emission divided by lowest emission observed, ($726\ 1 \times 10^{-4}$), taken as unity
°C		Amperes		
17	094	0 9400	1688×10^{-4}	2 324
	053	8106	1255 „	1 728
	034	·7675	1126 „	1 551
	019	7201	990 3 „	1 363
	013	6942	920 4 „	1 267
	011	6899	909 0 „	1 242
	0071	·6597	831 4 „	1 144
	·0046	6468	798 7 „	1 100
	00052	6339	767 4 „	1 056
	00019	6252	746 4 „	1 028
	*	6446	793 6 „	1 093
	*	6338	767 0 „	1 056
	*	·6252	746 4 „	1·028
		·6230	741 3 „	1 021
	†	·6209	736 2 „	1 014
		6187	731·1 „	1 007
		·6166	726 1 „	1·0

It only remains for me to give a brief account of experiments commenced on radiation from various surfaces in high vacuum. On this important subject I have, up to this time, been only able to touch very lightly by experiment; but I hope before long to be able to offer a communication on the subject. Figs. 5A, 5B, Plate 24, show an apparatus which I am using. A platinum wire, *ab*, is held, stretched between two spiral springs, in a glass tube. The outer ends of the spiral springs terminate in loops; and two pieces of glass rod, which are passed into tubes *cc*, *c′c′*, pass through the loops, so that the springs pull on these glass rods. After the rods have been passed into their places, the ends of the tubes *cc*, *c′c′*, are closed up, except one which is used for exhausting. Flexible copper electrodes are soldered to the loops, and are silver-soldered to stout multiple platinum terminals; and by means of these, which are fused with the help of some white enamel into the glass at *d*, *d*, the current is passed through the platinum wire. At *e*, *e*, *e*, platinum wires are brought through the sides of the tube and serve as potential electrodes, and it is to keep the platinum wire *ab* in the middle of the length of the tube, and to avoid pulling unduly on the potential electrodes, that the two spiral springs, one at either end of the tube, are employed.

The two halves of the platinum wire *ab* are differently treated as to surface, for example, I am at present experimenting on a wire, one half of which is brightly

* Pressure not measured, but gradually diminishing.

† Pressure coming down; during 1½ hour radiation gradually fell to lowest figure given, and then remained unaltered.

polished, and the other half coated with lampblack. I have also tried a wire extremely thinly coated, or rather stained in appearance, with platinum black. The same measured current passes through both halves of the wire, or, when I desire it, I can vary the amounts in the two halves by means of the electrodes *e*, *e*, *e*. By means of these electrodes, also, the differences of potentials at the extremities of the two halves are taken; and thus the energy expended in each part can be found, and the temperature ascertained at the same time · and with the glass envelope I am at present using, the condition of the wire and the light given off by it, if any, can be observed.

With this apparatus, and using a platinum wire No. 22 S.W.G., 0·7 millim. in diameter, one half thinly coated with lampblack and the other half bright and polished, I find that, with a current passing through the wire which keeps the polished wire at a strong red heat approaching to whiteness, the blackened portion is scarcely rendered luminous. I find, also, that an alteration in the surface of the wire, which, so far as appearance goes is very slight indeed, may yet give a very marked alteration in the emissive power of the surface. The merest staining of the platinum wire, by washing it over with solution of ammonio-chloride of platinum and then heating with the current, is sufficient to keep the stained half non-luminous when the polished half is heated to redness.

In the 'Proceedings of the Royal Society,' No. 243, 1886, p. 207, Mr. Mortimer Evans, C.E., has given an account of some remarkable observations on the light-giving properties of polished and unpolished carbon filaments in incandescent lamps. With the help of Mr. Evans I have been enabled to repeat some of his experiments; and the result has been a complete verification of his conclusions. In repeating these experiments careful measurement was made of the vacuum used. a point which was left doubtful in the original investigation.

An Edison A-lamp was taken for the purpose of experimenting, being chosen on account of the remarkable dulness of the surface of the filaments used in the Edison lamps. The filament in these lamps has almost the appearance of a lampblacked surface. This lamp was opened and the filament re-mounted in a form more convenient for the purpose, being somewhat shortened in the process. The filament was then placed in a bulb which was exhausted with the Sprengel pump down to a point measured by the McLeod gauge; all the necessary precautions for making a good incandescent lamp being strictly attended to. The lamp was sealed off from the pump, and was tested against a suitable Swan lamp, which was kept at moderate incandescence, as it is not desirable to incandesce Edison filaments very highly. The lamps were regulated, with resistances and a rheostat, in such a way that the brightnesses of the two filaments matched and the energy (current and potential) required in each lamp was determined. The Swan lamp using a definite amount of energy was taken as the standard to which return was to be made in subsequent trials. The candle-power of each lamp was also observed.

The EDISON lamp was now opened, and, the filament having been removed, it was re-"flashed" in such a way as to give it a beautiful polished metallic-looking surface. It was then placed in a fresh globe, and was exhausted down to the same point as before and sealed off from the pump; and the new lamp was re-tested against the SWAN lamp and standard candle. The result of these experiments was as follows:—

INCANDESCENT Lamp at same candle-power in two cases.

	(1.) Original EDISON filament—dull lustreless surface.	(2.) Same filament re-"flashed," to have brilliant surface
Potential (volts)	77 7	58 5
Current (amperes)	0 745	0 680
Watts	57·9	39·8

It thus appears that to maintain the same carbon in these two conditions, at the same candle-power, there is required 30 per cent. less energy in the second case than in the first.

On this important subject, of the influence on radiation of the nature of the surface, I hope at an early date to make a more complete investigation: and I propose, also, as soon as may be, to repeat and extend the observations of MACFARLANE, to which I have already more than once referred. The increase in the rate of loss of heat due to diminution of the radiating surface demands further examination.

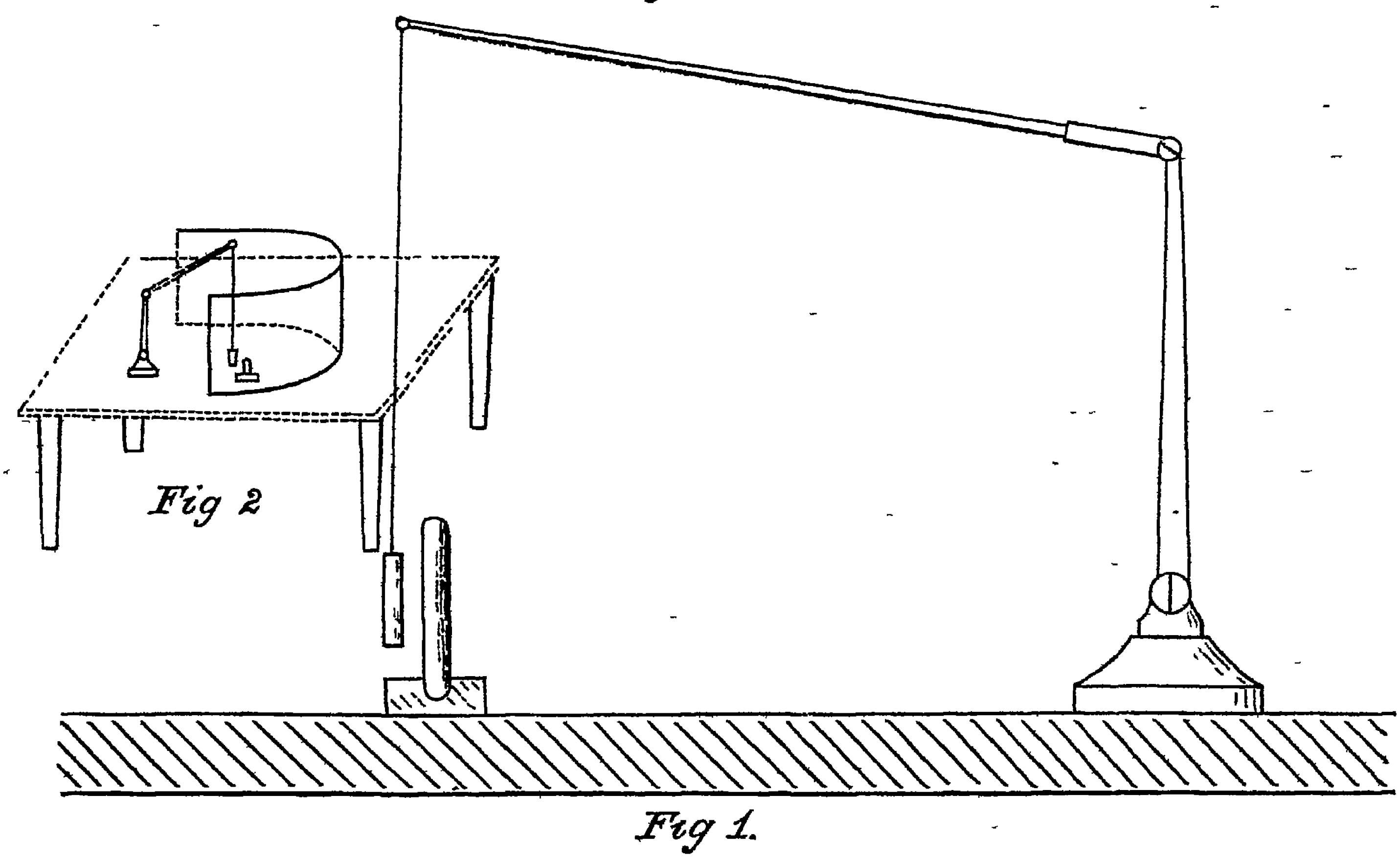
XV. *On the Supposed "New Force" of* M. J. Thore.*

By William Crookes, *F.R.S., Pres. C.S.*

Received May 5,—Read May 26, 1887.

On February 15 last M. J. Thore communicated to a scientific society at Dax a short paper describing some results he had obtained on the rotation of a delicately suspended cylinder of ivory. So remarkable were these results that in a private letter to myself, accompanying a printed copy of his paper, M. Thore said "they seem to demonstrate the existence of a new force inherent in the human organism."

Figs 1 and 2.



Figs. 1 and 2 are accurately copied from M. Thore's diagram. The following description of the apparatus and the experiments tried with it are translated from the printed paper.

"It consists simply of a cylinder of ivory, 24 mm. long and 5 mm. in diameter, suspended by a single fibre of cocoon silk, so that its axis is accurately in line with the suspending fibre. The fibre is fixed to a movable support allowing the cylinder to be

* 'Une Nouvelle Force?' Par J. Thore. Dax, 1887.

raised or lowered without sudden jerks which might rupture the fibre. The apparatus, in a word, is a small pendulum which hangs freely over the centre of a level table in the middle of a room having all the windows closed to avoid draughts.

"When thus arranged, if the cylinder is left to itself, after oscillating and rotating for some time, it becomes almost motionless. If desired, it can be steadied more quickly by lightly touching it with something When steady, if a second ivory cylinder is gently brought about a millimetre from the first cylinder, vertical and parallel to it, as shown in the accompanying figure, the hanging cylinder is observed to acquire a movement of rotation, accelerating, and apparently only limited by the torsion of the fibre.

"This rotation always takes place in the direction of the hands of a watch when the second cylinder is on the left of the first in relation to the observer, who is supposed to face the apparatus; and in the contrary direction when the second cylinder is on the right. This double movement always takes place, whatever be the position of the observer round the table, when he approaches the second cylinder.

"This law is of remarkable constancy, for during the course of my numerous experiments I have never met with a single failure when I carefully took all precautions to avoid interfering influences.

"The nature of the substance of the two cylinders has no influence on the production of movement, and the same may be remarked as to their mass. With liquids or solids, full or empty, the rotation is always the same. The second cylinder may even be replaced by a single stretched hair, or by a single fibre of silk, which is still more fine, without there being any sensible modification.

"The speed of rotation is a function (1) of the length of the two cylinders; (2) of their nearness; (3) of the diameter of the first cylinder. It is in direct proportion to this length; it appears to vary inversely as the diameter of the first cylinder, and to diminish much faster than would be required by the law of inverse squares.

"Flat screens interposed between the experimenter and the apparatus, or placed the other side (at least when 20 centims. from the cylinders), interfere with the movement. When, on the contrary, they are arranged laterally on the right or left, or placed above or below, they are without action. A hemicylindrical screen placed behind the cylinders, as shown in fig. 2, so that the observer is opposite the opening, has the singular action of reversing the direction of rotation.

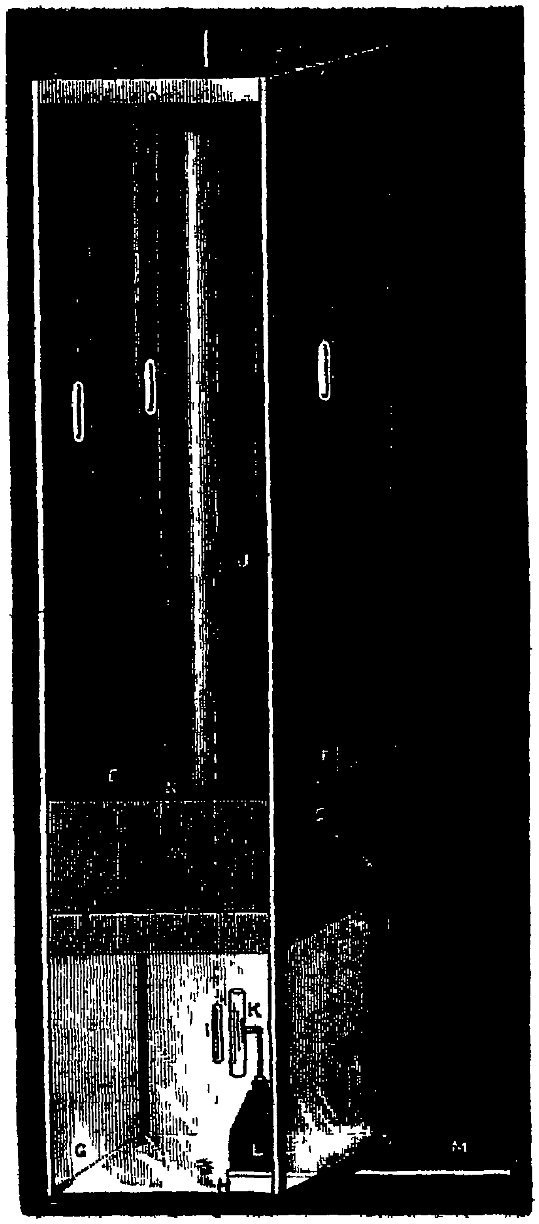
"I have ascertained that light is without action, whatever its nature, its intensity, or its direction. It is the same with heat. Neither can the action be attributed to electricity, for the cylinders, after immersion in water, are as active as before. The poles of a magnet are inert. The action of gravity is balanced by the tension of the suspending fibre. Lastly, air-currents cannot explain the remarkable uniformity of direction in which the rotation always takes place, nor its inversion in the case of the hemicylindrical screen.

"It is seen that the direction of rotation is closely connected with the position of

the observer; this seems to indicate that the origin of this force is in the observer himself. If so, what is its nature?"

Rough preliminary experiments having enabled me to verify the broad facts of rotation as described by M THORE, I fitted up a more accurate apparatus (fig. 3), consisting of a glass case, ABCDEF, six-and-a-half inches square and seven inches high, with a rising glass window, ABGH, in front, and similar windows at the sides. The top is of card, in the centre of which is a small hole. The cylinder, I, is suspended

Fig 3



in the middle of the case by a very fine cocoon silk fibre, 5 feet long, surrounded by a card tube, J, attached to the top of the glass box. K is a second cylinder attached to a support, LM, by a ball-and-socket joint for convenience of adjustment. The support, M, projects outside the case to admit of the second cylinder (which I shall call the *pillar*) being brought close to the suspended cylinder (which, for distinction, I shall call simply the *cylinder*) and transposed from one side to the other, &c., without opening the windows of the box. N is a cord attached to the front glass window, weighted

at the end, and passing over a pulley for convenience of raising and lowering the glass. A similar arrangement is attached to the other glass windows. Cotton-wool is put at the bottom of each glass window and round the support M, to keep out air draughts.

The cylinder is attached to the end of the fibre with a loop and hook, so that it can be rapidly changed for other cylinders. The fixed pillar screws to the arm, so that it also is easily changed. Ivory, ebonite, glass, and metal have been used for the cylinders, and ivory, ebonite, brass, and wood for the pillars. The pillars have also been made square, round, and wedge-shaped in section, and the surfaces have been bright and lampblacked. The apparatus was fitted up in a room free from draughts and quick changes of temperature, and during the course of the experiments no one but myself entered the room.

The cylinder mostly used was of ivory, 5 mm. in diameter and 25 mm. long. The first pillar experimented with was also ivory, 7 mm. diameter and 30 mm. long. The mode of experimentation was the following :—The cylinder being at rest, I sat down in front of the apparatus with my face 8 inches from the cylinder and pillar, taking precaution to keep the breath as much as possible away from cylinder and pillar. The pillar was always placed on the right of the cylinder. On raising the front glass the cylinder commenced to rotate in the opposite direction to the hands of a clock, the side nearest me moving to the right.* It made 4·5 complete revolutions, the maximum speed being one revolution in 12 seconds.

In several succeeding experiments a four-ounce flask of boiling water was used as the source of heat. It was coated externally with lampblack, and was placed exactly one inch from the pillar and cylinder.

In other experiments the ivory pillar was replaced by one of hollow brass, 9 mm. in diameter and 38 mm. long, the surfaces being brightly polished; by a wedge-shaped pillar of boxwood, 38 mm. high, 22 mm. broad, 9 mm. at the thickest end, and tapering off to a blunt edge at the thinnest part; by an ebonite cylinder 9 mm. in diameter and 38 mm. long; and by a brass parallelogram 38 mm. long by 7 mm square.

The results are given most conveniently in a tabular form. The first column of the following Table gives the numerical order of the experiment, the second column shows the material of which the pillar consists. In the third column is given the maximum speed in seconds of one revolution; the fourth column gives the number of revolutions performed by the cylinder before being stopped by torsion of the fibre and the exciting agent, the face or hot-water flask, is shown in the last column. In all cases the rotation was *negative*, namely, in the opposite direction to the hands of a clock. [illegible]

* [illegible] *negative* direction, and when the rotation is clockwise I will call it *positive*.

No of experiment.	Material of pillar	Maximum speed of one revolution.	No of revolutions.	Exciting agent.
		seconds		
1	Polished ivory .	12	4 5	Face
2	Polished ebonite	16	3 50	,,
3	Boxwood wedge, edge to cylinder .	41	3 25	,,
4	,, ,, flat side to cylinder	32	3 00	,,
5	Polished brass tube . . .	39	3 0	,,
6	Polished square brass rod, flat side to cylinder	24	3 5	,,
7	Polished square brass rod, edge to cylinder	21	4 0	,,

Instead of using the radiation from the face as the active agent, I now employed the four-ounce flask, lampblacked outside, full of boiling water. This was put 2 inches from the cylinder and pillar, and the following experiments were tried with it.—

No of experiment	Material of pillar	Maximum speed of one revolution	No of revolutions.	Exciting agent
		seconds		
8	Polished ivory	18	4 25	Hot water
9	Polished brass tube . .		1 0	,,

In these experiments a point which struck me as being remarkable was the greater action which took place when I held my face 8 inches off the cylinder than when the exciting agent was a lampblacked flask full of boiling water. M. Thore says that heat is without action, and that the origin of the force appears to lie in the observer himself. At first sight these results appear to favour this view. It must, however, be remembered that the circumstances are not such as would bring out in a marked manner any action due wholly, or in great part, to heat. White polished ivory, such as M. Thore used for the rotating body, is a very bad absorber of heat rays; and it is quite possible that the aggregate of heat rays absorbable by polished ivory, emitted by a few square inches of lampblacked glass at 100°, 2 inches off, might not be inferior in amount to those emitted by the much larger surface of moist skin 8 inches off. It seemed possible to put this action to a test by blackening the ivory cylinder. If the action was, as M. Thore seemed to think, one inherent in the human organism, and not an effect of heat, the effect of blackening the cylinder should not materially alter the relative effects of the face and the hot-water flask; the action of the boiling water should still be less than that of the face. If, on the other hand, the action was one in which radiant heat played the principal part, the effect of blackening the ivory cylinder would be to upset this ratio, and to give a decided preponderance in favour of the hot-water flask. The ivory cylinder was accordingly blackened by holding it over the

smoke of burning camphor. On re-suspension, the following series of experiments was tried :—

No of experiment.	Material of pillar.	Maximum speed of one revolution.	No. of revolutions.	Exciting agent
		seconds.		
10	Polished ivory	21	4·25	Face
11	,, ,, ...	18	4·25	Hot water
12	Boxwood wedge, edge to cylinder .	11	5·75	Face
13	,, ,, ,, . .	15	5·5	Hot water
14	Polished brass tube . .	20	4·25	Face
15	Polished square brass rod, flat side to cylinder	37	2·75	,,
16	Polished square brass rod, edge to cylinder	16	6·5	,,
17	Polished brass wedge, edge to cylinder	10	7·5	,,
18	,, ,, ,,	7	17·5	Hot water
19	Glass rod, 65 mm. long, 1·5 mm. thick	11	5	Face
20	,, ,, ,,	21	4·25	Hot water
21	Brass wire, 60 mm. long and 0·5 mm. diameter	20	2·25	Face
22	Brass wire, 60 mm. long and 0·5 mm. diameter	24	2·5	Hot water
23	Platinum wire, 65 mm. long and 0·4 mm diameter	61	2·25	,,
24	Platinum wire, 65 mm. long and 0·4 mm. diameter	34	4·5	Face
25	Fine glass fibre, 65 mm long . . .	. .	0·5	,,
26	,, ,, ,, . . .	46	1·5	Hot water
27	Single fibre of cocoon silk, 50 mm. long		0·75	Face
28	,, ,, ,,	29	2·0	Hot water

This series of experiments having shown a decided increase of motion due to blackening the cylinder, another series was tried with the pillars as well as the cylinder coated with the soot from burning camphor.

No of experiment.	Material of pillar.	Maximum speed of one revolution	No of revolutions	Exciting agent.
		seconds		
29	Ivory, lampblacked	18	4 0	Face
30	" "	7	10 25	Hot water
31	Ebonite, lampblacked .	14	4 75	Face
32	" "	7	11 75	Hot water
33	Boxwood wedge, lampblacked, edge to cylinder	19	3 5	Face
34	Boxwood wedge, lampblacked, flat side to cylinder	28	2 75	"
35	Boxwood wedge, lampblacked, flat side to cylinder	4	16 75	Hot water
36	Brass tube, lampblacked . .	10	7 25	Face
37	" "	10	10·5	Hot water
38	" "	28	2 5	Candle 8 inches from cylinder
39	Square brass bar, lampblacked, edge to cylinder	16	6 5	Face
40	Square brass bar, lampblacked, edge to cylinder	8	8 0	Hot water
41	Square brass bar, lampblacked, flat side to cylinder	18	3 75	Face
42	Square brass bar, lampblacked, flat side to cylinder	8	8 25	Hot water
43	Brass wedge, edge to cylinder, lampblacked	8	11 5	"
44	Glass rod, 65 mm long, 1 5 mm thick, lampblacked	17	3	Face
45	Glass rod, 65 mm long, 1 5 mm. thick, lampblacked	16	4 5	Hot water

These results leave little doubt that the action exerted by the face was due to the radiant heat emitted by it M. THORE, however, considers that heat is without action. This being a most important point, and one which must be settled beyond all doubt, I devised the following experiments :—A large sheet of thick cardboard had an oval aperture 5 inches by 3 5 inches cut in the centre. This was fixed in front of the suspended cylinder, about 8 inches off. A Winchester quart-bottle covered with slightly damped brown paper was arranged on a stand close behind the aperture, and easily removable. By this arrangement I could compare the effect of a given surface of the face with that of the same surface of radiation from moist brown paper heated to the temperature of the face, which for this purpose was taken as 33° C. The cylinder in each case was of blackened ivory. The following Table gives the mean of a number of experiments :—

No of experiment	Material of pillar	Maximum speed of one revolution	No of revolutions.	Exciting agent.
		seconds.		
46	Blackened brass tube . .	17	4 0	Face
47	" " . .	15	3 5	Bottle

Considering that it would be almost impossible to get absolute equality between the radiating power of the face and any other substance which could be used in the comparison, I think these results are quite near enough, especially when taken with those already tried, to prove that there is nothing special in the human organism, beyond the heat it radiates, to produce rotation of the cylinder.

I attempted to verify the experiment of M. THORE's in which he got reversed rotation by putting a hemicylinder behind the rotating cylinder. A half-cylinder of glass, $4\frac{1}{4}$ inches across and $4\frac{3}{4}$ inches high, was put behind so that the suspended ivory cylinder was in the centre of the curve. The following experiments were then tried, the cylinder, as in the other cases, being of lampblacked ivory:—

No of experiment.	Material of pillar.	Maximum speed of one revolution.	No. of revolutions.	Exciting agent.
		seconds.		
48	Polished ivory	18	3·75	Face. With glass cylinder
49	"	12	4·5	Face. Without glass cylinder
50	"	..	1·25	Hot water. With glass cylinder
51	"	18	4·25	Hot water. Without glass cylinder
52	Polished brass tube	..	0·5	Face. With glass cylinder
53	" "	19	4·5	Face. Without glass cylinder

These results show that the effect of putting a half-cylinder as a screen behind the suspended cylinder does not produce exactly the effect described by M. THORE. It does, however, cause a marked diminution of action, and, had the cylinder and pillar been freely exposed to the air as in M. THORE's experiment, reversal of movement might have taken place. I found it difficult to try accurate experiments in the free air of a room, owing to the interference of air-currents. The deadening of motion in my experiments, and its reversal in the experiments of M. THORE, I attribute to the reflection of heat rays from the concave surface of the cylinder, and their concentration to a focus on the further side of the suspended cylinder.

The following experiments were made with the object of ascertaining what would really be the effect of an upward current of air on the suspended cylinder. A glass jet about 1 mm. in diameter was fixed vertically in front of the cylinder. To the jet was attached a long india-rubber tube, connected at the other end with a system of water-bottles in such a way that, by raising one of them, a gentle stream of air rose from the jet in front of the cylinder. The amount of air ascending could be varied at will from a scarcely perceptible current to a strong blast. A telescope was fixed near the bottles, some distance away from the cylinder, so that observations could be taken uninfluenced by the heat of the body. The blackened brass wedge, with its edge towards the cylinder, as in Expt. 43, was used as the pillar, and the lampblacked

ivory was used as the cylinder. The jet was half an inch from the cylinder, and a moderate stream of air issued from it. The cylinder revolved three times to the right and then two and a-half to the left.

Similar experiments were tried, placing the jet at different distances from the cylinder, altering the velocity of the air, and increasing the size of the orifice, but the results in all cases were of the same kind, the cylinder first rotating once or twice to the right, and then about the same amount to the left. No permanent movement of eight or ten revolutions could be got, neither by any modification of the draught could I see my way to produce any of the strong rotations easily obtained with hot bodies.

There is a general accord between these experiments, but the agreement between repetitions of the same individual experiment is not so close as I should like. All were performed in duplicate or triplicate, and the mean taken. Much of the discrepancy may be accounted for by great variations of zero owing to the silk fibre becoming warmed or absorbing moisture, and part may be due to the impossibility of bringing the pillar and cylinder exactly 1 mm. apart in every case.

Another noteworthy point is the non-accord between the maximum speed of one revolution and the total number of revolutions performed by the cylinder.

All the experiments tried so far show that the rotations are produced by radiant energy falling on the cylinder and pillar. Radiant heat (and in less degree light) falling on the lampblacked surfaces is absorbed, and increases the surface-temperature. There are two ways in which this increase of temperature may act.—

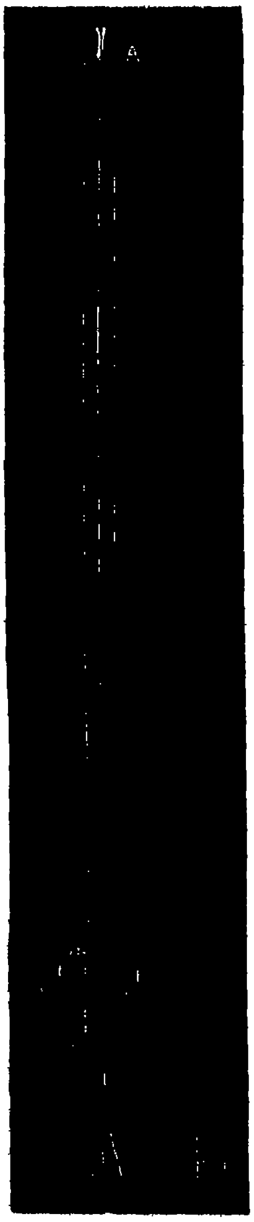
1. It may produce a current of warm air, rising in front of the surfaces of the moving body; to replace this, cold air will come in from all sides, and, striking against the delicately suspended cylinder, caused it to rotate. If, however, the source of heat is of considerable surface, such as the face, or a Winchester quart-bottle full of warm water, it is difficult to imagine that there would be much tendency to rotate in one direction rather than in the other.

2. An increased surface-temperature of the cylinder and pillar may produce an increase of molecular pressure between the two bodies, and thus give rise to motion, after the manner of the radiometer. In this, as in the former case, the movement should be in the opposite direction to what it is in reality, as it would be produced by mutual repulsion acting between the sides nearest the source of heat.

It seemed likely that information decisive as regards one or other of these two theories might be gained by suspending the cylinder in a glass tube attached to a SPRENGEL pump, and taking observations at different degrees of exhaustion. An apparatus was accordingly fitted up as shown in the accompanying figure (fig. 4). ABC is a glass tube, 39 inches long, expanded into a bulb, BC, at the lower end, and connected with a SPRENGEL pump at the upper part. An ivory cylinder, D, is suspended from A by a single fibre of cocoon silk, and at E is attached to the glass bulb a hollow brass tube, inside which, but not touching it, is a platinum spiral with the two ends sealed through the glass. By making battery contact between the

extremities of this spiral, the brass tube E can be heated. The upper end of the tube is clamped at A, and the lower end rests in a socket, F, capable of a little lateral adjustment by a screw. By this means the brass tube (the pillar) can be adjusted in respect to its distance from the cylinder D. Both E and D are lampblacked.

Fig 4



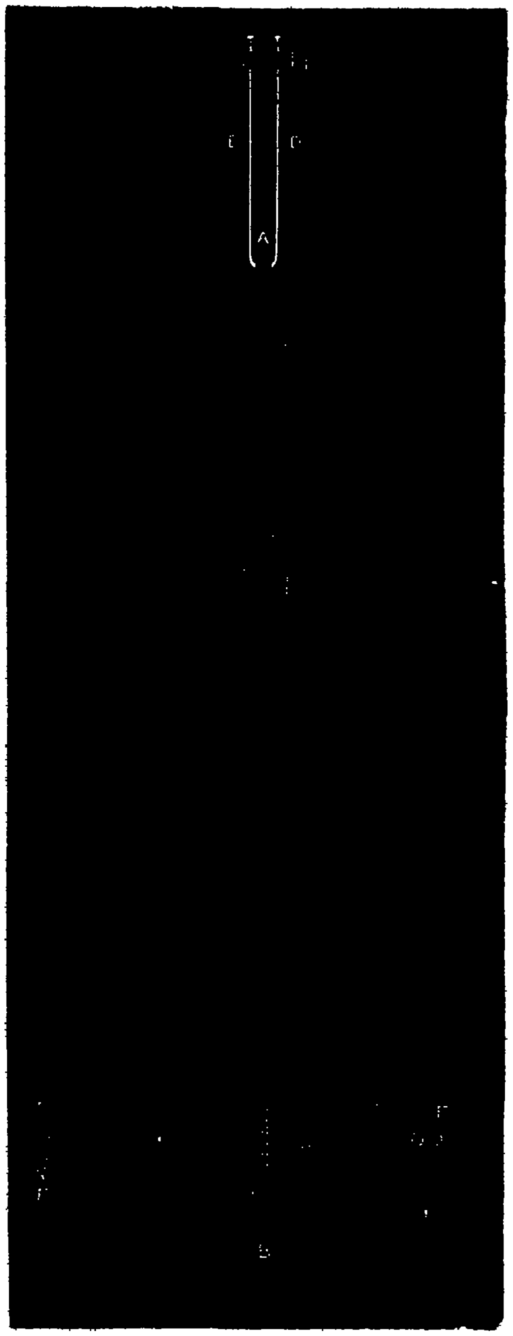
This apparatus is almost identical with one described and figured in my paper "On Repulsion resulting from Radiation" * (Part II., pars. 99, 100), only there the cylinder was of magnesium, and the platinum spiral was bare. With my old apparatus the noteworthy fact was ascertained that the incandescent spiral attracted the suspended cylinder to a moderate extent at normal atmospheric pressure; the attraction diminished to a minimum between a tension of 50 mm. and 150 mm., then rose as the pressure diminished, until, at a tension of 1·15 mm., the attraction was nearly four times what it was in dense air. Above this exhaustion the attraction suddenly dropped and changed to repulsion, and at the best vacuum I could get the repulsion was nearly thirteen times stronger than the attraction in air.

With the present apparatus I was able to verify the broad phenomena formerly [illegible]

* 'Phil. Trans.,' 1875, pp. 528-532.

obtained of attraction in air and stronger repulsion in high vacua, although, owing probably to the blackening of the cylinder and pillar, or to the diminished sensitiveness of the apparatus, I could detect no repulsion at intermediate pressures. Igniting the spiral, so as to make the brass tube E hot, produced attraction or repulsion according to the degree of exhaustion, but it produced no rotation. Rotation, however, could easily be obtained by placing a flask of boiling water in front of the bulb, close

Fig. 5.



to a point equidistant from E and D; and applying a gas-flame to this part of the bulb produced a still stronger effect. Experiments were carried out with this apparatus, gradually raising the exhaustion to a very high point, and noting the repulsion by the spiral, and the rotation by externally applied heat. The two phenomena ran absolutely in parallel lines; when there was attraction of D to E, I could also produce negative rotation of D; when the exhaustion was such that the attraction was *nil*, the rotation was *nil* also; when the attraction changed to repulsion,

the rotation changed from negative to positive; and when the vacuum was so good that the repulsion between the two heated bodies was at its maximum, then also the positive rotation was the strongest. It was impossible to resist the conclusion that the two sets of phenomena were due to the same cause, and that, as air-currents did not produce the old attractions of the magnesium pendulum, so likewise were they equally inoperative in giving rise to the present rotations of the suspended cylinder. I will not give in a tabular form the observations taken with this apparatus, as more decided results were obtained with a modified form of apparatus which I will now describe, and it is not worth while to record observations beyond what are needed to prove the case under discussion.

If the rotation is produced by a reaction between the suspended and fixed body, it follows that, were both free to move, each would rotate, but in opposite directions. A modification of the form of apparatus last used was therefore devised; it is shown in fig. 5. It consists of a long glass tube, AB, having a bulb, C, blown near the lower part. At the top two narrower glass tubes, D, E, are blown on; these contain glass rods sealed to the tubes at the upper ends, but, in other respects, loose in the tubes. To the ends of these rods are attached two fine silk fibres, each having a cylinder of blackened ivory, 1, 2, suspended to it. F is a platinum spiral, equidistant from the two cylinders. The small tubes, D and E, are clamped at their upper ends by a brass band having a screw at one side. By tightening or liberating this screw the tubes are more or less inclined to one another, and the cylinders, 1, 2, can thus be adjusted to any desired distance apart. Exhaustion was effected through a lateral tube. Observations were taken at intervals during exhaustion, the source of heat being either a flask of hot water or a non-luminous gas-flame applied to the glass bulb at the place marked by an asterisk. In all cases when rotation was obtained the two cylinders moved in opposite directions. Thus, in air of ordinary density, cylinder No. 1 rotated counter-clockwise, while No. 2 rotated clockwise; I shall designate this movement as *negative*, and the opposite rotations, where No. 1 rotated clockwise and No. 2 counter-clockwise, I shall call *positive*.

The following Table exhibits, in a convenient form, the results obtained with this apparatus:—

BAROMETER = 767 mm

No of experiment.	Pressure	Direction	Maximum speed of one revolution	No of revolutions	Exciting agent.
	mm.				
54	400	Negative	47 seconds	2·25	Hot water
55	338	„	50 „	2·15	„
56	220	„	52 „	1·75	„
57	178	„	61 „	1·50	„
58	129	„	130 „	1·00	„
59	100	„	160 „	0·75	„
60	80	„	Slow	0·75	„
61	50	„	„	0·25	„
62	30	„	Very slow	0·20	„
63	20	„	Still slower	0·10	„
64	14	„	„	..	Gas flame
65	14	„	Just visible	..	Hot spiral
66	14	0	No movement	..	Hot water
67	8	0	0	0	Gas flame, Hot spiral, Hot water
68	4	0	0	0	„
69	3	0	0	0	„
70	1·5	Positive	50 seconds	2·0	Gas flame
71	0·75	„	20 „	3·00	„
72	0·50	„	.	7·00	„
73	0·30	„	..	10·00	Hot spiral
74	0·129	„	.	13·00	„
75	0·0495	„	..	13·00	„

It will be observed, on comparing these results with those obtained in 1875,* that the neutral point here is between 8 mm. and 3 mm., whereas in the former case it was between 0·8 mm. and 0·3 mm. I, however, attach little importance to this, as the older apparatus was much more sensitive than the one now used.† The important fact is that in each case the direction changes at a high exhaustion, and then the movement becomes five times as strong as it was originally.

The motive force producing these rotations is, at high exhaustions, the molecular impacts between adjacent surfaces of the suspended cylinders excited by the radiation falling on them from the hot water, hot spiral, or a candle (which is equally effective).

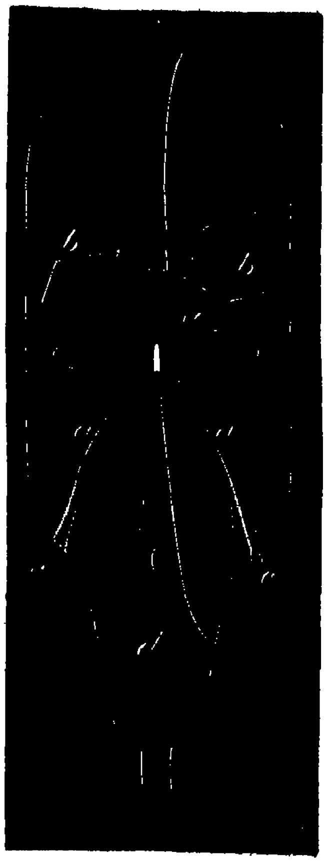
* *Loc. cit*

† "The barometric position of the neutral point dividing attraction from repulsion varies according to circumstances, among these may be mentioned the density of the substance on which radiation falls, the ratio of its mass to its surface, its radiating- and conducting-power for heat, the physical condition of its surface, the kind of gas filling the apparatus, the intensity of radiation, and the temperature of the surrounding atmosphere. When the surface exposed to radiation is pith, the neutral point is somewhat low I have had it vary between 50 millims and 7 millims. below a vacuum. It is, however, impossible to ascertain exactly, for a point of rarefaction can be obtained at which the warm fingers repel and incandescent platinum attracts. With a heavy metal in the form of a sphere, so as to expose the smallest surface in proportion to the mass, I have not attained the neutral point until the exhaustion was within a very small fraction of a millimetre; whilst, if the metal is in the form of thin foil, the neutral point may easily be got lower than with pith." "On Repulsion resulting from Radiation. Part II." (March, 1875.) 'Phil. Trans.,' vol. 165, p. 540.

But what produces the negative rotation at ordinary atmospheric pressure? *Air-currents* are the obvious explanation, but there are grave reasons for believing this explanation to be inadequate. In the first place, actual air-currents, when tried, did not produce the desired result Secondly, it is most logical to assume that, as the present set of experiments are strictly paralleled with those tried in 1875, and as the results at high exhaustions are in each case due to molecular bombardment, so also must the similar results at low exhaustions be due to the same cause.

My series of papers on "Repulsion resulting from Radiation"* contain numerous observations of attraction in air of ordinary density or at low exhaustions; and in the

Fig. 6.



Bakerian Lecture for 1878 I described an apparatus devised with the object of distinguishing between the action of air-currents, and of attraction in air of low exhaustion, and the repulsion in air at high exhaustions. In the following description I have condensed the experiments tried in 1878, and have added other results obtained subsequently with a similar piece of apparatus.

The apparatus is shown in fig. 6†; it consists of a cylindrical glass vessel sealed at the top; drawn off narrow at the other end, and having a stem, *d*, sealed in to hold a needle-point. The vessel is connected with the Sprengel pump by the narrow tube

* 'Phil. Trans.,' 1873, pp. 501–527; 1875, pp. 519–547; 1876, pp. 325–376; 1878, pp 243–318; 1879, [illegible]

† [illegible] unnecessary complication, I have omitted from the drawing parts not used in the present series of experiments.

at the lower end. Round the needle is a ring of platinum wire, *a, a, a,* sealed into the glass and connected with outside terminals, *c, c.* A current of electricity from two Grove cells, turned on and off by a contact key, gives the power of making the wire red-hot when required The fly consists of four thin vanes of clear mica, *b, b,* supported on light aluminium arms, and has in the centre a small glass cap, which rests on the needle-point. The vanes are inclined at an angle of 45° to the horizontal plane They are in such a position that, when rotating, the centres of the vanes pass along the platinum ring and keep about 5 mm. distant from it.

In describing the direction of rotation of the fly, I shall consider the observer's eye to be on a level with the plane in which the fly rotates, and the direction recorded will be that taken by each vane as it passes in front. Assuming that the fly is rotating in the direction of the hands of a watch held face upwards on the top of the apparatus, each vane will be foreshortened, and, passing the observer, will have the appearance of /. The direction of rotation in this case will be considered as *positive*, *i e.*, as the direction followed by the fly, were molecular pressure or a molar wind to proceed from the platinum wire

In air of ordinary pressure (Bar. = 760 mm.), on igniting the platinum ring to redness by a current from two Grove cells, the vanes rotate in the positive direction, such as would be produced either by air-currents or by molecular pressure from the platinum ring.

The following experiments were tried .—

No of experiment.	Pressure	Direction of rotation of vanes.	Number of revolutions per minute.
	millims.		
76	760	Positive ←	13
77	200	Positive ←	7
78	100	Positive ←	2
79	30	Positive ←	Very slight when tapped
80	20	0	No movement
81	10	0	No movement
82	1	0	No movement
83	700 M*	Negative →	40
84	450	Negative →	30
85	300	Negative →	20
86	200	0	No movement
87	138	0	No movement
88	110	Positive ←	18
89	98	Positive ←	33
90	54	Positive ←	60
91	17	Positive ←	150
92	10	Positive ←	600
93	5	Positive ←	1000
94	2 5	Positive ←	Increasing, but too quick to count
95	0 5	Positive ←	,, ,, ,,
96	0·1	Positive ←	,, ,, ,,

Some points in this series of experiments are noteworthy. The vanes were arranged at such a slope in relation to the heated ring that the effect of the rising current of hot air should be a maximum. Owing to this, the first action of heat is to drive the vanes round in the positive direction, in opposition to the tendency to negative rotation which is almost always observed with air of moderate density. This antagonism lasts until a pressure of about 25 mm. is reached, when the two opposing forces balance and no movement is observed. After this point is reached the negative rotation continues till a pressure of about 250 M is reached, when it dies out, to be succeeded at an exhaustion of about 100 M by the positive rotation which all experiments show to be the natural direction for high vacua The neutral point arrived at in Experiments 86 and 87 is, I believe, the analogue of the neutral

* M = one millionth of an atmosphere, or 0·00015 mm. At low exhaustions I speak of millimetres of pressure, but at high exhaustions I prefer to count in millionths of an atmosphere. The inconvenience of using two systems of measure is less than that of employing one system for both ends of the scale

point seen at Experiments 66 to 69 when working with M. THORE's apparatus; in each case the negative rotation is slight before the neutral point is reached, while the positive rotation observed after neutrality increases rapidly in each case, until it eventually far exceeds the original movement

In conclusion, I think I may consider as established by these experiments the following results:—

1. The broad facts of rotation as observed by M. THORE are abundantly confirmed.

2. The numerous experiments in which the face and hot water are tested under the same circumstances, and especially Experiments 46 and 47, prove that the action is due to radiation alone.

3. Blackening the cylinder increases the action. This is especially shown in Experiments 11, 12, 13, 14, and 15.

4. The action is slightly increased by blackening both cylinder and pillar.

5. The remarkable fact observed by M. THORE that a fine fibre of silk brought near the suspended cylinder produces rotation has been verified in Experiments 27 and 28, and with other fine fibres in Experiments 19 to 26.

6. That the rotation is produced by a reaction between the cylinder and pillar, and not between the cylinder and the source of radiation, is shown in Experiments 54 to 75.

7. The hypothesis that the rotations are produced by air-currents is disproved partially by the experiments in which the effect of an ascending current of air is shown to be almost without action, and it is entirely disproved by Experiments 66 to 75, and 86 to 96, in which the movements become more energetic in proportion as the space in which they occur is exhausted of air.

8. The rotation takes place negatively in dense air, and positively in high vacua. It is proved beyond a doubt by Experiments 66 to 75 and 86 to 96 that the positive rotations are due to the same cause which produces rotation of the radiometer, *i.e.*, to molecular pressure caused by radiation falling on the blackened surfaces.* In all cases there is noticed strong action in very high vacua, diminishing as the vacuum gets less perfect, until a point is reached where there is no action. Below this neutral point movement recommences, but in the opposite direction to that observed at high exhaustions. This negative movement is common to M. THORE's phenomena, and to the whole series of phenomena investigated in my researches on "Repulsion resulting from Radiation." The explanation of the negative motion is, however, not clearly made out. But, from the strict parallelism between the two sets of phenomena, I have no doubt that the explanation which will account for the one will be equally adequate to account for the other.

In my sixth paper on "Repulsion resulting from Radiation,"† pars. 415, 416, I described apparatus in which negative rotation was produced at an exhaustion of 117 M, and positive rotation at an exhaustion of 0·18 M. This phenomenon is,

* 'Phil Trans.,' 1876, pp. 375–376; 'Roy Soc. Proc,' vol. 25, 1876, p. 308.

† 'Phil Trans,' 1879, pp 101–103.

I consider, perfectly explained in the paper by the "molecular bombardment" theory, and I, therefore, am justified in assuming that the negative rotations in M. THORE's apparatus will equally well be explained by the same theory.

ADDENDUM.

(Added May 24, 1887.)

I sent M. THORE a detailed account of my experiments, asking him to favour me with any comments or remarks he might wish to make, and offering to communicate them, if desirable, to the Royal Society. I have just received a long communication, partly printed and part in MS., in which he describes many fresh experiments, and adduces arguments to show that my dynamical explanation is not sufficient to account for more than a few of the facts he describes, and saying that he "persists in still believing that this force emanates from the observer, or else that the observer is the indispensable intermediary for its manifestation."

The experiments are numerous, and are devised with great ingenuity. It is impossible in the space of a brief abstract to do more than refer to a few of the principal facts here brought forward. M. THORE commences by objecting to my having experimented in an enclosed space, saying that he always operates in free air. He thinks that enclosure may almost or quite suppress his force. To this I can reply that I have myself verified nearly all M. THORE's facts of rotation (including those just now communicated) when working in the free air of a large room, and it was only when I found the delicacy of the observations was impeded by draughts and currents that I put screens round the apparatus. I have not found glass screens interfere materially with any of the rotations. M. THORE now says that it is necessary to hold the pillar or the exciting body in contact with the hand during the whole duration of the experiment. I was not aware that importance was attached to this point, but I have since repeated many of my former observations, holding the pillar in the hand. The results are certainly stronger, but the extra heat imparted to the apparatus is, in my opinion, sufficient to account for this. M. THORE brings forward many new and ingeniously devised experiments to prove that heat cannot be considered the cause of the movement. He exposes the instrument to the full sun, and then brings it into a cool dark room; he suspends it over boiling water; he places a large block of ice between the cylinder and the observer; he similarly interposes metallic vessels full of boiling water between the cylinder and observer (the observer not moving from his place in front), and he tries the experiment in a hot chamber, alternately moist and dry, without finding the regularity of the movements interfered with. I have tried most of these, and obtained results corroborating M. THORE's, but I have also tried the experiment of quietly bringing near to the stationary cylinder a bottle of hot water, and observing the movement from a safe

distance through a telescope, and I find that the hot bottle is able to effect rotation as well as the observer.

Among the curious observations mentioned by M. THORE is this —Placing the pillar in front of the cylinder (between it and the observer), if the pillar is held with the right hand the movement is clockwise, and if the left hand is used the rotation is counter-clockwise The right hand is stronger in its effects than the left hand in the proportion of 2 to 1.

M. THORE has given, in addition, a large number of curious and interesting observations, using two, three, and more movable cylinders, and recording their movements under a great variety of circumstances I admit I do not see at once how all these are to be explained on the molecular bombardment theory. But this theory has not yet explained all the anomalous results I have recorded in my papers on "Repulsion resulting from Radiation," although I believe it capable of doing so, and I therefore think that it is not necessary to call upon a new force to explain any of M. THORE's results which radiation does not yet seem able to account for.

XVI. *Some Applications of Dynamical Principles to Physical Phenomena.*—Part II.

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§ 1. THE two laws of Thermodynamics have proved by far the most powerful, indeed almost the only, means we possess of connecting the phenomena in one branch of Physics with those in another. Though the two laws are usually grouped together, it should not be forgotten that they differ essentially in character. The First Law is a direct application to Physics of one of the most important dynamical principles, that of the Conservation of Energy; while the Second Law, which for the purpose of connecting various physical phenomena is even more important than the first, is not, strictly speaking, a dynamical principle at all, since its statement involves a reference to quantities which never occur in abstract Dynamics.

CLAUSIUS and Sir WILLIAM THOMSON, the two physicists to whom the Second Law owes its importance, have connected it with other principles which seem more axiomatic.

Thus CLAUSIUS bases the Second Law on the principle that "heat cannot by itself pass from a colder to a hotter body." In this statement too much depends upon the meaning to be attached to the words "by itself" for it to be regarded as axiomatic, and the following extract from CLAUSIUS seems to show that in his view it is the statement of a new physical principle, and not the necessary consequence of previously recognised ones. He says ('Mechanical Theory of Heat,' English translation, by W. R. BROWNE, p. 342), "If, however complicated the processes may be, it is maintained that without some other permanent change, which may be looked upon as a compensation, heat can never pass from a colder to a hotter body, it would seem that this principle ought not to be treated as one altogether self-evident, but rather as a newly propounded fundamental principle on whose acceptance or non-acceptance the validity of the proof depends."

Again Sir WILLIAM THOMSON has connected the Second Law with the principle that "it is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects." To follow out the connection between this principle and the Second Law, it is convenient to divide the energy of a body into two kinds, the one kind depending upon circumstances over which we have complete control, the other

lepending on circumstances which we cannot completely control. Take as an example he energy possessed by a stretched spring · part of the energy depends upon the xtension of the spring, we have complete control over this, but another part of the nergy (the heat energy) depends upon the motion of the molecules of the spring, and, lthough we have some control over the average motion of all the molecules, we have one over the motion of individual molecules. Let us for the moment call the energy f the first kind "controllable energy," that of the second "intrinsic energy." We may ook on an engine as a means of converting intrinsic into controllable energy. Then, f we follow the connection between the axiom and the principle that the efficiency of perfectly reversible engine is a maximum (which we may take as equivalent to the Second Law), we shall see that if the axiom is to cover all the cases to which the Second Law has been applied it must be equivalent to the statement that it is impossible to lerive mechanical effect by abstracting intrinsic energy from the refrigerator. Now he intrinsic energy consists, in addition to sensible heat, of what CLAUSIUS calls the nternal energy of the body, that is, energy depending upon the arrangement of the nolecules, and, it may be, also upon their motion. If we consider the various forms which this intrinsic energy can take, the statement that it is impossible to derive nechanical effect by abstracting intrinsic energy from the refrigerator would seem to be hardly more axiomatic than the Second Law itself.

The Second Law of Thermodynamics, like the Law of Gravitation, seems then to be proved rather by the truth of its consequences than by any *à priori* considerations.

For this, among other reasons, I have thought it might perhaps be interesting to deduce by the use of purely dynamical principles many results which are usually obtained by the aid of the Second Law of Thermodynamics, as well as some others which, so far as I know, have not previously been obtained. This I have endeavoured to do in the following paper, as I did in one previously published under the same title in the 'Philosophical Transactions,' 1885, Part 1. In the first paper I considered the relation between thermal, elastic, and magnetic phenomena, but did not consider any phenomena in which chemical or quasi-chemical processes were concerned, such as dissociation, evaporation, solution, chemical combination: or any effects which are not reversible, such as those produced by the electric resistance of metals and electrolytes. In this paper I shall endeavour to apply the same or analogous principles to the phenomena mentioned above, as well as to a few additional phenomena of the kind discussed in the first paper.

Though the dynamical method is open to the objection that the quantities made use of are those which occur in abstract Dynamics, such as mass, velocity, energy, acceleration, and so require further knowledge before we can connect them with such things as temperature, electric current, resistance, and so on—a knowledge which, in many cases, we do not possess—while, in the Second Law, the results are expressed [illegible] which can be readily measured; still it has advantages which [illegible] of consideration.

In the first place, there is the mental satisfaction to be got by explaining things on dynamical principles; and, again, there is the certainty that the method is capable of completely solving the question (whether we can make it do so is another matter), while we have no certainty that all possible information is given by the two laws of Thermodynamics, or that some unknown third law might not enable us to arrive at results beyond the powers of the first and second.

§ 2. The researches of CLAUSIUS, SZILY, and BOLTZMANN have shown that the Second Law of Thermodynamics is closely connected with the principle of Least Action, and it might therefore be thought that the Second Law was only a more convenient way of stating this principle, so that no advantage could be gained by the direct use of dynamical principles. In the investigations on the connection between the Second Law and the principle of Least Action there are, as I shall endeavour to show later on, a good many assumptions implicitly made, so that it seems to be much preferable to proceed, if possible, in any special case by the direct use of dynamical principles.

Again, there can, I think, be no question that the principle of Least Action and the Second Law of Thermodynamics are not equivalent; for, in the first place, as is well known (see ROUTH's 'Advanced Rigid Dynamics,' p 257), the principle of Least Action includes that of the Conservation of Energy, so that, if the Second Law of Thermodynamics included all that could be got from the principle of Least Action, it ought to include the First Law as a particular case.

Again, in the most general case, the principle of Least Action will for a system fixed by n coordinates give n equations; but the Second Law of Thermodynamics, which asserts that a certain function is a perfect differential, would, in the most general case, give rise to $\frac{1}{2} n (n-1)$ equations, as that is the number of conditions to be satisfied if

$$P_1 dx_1 + P_2 dx_2 + \ldots$$

is a perfect differential.

§ 3. The dynamical methods we shall most frequently use in the following paper are the Hamiltonian principles expressed by the equations

$$2\delta \int_0^t \mathrm{T}\, dt = t\delta (\mathrm{T} + \mathrm{V}) + \left[\Sigma \frac{d\mathrm{T}}{dq} \delta q \right]_0^t, \quad \ldots \ldots \quad (1)$$

$$2\delta \int_0^t (\mathrm{T} - \mathrm{V})\, dt = (\mathrm{T} + \mathrm{V})\, \delta t + \left[\Sigma \frac{d\mathrm{T}}{dq} \delta q \right]_0^t, \quad \ldots \ldots \quad (2)$$

where T and V are respectively the kinetic and potential energies of a system, q a typical coordinate helping to fix the configuration of the system, and t the time.

The first of these equations has been used to show the connection between the principle of Least Action which it expresses and the Second Law of Thermodynamics;

and it will be convenient to begin by considering the investigations which have been made on this connection.

The proof usually given is as follows (see ROUTH's 'Advanced Rigid Dynamics,' p. 254) —

Let us suppose that no external work is done by the system; then—

$$\delta(\mathrm{T} + \mathrm{V}) = \delta\mathrm{Q}, \qquad (3)$$

where $\delta\mathrm{Q}$ is a small quantity of work supplied to the system. The quantity $\delta(\mathrm{T} + \mathrm{V})$ occurs on the right-hand side of equation (1) let us now consider the term $[\Sigma\, \delta q\, d\mathrm{T}/d\dot{q}]_0^i$, which also occurs on the same side of the equation. If the motion be oscillatory, and i a period of complete recurrence, $\delta q\, d\mathrm{T}/d\dot{q}$ will have the same value at the lower as it has at the upper limit of the integral, and therefore the difference of the values will vanish. The case when the motion is oscillatory is not, however, the only, nor indeed the most important, case in which this term may be neglected. Let us suppose that the system consists of a great number of secondary systems, or, as they are generally called, molecules, and that the motion of these molecules is in every variety of phase, then the term $[\Sigma\, \delta q\, d\mathrm{T}/d\dot{q}]$, the sum being taken for all the molecules, will be small, and will not increase indefinitely with the time, but will continually fluctuate within narrow limits. This is evidently true if we confine our attention to those coordinates which fix the configuration of the molecule relatively to its centre of gravity; and, if we remember that the motion of the centre of gravity of the molecules is by collision with other molecules and with the sides of the vessel which contain them continually being reversed, we can see that the above statement remains true even when coordinates fixing the position of the centres of gravity of the molecules are included. Thus, if the time over which we integrate is long enough, we may neglect the term $[\Sigma\, \delta q\, d\mathrm{T}/d\dot{q}]_0^i$ in comparison with the other terms which occur in equation (1), as these terms increase indefinitely with the time, so that in this case, even though the motion is not entirely periodic, equation (1) may be written—

$$2\delta \int_0^i \mathrm{T}\, dt = i\, \delta\mathrm{Q},$$

or

$$2\delta(i\mathrm{T}_m) = i\, \delta\mathrm{Q}, \qquad (4)$$

where T_m is the mean kinetic energy. This equation may be written—

$$2\delta \log(i\mathrm{T}_m) = \delta\mathrm{Q}/\mathrm{T}_m, \qquad (5)$$

so that $\delta\mathrm{Q}/\mathrm{T}_m$ is a perfect differential.

One of the ways of stating the Second Law of Thermodynamics is that $\delta\mathrm{Q}/\theta$ is a perfect differential, θ being the absolute temperature; thus, if θ is a constant

multiple of the mean kinetic energy, the Second Law of Thermodynamics can be deduced from the principle of Least Action. Thus even in the simplest case, when the system does no external work, we require the additional assumption that the absolute temperature is proportional to the mean kinetic energy. Now, in the only case in which the theory has been completely worked out, that of the kinetic theory of gases, the absolute temperature is measured, not by the mean total kinetic energy directly, but by the mean kinetic energy due to the translatory motion of the centres of gravity of the molecules This is shown by the way in which BOYLE's Law is deduced from the kinetic theory. If the temperature depended upon the vibratory energy, we could not explain why the relation between pressure, density, and temperature is practically the same for all gases, while the ratio of the vibratory energy to the translatory energy varies from an exceedingly small fraction in the case of mercury vapour to more than half in the case of hydrogen, oxygen, and nitrogen. Thus in the case of gases we have strong reasons for supposing that the temperature is measured by the mean translatory energy, the mean being taken for all the molecules. In the case of solids and liquids this is not so clear, but even here there seem to be reasons for believing that the temperature is measured by the mean of some particular kind of energy rather than by the mean total kinetic energy. From the continuity of the solid, liquid, and gaseous states of matter we should expect the temperature to depend upon the kinetic energy in the solid or liquid as well as in the gaseous state. But, if in the case of a solid the temperature were measured by the mean total kinetic energy and not by the mean of some special kind of energy, then, if we have a gas and a solid at the same temperature, the mean total kinetic energy of the gas will be greater than that of the solid, for by our supposition the mean translatory energy of the molecules of the gas equals the mean total kinetic energy of the molecules of the solid. Now, the specific heat of water in the solid state is about the same as that of the same body when in the gaseous state, while for some substances it is double, as it would be if the kinetic energy in the solid state were equal to that in the gaseous, and if, as we should expect *à priori*, the work supplied to a solid is equally divided between the kinetic and potential energies. For this reason, we conclude that the mean kinetic energy of the molecules of a solid is not less than the mean kinetic energy of the molecules of a gas at the same temperature; and hence, that the temperature in the solid state is measured by the mean of some particular kind of energy. It would seem most probable that this particular kind would be the energy due to the translatory motion of the molecules; and that the temperature is measured by the mean energy due to the translatory motion of the molecules in the solid and liquid as well as in the gaseous states.

In the simple case we are considering, we have seen that it follows from the principle of Least Action that $\delta Q/T_m$ is a perfect differential.

If this is identical with the Second Law of Thermodynamics, then T_m must either be a constant multiple of θ, the absolute temperature, or T_m/θ must be a function of ϕ,

where ϕ is given by the equation $d\phi = dQ/T_m$. Making the first supposition, and remembering that the absolute temperature measures the mean energy due to the translatory motion of the molecules, we see that it is equivalent to supposing that the mean total kinetic energy of the molecules is a constant multiple of their mean translatory energy. This is the assumption which was originally made by CLAUSIUS, and we see that it must be made if we are to derive the Second Law of Thermodynamics from the principle of Least Action

BOLTZMANN, in his celebrated investigation* of the distribution of energy among the molecules of a gas, each molecule of which possesses n degrees of freedom, arrives at a much more definite result. According to this investigation the mean kinetic energy corresponding to each degree of freedom is the same, so that the mean kinetic energy due to the translatory motion of the centres of gravity of the molecules is only $3/n$ of the mean total kinetic energy of the molecules. The proof of this theorem given by the author seems to me to be open to grave objection, and the results to which it leads have certainly not been reconciled with the properties possessed by actual gases. According to this theorem, the result is the same, whatever be the constitution of the molecule, and whatever the forces exerted by one molecule on another when they come so close together as to be within the range of each other's action. BOLTZMANN shows that, if the number of molecules which have the coordinates $q_1, q_2, \ldots \ldots q_n$ and the corresponding momenta $p_1, p_2, \ldots \ldots p_n$ between the limits $q_1, q_1 + \delta q_1, q_2, q_2 + \delta q_2, \ldots q_n, q_n + \delta q_n, p_1, p_1 + \delta p_1, \ldots p_n, p_n + \delta p_n$, is

$$Ce^{-h(T+\chi)}\, dq_1\, dq_2 \ldots dq_n \,.\, dp_1\, dp_2 \ldots dp_n,$$

where C and h are constants, and T and χ the kinetic and potential energies of such a molecule; then the number of such molecules will remain constant, as in a given time as many molecules pass out of that state as enter it. Thus, if this distribution is ever established, it will be a steady distribution, *i.e.*, the state of the gas will not change. To prove the theorem we have quoted above, BOLTZMANN integrates this expression, assuming that each velocity may have all values from *plus* infinity to *minus* infinity. It seems to me, however, that this assumption is not legitimate, and that before we can fix the limits of the velocity we must know the nature of the molecule and the forces between two molecules when they come within the sphere of each other's action. We can easily imagine cases in which the assumption is not true. Take, for example, the case of two bodies describing orbits about their centre of gravity under each other's attraction. If the relative velocity of the bodies exceeds a certain value, which depends upon the distance between them and the law of attraction, the two bodies will not remain together, but will separate until they are finally at an infinite distance apart. Thus, if the two bodies represent the atoms in a

* [illegible] 'Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften,' vol 63 (Abth. 2), [illegible] MAXWELL, 'Transactions of the Cambridge Philosophical Society,' vol. 12, 1879.

molecule, the molecule will be split up when the relative velocity of the atom exceeds a certain value; so that in this case the limits of the relative velocity would be functions of the coordinates fixing the position of the atoms, and not *plus* and *minus* infinity, as in BOLTZMANN'S investigation. We can, moreover, imagine a kind of molecule for which we can prove that the theorem itself is not true. We know that many dynamical theorems have their most elegant applications to systems of electric currents flowing through neighbouring circuits, and that if any dynamical theorem is true at all it must be true when interpreted in an electrical sense as well as in the mechanical one. This is evident, because we can apply the same method, that of LAGRANGE'S equations, to both the electrical and mechanical problems. Thus BOLTZMANN'S theorem, if it is true at all, must be true when some of the coordinates fixing the configuration of the molecule are coordinates which fix the distribution of electric currents flowing through circuits attached to the molecule Let us suppose that these coordinates, which we will call x_1, x_2, . . x_n, fix currents flowing through perfectly conducting circuits in parallel planes in the molecule, the circuits being so close together that the magnetic force due to the currents round the other molecules, or to any other external source, may be taken as constant over the circuits. The kinetic energy due to the currents $\dot{x}_1, \dot{x}_2, \ldots$ circulating through these conductors is of the form

$$\tfrac{1}{2}(L_{11}\dot{x}_1^2 + 2L_{12}\dot{x}_1\dot{x}_2 + \ldots).$$

Let y_1, y_2, be the "principal" coordinates, fixing the same configuration as that fixed by $x_1, x_2, \ldots$; then, when the kinetic energy is expressed in terms of these coordinates, it is of the form

$$\tfrac{1}{2}(L_1\dot{y}_1^2 + L_2\dot{y}_2^2 + \ldots).$$

The electrical equations are :—

$$\frac{d}{dt}(L_1\dot{y}_1) = \text{rate of diminution in the number of lines of force passing through the circuit corresponding to } y,$$

$$\frac{d}{dt}(L_2\dot{y}_2) = \text{the same thing for the circuit } y_2,$$

Now, since all the circuits are parallel, and so close together that the magnetic force may be considered constant over them, the number of lines of force passing through y_1 will be in a constant ratio to the number passing through y_2. Let this ratio be λ; then

$$\frac{d}{dt}(L_1\dot{y}_1) = \lambda\frac{d}{dt}(L_2\dot{y}_2),$$

or, if $\dot{y}_1$ and $\dot{y}_2$ are each initially zero,

$$L_1\dot{y}_1 = \lambda L_2\dot{y}_2.$$

Now, the kinetic energy corresponding to $\dot{y}_1$ is

$$\tfrac{1}{2} L_1 \dot{y}_1^2,$$

that corresponding to y_2 is

$$\tfrac{1}{2} L_2 \dot{y}_2^2.$$

The ratio of these is $\lambda^2 L_2/L_1$; this is a quantity depending only on the configuration of the circuits, and, if the molecules are geometrically similar, will be the same for each molecule, thus the ratio of the mean kinetic energy corresponding to $\dot{y}_1$ to that corresponding to $\dot{y}_2$ is $\lambda^2 L_2/L_1$, and, by properly choosing the configuration of the circuits, this quantity may be made to have any positive value we please, whereas, if BOLTZMANN's theorem were true, the ratio ought always to be equal to unity. Hence we conclude that BOLTZMANN's theorem is not true. It ought to be noticed that in this case the ratio is constant, though not unity, and this is all that is assumed by CLAUSIUS.

The consideration of the collision of two vortex rings, according to the vortex ring theory of gases, would, I think, lead us to the conclusion that the energy corresponding to each mode of vibration is, when the gas is in a steady state, a function of the mean translatory energy of the molecules of the gas, the function being of such a kind that, the higher the mode of vibration, the smaller the ratio of the corresponding energy to the mean translatory energy.

The application of the Second Law to the case we have just discussed, which is the one investigated by SZILY and CLAUSIUS, does not include the application to the case (almost the only one of importance in the applications of the Second Law) when the system absorbs or expends work when heat is communicated to it. It would not include, for example, the case when heat is applied to a gas at constant pressure.

Let $P_1, P_2, \ldots$ be the external forces of type p_1, p_2, respectively, acting on the system, and let the points of application of these forces move through $\delta p_1, \delta p_2 \ldots$ Then, if δQ be the quantity of undirected energy supplied to the system, that is, energy that is not supplied by moving the system against definite external forces, δT and δV the increments in the kinetic and potential energies of the system,

$$\delta Q = \delta T + \delta V - P_1 \delta p_1 - P_2 \delta p_2 \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus, for example, in the case of a gas contained in a cylinder with a movable piston,

$$\delta Q = \delta T + \delta V + p\,\delta x,$$

where p is the total pressure on the piston, and x the distance of the piston from the bottom of the cylinder.

[* In considering this case it will be convenient to divide the kinetic energy into two parts, one part, which we shall denote by T_1, depending on the velocities of coordinates fixing the position of the molecule, the other, which we shall denote by T_2, depending on the velocities of coordinates of the type p, which we have completely under our control. The coordinates fixing the configuration of the system with respect to strain, electrification, magnetisation, &c., are coordinates of this class. T_2 will be a quadratic function of the velocities of the p coordinates, since the total kinetic energy of the system cannot involve the product of the velocity of a p coordinate with that of one fixing the position of a molecule; otherwise the kinetic energy of the system would be altered by reversing the motion of all the molecules.

We have, by LAGRANGE's equation,

$$\frac{d}{dt}\frac{dT}{dp_1} - \frac{dT}{dp} + \frac{dV}{dp} = P;$$

now

$$T = T_1 + T_2,$$

and by definition

$$\frac{dT_1}{dp_1} = 0,$$

we have also, by the Conservation of Energy,

$$\delta Q = \delta T_1 + \delta T_2 + \delta V - \Sigma P\,\delta p; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6^*)$$

now

$$\delta T_2 = \Sigma\left(\frac{dT}{dp}\,\delta p + \frac{dT}{d\dot{p}}\,\delta\dot{p}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

since T_2 is a homogeneous quadratic function of the velocities of the "p" coordinates,

$$2T_2 = \Sigma\,\dot{p}\,\frac{dT_2}{d\dot{p}},$$

and therefore

$$2\,\delta T_2 = \Sigma\left(\delta\dot{p}\,\frac{dT_2}{d\dot{p}} + \dot{p}\,\delta\,\frac{dT_2}{d\dot{p}}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

subtracting (7) from (8), we have

$$\delta T_2 = \Sigma\left(\dot{p}\,\delta\,\frac{dT_2}{d\dot{p}} - \delta p\,\frac{dT_2}{dp}\right).$$

If the change in the configuration is that which actually takes place, then we have

$$\dot{p}_1\,\delta t = \delta p_1,$$

so that

$$\delta T_2 = \Sigma\,\delta p\left(\frac{d}{dt}\frac{dT_2}{d\dot{p}} - \frac{dT_2}{dp}\right),$$

* This portion within brackets re-written October 17, 1887.

so that equation (6) becomes

$$\delta Q = \Sigma\, \delta p \left(\frac{d}{dt}\frac{dT_2}{d\dot{p}} - \frac{dT_2}{dp} + \frac{dV}{dp} - P\right) + \delta V_{p \text{ constant}} + \delta T_1\,;$$

substituting for P from (5), we have

$$\delta Q = \Sigma \frac{dT_1}{dp}\delta p + \delta T_1 + \delta V_{p \text{ constant.}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now, if p enters into the expression for the kinetic energy due to the motion of the molecules of the body, it must enter as a factor into all the terms expressing this energy, otherwise the phenomenon symbolised by p would be more affected by the motion of particular molecules than by that of others. Thus T_1 must be of the form

$$f(p)\,T_3$$

when T_3 does not involve p.

Thus

$$\begin{aligned}\frac{dT_1}{dp} &= f'(p)\,T_3 \\ &= \frac{f'(p)}{f(p)}T_1,\end{aligned}$$

and therefore, by (9),

$$\delta Q = \Sigma \frac{f'(p)}{f(p)} T_1\, \delta p + \delta T_1 + \delta V_{(p \text{ constant})},$$

so that

$$\frac{\delta Q}{T_1} = \Sigma \frac{f'(p)}{f(p)}\delta p + \delta \log T_1 + \frac{\delta V_{(p \text{ constant})}}{T_1}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Since T_1 is assumed to be proportional to the absolute temperature, and since the first two terms on the right-hand side of the equation are perfect differentials, we see that, in order that $\delta Q/\theta$ should be a perfect differential,

$$\frac{\delta V_{(p \text{ constant})}}{\theta}$$

must be one too.

The state of the body is determined if we know the value of the p coordinates and the temperature, so that δV, when p is constant, may be written as

$$\frac{dV}{d\theta}\cdot\delta\theta,$$

where $dV/d\theta$ is very large when the temperature is near the melting or the boiling points of the substance.

Making this substitution for δV in equation (10), we have

$$\delta Q = \Sigma \frac{dT_1}{dp}\delta p + \delta T_1 + \frac{dV}{d\theta}\cdot\delta\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now let us suppose that only one coordinate, p_1, changes, and that just as much heat is supplied or absorbed as is sufficient to prevent the temperature from changing; then, since the temperature is constant, δT_1 and $\delta\theta$ both vanish, and we have

$$\delta Q = \frac{dT_1}{dp_1}\delta p_1.$$

Now, if P_1 be the force which is required to keep p_1 constant when the system is in a steady state,

$$\begin{aligned} P_1 &= \frac{dV}{dp_1} - \frac{dT_1}{dp_1} - \frac{dT_2}{dp_1} \\ &= \frac{dV}{dp_1} - \frac{f'(p)}{f(p)}T_1 - \frac{dT}{dp_1}. \end{aligned}$$

Now, since dV/dp_1 and dT_2/dp_1 do not explicitly involve θ, and since T_1 is proportional to θ, we have

$$\begin{aligned} \theta\left(\frac{dP_1}{d\theta}\right) &= -\frac{f'(p)}{f(p)}T_1 \\ &= -\frac{dT_1}{dp_1}, \end{aligned}$$

where, in finding $\left(\frac{dP_1}{d\theta}\right)$, θ is the only quantity which is supposed to vary.

Thus equation (11) becomes

$$(\delta Q)_{\theta \text{ constant}} = -\theta\left(\frac{dP_1}{d\theta}\right)\delta p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)\,]$$

This result can be obtained from the Second Law of Thermodynamics; it was so obtained by VON HELMHOLTZ, and applied by him to the very important case of the heat produced in the voltaic cell in his paper "Die Thermodynamik chemischer Vorgänge" ('Wissenschaftliche Abhandlungen,' vol. 2, p. 962).

§ 4. It will be seen from the preceding work that the Second Law of Thermodynamics cannot be deduced from the principle of Least Action, unless we know a good deal about the distribution of energy among the molecules, and unless we make in addition a good many assumptions. For this reason, in discussing the applications of Dynamics to Physics, I prefer to apply the principle of Least Action directly to the various problems, and not to start from the Second Law as an intermediate stage. In the rest of the paper I shall endeavour to show how this can be done.

The Application of Dynamical Principles to Phenomena which are in a Steady State

§ 5 The most convenient principle for this purpose is the Hamiltonian one, according to which, if i be constant,

$$\delta \int_0^i \mathrm{L}\, dt = \Sigma \left(\frac{d\mathrm{L}}{dq} \delta q\right)_0^i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12^*)$$

where q is one of the coordinates helping to fix the configuration of the system, and L the Lagrangian function or ROUTH's modification of it, according as it is or is not expressed entirely in terms of the velocities of the coordinates.

If

$$\int_0^i \mathrm{L}\, dt = \mathrm{S},$$

and if S be expressed in terms of i, and the coordinates at the times 0 and i, then, if q be a coordinate at the time i, we see from equation (12*) that

$$\frac{d\mathrm{T}}{d\dot{q}} = \frac{d\mathrm{S}}{dq},$$

and, by LAGRANGE's equation,

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\dot{q}} = \frac{d}{dq}(\mathrm{T} - \mathrm{V});$$

hence we see that the momentum corresponding to any coordinate and the rate of change of the momentum can both be expressed as the differential coefficients of functions with respect to that coordinate.

We shall now proceed to show that for Steady Motion

$$\delta \overline{\mathrm{L}} = 0,$$

where $\overline{\mathrm{L}}$ is the mean value of L, and where δ is to be interpreted in the following way.

All, or nearly all, the systems we shall have to deal with are those which consist of a large number of molecules, and we may conveniently for our purpose divide the coordinates, fixing the configuration of such a system into two kinds :—

(*a*) Molar coordinates, which fix the configuration of the system as a whole, and whose value we may by various physical processes alter at our pleasure. When we say that the system is in a steady state, all that we mean is that the configuration as fixed by the molar coordinates is steady.

(*b*) Molecular coordinates, which fix the position of individual molecules. The values of these coordinates are quite beyond our control.

Now, if we consider the molar coordinates, we shall see that they are of two kinds: the first kind, which I called in my first paper kinosthenic coordinates, only enter into the expressions for the energy through their differential coefficients, and do not occur explicitly themselves; the molar coordinates of the second kind enter explicitly into the expressions for the energies, and do not occur merely as differential coefficients.

A good example of the two classes of coordinates is afforded by the coordinates required to fix the position of a rod suspended by one extremity. We may fix it by the angle θ which the rod makes with the vertical, and the azimuth ϕ of the plane through the rod and the vertical line through its fixed extremity. The expression for the kinetic energy of the rod in terms of these coordinates is of the form

$$A\dot{\theta}^2 + B \sin^2 \theta\dot{\phi}^2 ;$$

the potential energy is of the form

$$C \cos \theta,$$

where A, B, C, are constants. We see that ϕ is a coordinate of the first kind, since it only enters the expression for the kinetic energy through its differential coefficient, while θ is a coordinate of the second kind, as functions of θ occur in the expressions for the kinetic and potential energies. When the system is in a steady state the velocity of the first kind of coordinate is constant, while that of the second kind is zero. In the variations which we shall suppose $\overline{L}$ to suffer we shall suppose that the velocities of the kinosthenic coordinates remain unaltered, while the coordinates of the second kind are varied. In calculating the mean value of L for a system in a steady state, we may suppose that all the terms in the kinetic energy which involve a differential coefficient of a coordinate of the second kind are omitted, since in the steady state these differential coefficients vanish. We may, therefore, for our purpose, without loss of generality, suppose that $dL/d\dot{q}_2 = 0$, where q_2 is a molar coordinate of the second kind.

The equation

$$\delta \overline{L} = \Sigma \left(\frac{dL}{d\dot{q}} \delta q\right)_0^1$$

may conveniently be written

$$\delta \overline{L} = \Sigma \left(\frac{dL}{d\dot{q}_1} \delta q_1\right)_0^1 + \Sigma \left(\frac{dL}{d\dot{q}_2} \delta q_2\right)_0^1 + \Sigma \left(\frac{dL}{d\dot{q}_3} \delta q_3\right)_0^1 ;$$

where q_1 and q_2 are molar coordinates of the first and second kinds respectively, and q_3 is a molecular coordinate. We may disregard the last term by the reasoning, due

to CLAUSIUS, which we have already given on page 474, and we have just seen that we may suppose $d\mathrm{L}/d\dot{q}_2 = 0$, so that the equation becomes

$$\delta\,\overline{\mathrm{L}} = \Sigma\left(\frac{d\mathrm{L}}{d\dot{q}_1}\,\delta q_1\right)_0^1;$$

but, since the motion is steady, $d\mathrm{L}/d\dot{q}_1$ is constant, so that this may be written

$$\delta\,\overline{\mathrm{L}} = \Sigma\,\frac{d\mathrm{L}}{d\dot{q}_1}\,(\delta q_1)_0^1.$$

Now we have supposed that the variation is of such a kind that $\dot{q}_1$ remains unaltered, in this case δq_1 remains constant throughout the motion, and therefore $(\delta q_1)_0^1$ vanishes, so that we have

$$\delta\,\overline{\mathrm{L}} = 0, \qquad \qquad (13)$$

or $\overline{\mathrm{L}}$ has a stationary value for all changes which leave the velocities unchanged

It is convenient to work with the mean values of L, because, as we shall see later on, it is possible in many cases involving the motion of great numbers of molecules to calculate $\overline{\mathrm{L}}$ from data given by experiment, when it would not be possible to calculate L.

The expression for the energy of a system consisting of a great number of molecules will contain terms of three kinds. (1) terms depending entirely on molar coordinates; (2) terms depending partly upon molar and partly upon molecular coordinates; and (3) terms depending entirely upon molecular coordinates. The energy expressed by the terms of the first kind can be entirely converted into mechanical work, while that expressed by the terms (2) and (3) can only be partially converted, the extent of the conversion depending on the distribution of kinetic energy throughout the system. VON HELMHOLTZ* calls the first kind of energy free energy, the other he calls bound energy.

Since the velocities are supposed to remain constant in the variations we contemplate in equation (13), it is evident that the only terms in the kinetic energy which are affected are those which involve the coordinates themselves. The energy expressed by such terms we may call the positional kinetic energy, and it is the only part of the kinetic energy which we need consider, or which has any influence on the way in which any transformation of energy takes place. We shall now go on to apply the principle that

$$\delta\,\overline{\mathrm{L}} = 0$$

to some special cases.

* VON HELMHOLTZ, "Die Thermodynamik chemischer Vorgänge," 'Wissenschaftliche Abhandlungen,' vol. 2, p. 958.

Evaporation.

§ 6. The first case we shall take is that of evaporation. Let us suppose that we have a liquid and its vapour in a closed vessel, and endeavour to find an expression for the density of the vapour when it is in equilibrium with the liquid. We have here two systems for which we have to find expressions for $\overline{\mathrm{L}}$ when in a steady state, the first being the gas, the second the liquid.

The variation we shall consider is that which would be produced if a small quantity of the liquid were vaporised, keeping the velocities of the molecules the same as in the liquid condition, and thus keeping the temperature of the liquid and gas constant. We must find the effect of this change on the value of $\overline{\mathrm{L}}$ for the gas and the liquid. To do this for the gas, let us consider the case of a cylinder furnished with a piston and containing a given quantity of gas. Let x denote the distance of the piston from the base of the cylinder, and let us look on the gas as a dynamical system defined by the coordinate x.

We have, by LAGRANGE's equations,

$$\frac{d}{dt}\frac{d\mathrm{L}}{d\dot{x}} - \frac{d\mathrm{L}}{dx} = \text{external force tending to increase } x\,;$$

or, when there is equilibrium,

$$-\int_0^1 \frac{d\mathrm{L}}{dx}\,dt = \text{average external force tending to increase } x.$$

Since x does not enter into the limits of integration,

$$\int_0^1 \frac{d\mathrm{L}}{dx}\,dt = \frac{d\overline{\mathrm{L}}}{dx}.$$

The average external force tending to increase x is $-p\mathrm{A}$, where p is the pressure per unit area, and A is the area of the piston.

Thus

$$\frac{d\overline{\mathrm{L}}}{dx} = p\mathrm{A}.$$

If the gas obeys BOYLE's Law,

$$p = \mathrm{R}\rho\theta,$$

where ρ is the density of the gas and θ the absolute temperature. If v be the volume of the gas, and if its mass be unity, we have

$$\frac{dv}{dx} = \mathrm{A}, \quad \text{and} \quad \rho = \frac{1}{v},$$

so that

$$\frac{d\overline{\mathrm{L}}}{dx} = -\mathrm{R}\theta\frac{1}{\rho}\frac{d\rho}{dx}$$

$$= -\mathrm{R}\theta\frac{d}{dx}\log\rho.$$

Now, during the changes that we contemplate, θ remains constant; hence we see that the change in $\overline{\mathrm{L}}$ is the same as the change in

$$\mathrm{R}\theta\log\frac{\rho_0}{\rho},$$

where ρ_0 is some constant density, so that we may put for the positional part of the kinetic energy, and that part of $\overline{\mathrm{V}}$ which depends on the density,

$$\overline{\mathrm{L}} = \overline{\mathrm{L}}_0 + \mathrm{R}\theta\log\frac{\rho_0}{\rho},$$

where $\overline{\mathrm{L}}_0$ is the value of $\overline{\mathrm{L}}$ when the density is ρ_0. As the energy vanishes at the zero of absolute temperature, $\overline{\mathrm{L}}_0$ will contain θ as a factor, so that we may put for the mean kinetic energy, and that part of V which depends on the density,

$$\overline{\mathrm{L}} = \theta\left(\mathrm{A} + \mathrm{R}\log\frac{\rho_0}{\rho}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where A may be a function of θ, but not of ρ

This is the value of the aforesaid part of $\overline{\mathrm{L}}$ for unit mass of the gas; if the mass of the gas were m, the value of this part of $\overline{\mathrm{L}}$ would be

$$m\theta\left(\mathrm{A} + \mathrm{R}\log\frac{\rho_0}{\rho}\right),$$

and we may treat the gas as if it were a dynamical system whose positional Lagrangian function contained the term

$$m\theta\left\{\mathrm{A} + \mathrm{R}\log\frac{\rho_0}{\rho}\right\},$$

all the variations being made at constant temperature.

We have next to consider the liquid. The expansion of solids and liquids by heat shows that there must be some terms in the expression for the energies of a solid or liquid which indicate the existence of a stress depending on the temperature. In order to find such terms, let us suppose that v is the volume of the solid or liquid at the absolute temperature θ. The dilatation per degree of temperature is

$$\frac{1}{v}\frac{dv}{d\theta}.$$

If the rate of dilatation be uniform and equal to β, the dilatation for θ degrees is $\beta\theta$, the stress required to produce this dilatation is $\kappa\beta\theta$, where κ is the bulk modulus. Now, we can show by HAMILTON's principle that there will be a stress of this amount if, in the expression for the positional kinetic energy, there is the term

$$\int_{v_0}^{v} \kappa\beta\theta\, dv,$$

where v_0 is a constant volume.

Since

$$\beta = \left(\frac{1}{v}\frac{dv}{d\theta}\right)_{p \text{ constant}},$$

$$\kappa = -\left(v\frac{dp}{dv}\right)_{\theta \text{ constant}},$$

p being the pressure to which the surface of the solid or liquid is exposed,

$$\beta\kappa = -\frac{dv}{d\theta}\frac{dp}{dv} = \left(\frac{dp}{d\theta}\right)_{v \text{ constant}};$$

so that the term in the positional kinetic energy may be written

$$\int_{v_0}^{v} \theta\left(\frac{dp}{d\theta}\right)_{v \text{ constant}} dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Thus a dynamical system with this term in the expression for the positional kinetic energy will behave like the solid or liquid so far as expansion by heat goes.

We may add to this the term

$$\mathrm{M}c_1\theta,$$

where M is the mass of the solid or liquid, and c_1 a constant, as this term will not give rise to any stresses tending to strain the body. Unless this term has different values for the different states in which the body can exist, the temperature being kept constant, it will disappear from the variation equation.

We can now solve the problem of finding the density of the saturated vapour of a liquid at any temperature.

Let us suppose that we have a mass N of the liquid and its vapour in a closed space: let ξ be the mass of the vapour, $\mathrm{N} - \xi$ that of the liquid, w_1, w_2, the mean intrinsic potential energies of unit mass of the vapour and liquid respectively, Q the volume occupied by the vapour. Then, if there is no energy due to surface-tension, electrification, and so on, the value of $\overline{\mathrm{L}}$ for the vapour, using the same notation as before, is

$$\xi\left(\mathrm{A}\theta + \mathrm{R}\theta\log\frac{\rho_0}{\rho}\right) - \xi w_1,$$

for the liquid

$$(\mathrm{N} - \xi)\, c_1\theta + \int_{v_0}^{v} \theta\frac{dp}{d\theta}dv - (\mathrm{N} - \xi)\, w_2.$$

If σ be the density of the liquid

$$v = \frac{\mathrm{N} - \xi}{\sigma};$$

we have also

$$\rho \mathrm{Q} = \xi;$$

hence the value of $\bar{\mathrm{L}}$ for the solid and liquid equals

$$\xi\left(\mathrm{A}\theta + \mathrm{R}\theta \log \frac{\rho_0 \mathrm{Q}}{\xi}\right) + (\mathrm{N} - \xi)\, c_1 \theta + \theta \int_{v_0}^{v} \frac{dp}{d\theta}\, dv - \xi w_1 - (\mathrm{N} - \xi)\, w_2.$$

Let us suppose that the mass of the vapour is increased by $\delta\xi$; then, since $\bar{\mathrm{L}}$ is stationary,

$$\frac{d\bar{\mathrm{L}}}{d\xi} = 0,$$

the temperature remaining constant. Hence we have

$$\mathrm{A}\theta + \mathrm{R}\theta \log \frac{\rho_0 \mathrm{Q}}{\xi} - \mathrm{R}\theta + \xi \mathrm{R}\, \theta \frac{1}{\mathrm{Q}} \frac{d\mathrm{Q}}{d\xi} - c_1 \theta - \frac{1}{\sigma} \frac{dp}{d\theta} - w_1 + w_2 = 0. \quad . \quad (16)$$

When the mass of the vapour increases by $\delta\xi$, the mass of the liquid diminishes by the same amount, so that the volume of the liquid diminishes by $\delta\xi/\sigma$, and therefore, since the liquid and vapour are supposed to be contained in a vessel of constant volume, the volume of the liquid increases by the same amount, so that

$$\frac{d\mathrm{Q}}{d\xi} = \frac{1}{\sigma},$$

and equation (16) becomes

$$\mathrm{A}\theta + \mathrm{R}\theta \log_\epsilon \frac{\rho_0}{\rho} - \mathrm{R}\theta\left(1 - \frac{\rho}{\sigma}\right) - c_1\theta - \frac{1}{\sigma}\, \theta \frac{dp}{d\theta} - w_1 + w_2 = 0; \quad . \quad . \quad (17)$$

or if, for brevity, we write

$$s = -\frac{1}{\mathrm{R}\sigma} \frac{dp}{d\theta} + \frac{\mathrm{A}}{\mathrm{R}} - 1 - \frac{c_1}{\mathrm{R}},$$

$$\rho = \rho_0\, \epsilon^s\, \epsilon^{-\rho/\sigma}\, \epsilon^{w_2 - w_1/\mathrm{R}\theta}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

or, since ρ/σ is very small, we have approximately, writing ρ_0' for $\rho_0 \epsilon^s$,

$$\rho = \rho_0'\, \epsilon^{w_2 - w_1/\mathrm{R}\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

[The quantity most closely related to $w_1 - w_2$ is the latent heat of evaporation, but the two quantities are not necessarily identical, for $w_1 - w_2$ is the excess of the intrinsic potential energy of unit mass of the vapour over that of unit mass of the liquid at the same temperature. The latent heat, however, at this temperature is

the amount of work required to convert unit mass of liquid into vapour. If the kinetic energy of unit mass of the liquid is the same as that of unit mass of the gas at the same temperature, the latent heat will equal $w_1 - w_2$, but if the kinetic energies are different, then, since the latent heat equals the difference between the sum of the kinetic and potential energies, it will not equal $w_1 - w_2$]*

In the above work we have assumed that the vapour obeys BOYLE's Law. If we assume that the relation between pressure, density and temperature is that given by VAN DER WAALS' formula

$$p = \frac{R\theta}{v - b} - \frac{a}{v^2},$$

where v is the reciprocal of the density, and a and b constants, we may show, in a way similar to that by which we established equation (16), that, when the vapour is in equilibrium with the fluid,

$$R\theta \log \left(\frac{\frac{1}{\rho} - b}{\frac{1}{\rho_0} - b} \right) + a\rho \left(2 - \frac{\rho}{\sigma} \right) - \left(1 - \frac{\rho}{\sigma} \right) \frac{R\theta}{1 - b\rho} - c_1\theta - \frac{1}{\sigma}\theta \frac{dp}{d\theta} - w_1 + w_2 = 0\,;$$

or, since ρ/σ is very small, we may write this equation as

$$R\theta \log \left(\frac{\frac{1}{\rho} - b}{\frac{1}{\rho_0} - b} \right) + 2a\rho - R\theta\, b\rho - L\theta - (w_1 - w_2) = 0, \quad . \quad . \quad . \quad (20)$$

neglecting b^2, and writing L for a number of constant terms.

§ 7. The method just given enables us to calculate readily the effect of slight changes in the physical conditions on the vapour-pressure. Let us take, first, the effect of surface-tension If the shape of the liquid is such that the area of its free surface changes when any of it evaporates, then we must take into account the energy due to the surface-tension. Let us suppose that the liquid is a spherical drop, whose radius is a; then we must add to the expression for the mean potential energy of the liquid the term $4\pi a^2T$, where T is the surface-tension of the liquid. In this case, using the same notation as before, and assuming BOYLE's Law, we have

$$\overline{L} = \xi \left(A\theta + R\theta \log \frac{\rho_0 Q}{\xi} \right) + (N - \xi)\, c_1\theta + \int_{v_0}^{v} \theta \frac{dp}{d\theta} dv - \xi w_1 - (N - \xi)\, w_2 - 4\pi a^2 T\,;$$

and the equation got by making the value of $\overline{L}$ stationary for a small change in ξ is

$$A\theta + R\theta \log \frac{\rho_0}{\rho} - R\theta + R\theta \frac{\rho}{\sigma} - c_1\theta - \frac{1}{\sigma}\theta \frac{dp}{d\theta} - w_1 + w_2 - 8\pi a T \frac{da}{d\xi} = 0\,; \quad (21)$$

* Paragraph substituted October, 1887

but

$$d\xi = -4\pi a^2 \sigma\, da,$$

so that

$$\frac{da}{d\xi} = -\frac{1}{4\pi a^2 \sigma},$$

and equation (21) becomes

$$A\theta + R\theta \log \frac{\rho_0}{\rho} - R\theta + R\theta \frac{\rho}{\sigma} - c_1\theta - \frac{1}{\sigma}\theta\frac{dp}{d\theta} - w_1 + w_2 + \frac{2T}{a\sigma} = 0. \quad . \quad (22)$$

Comparing this with equation (17), we see that, if $\delta\rho$ be the change in the vapour-pressure due to the curvature,

$$-R\theta\frac{\delta\rho}{\rho} + R\theta\frac{\delta\rho}{\sigma} + \frac{2T}{a\sigma} = 0,$$

or

$$R\theta\,\delta\rho = \frac{2\rho}{\sigma - \rho}\frac{T}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This coincides with the formula given by Sir WILLIAM THOMSON ('Proceedings of the Royal Society of Edinburgh,' Feb. 7, 1870; quoted in MAXWELL's 'Theory of Heat,' p. 290).

We can also prove that the density of the saturated vapour will be altered by charging drops of the liquid with electricity. For suppose the drop to be spherical and charged with a quantity e of electricity. The potential energy due to the electrification of the drop is $\frac{1}{2}e^2/a$; subtracting this from the value previously given for $\overline{L}$, we may easily prove, in the same way as before, that the change $\delta\rho$ in the vapour-density, due to the electrification, is given by the equation

$$R\theta\,\delta\rho = -\frac{1}{8\pi}\frac{e^2}{a^4}\frac{\rho}{\sigma - \rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

assuming that as the drops evaporate the electricity remains behind on the drop. This seems to be proved by BLAKE's experiments on the evaporation of electrified liquids (WIEDEMANN's 'Lehre von der Elektricitat,' vol. 4, p. 1212). We see from equation (24) that electrification diminishes the density of the saturated vapour, so that moisture might condense on a drop of water when electrified, though the same drop would evaporate if not charged with electricity. This would tend to make electrified drops of rain larger than unelectrified ones, and would probably tend to increase the size of the rain drops in a thunderstorm.

The maximum value of e/a^2 in air at atmospheric pressure is about 130 C.G.S. units in electrostatic measure, so that the maximum change in the density of the saturated vapour is given by the equation

$$R\theta\,\delta\rho = \frac{169 \times 10^2}{8\pi}\frac{\rho}{\sigma - \rho}.$$

$\rho/(\sigma - \rho)$ for water at atmospheric pressure is about $\frac{1}{800}$, so that δp, the alteration in pressure, which equals $R\theta\,\delta\rho$, is equal to

$$\frac{-169 \times 10^2}{8\pi \times 800},$$

which is roughly about ·87, so that the maximum change in the vapour-pressure which can be produced by electrification is about $\frac{1}{14000}$ of the vapour-pressure of water at 15° C. In sulphuric acid the ratio would be very much greater.

We can calculate in a similar way the alteration in the vapour-pressure produced by any alteration in the state of the liquid. All we have to do is to calculate the change in the value of $\overline{L}$ due to this alteration. If this change be χ, then we may prove, just as before, that

$$R\theta\,\delta\rho = \frac{d\chi}{d\xi}\frac{\rho\sigma}{\sigma - \rho}. \qquad (25)$$

It should be noticed that $\delta\rho$ and $d\chi/d\xi$ are of the same sign, so that the presence of any kind of energy which causes $\overline{L}$ to increase as evaporation goes on will facilitate the evaporation. Thus, in the case of surface-tension, the potential energy due to the surface-tension diminishes as evaporation goes on . this corresponds to an increase in $\overline{L}$, so that the surface-tension will facilitate the evaporation; again, in the electrical case, the potential energy due to the electrification increases as evaporation goes on . this corresponds to a decrease in $\overline{L}$, so that the electrification will tend to stop the evaporation. These are only special cases of a general principle, of which we shall find frequent illustrations in the subsequent work.

Dissociation.

§ 8. Another problem to which the method can be applied is that of a gas partly dissociated into two components. Let us suppose that we have a quantity of gas contained in a vessel whose volume is V, and that part of it is in its normal condition, which we shall call A, while the molecules of the rest of the gas have been split up . we shall call this state of the gas B. Let ξ, η, be the masses of the gases in the states A and B respectively. Then the value of $\overline{L}$ for the gas in the state A is, by the investigation on p. 487,

$$\xi\left(A_1\theta + R_1\theta\log\frac{\rho_0}{\rho} - v_1\right),$$

where A_1 is a constant, ρ the density of the gas in the state A, θ the absolute temperature, v_1 the mean potential energy of unit mass of the gas, $R_1 = p/\rho\theta$, p being the pressure of the gas.

Since $\rho v = \xi$, we may write this as

$$\xi\left(A_1\theta + R_1\theta \log \frac{\rho_0 V}{\xi} - v_1\right). \qquad (26)$$

The value of $\overline{L}$ for the gas in state B is similarly

$$\eta\left(A_2\theta + R_2\theta \log \frac{\rho_0' V}{\eta} - v_2\right), \qquad (27)$$

where letters with the suffix 2 denote for gas in the state B quantities corresponding to those denoted by the same letters with the suffix 1 for gas in the state A. The value of $\overline{L}$ for the whole system equals the sum of (26) and (27).

Let us suppose that a mass $\delta\xi$ of the gas in the state A gets decomposed; then, if $\delta\eta$ be the increase in the mass of the gas in the state B, we have $\delta\eta = \delta\xi$. The change in the positional part of $\overline{L}$ for the system is

$$-\delta\xi\left(A_1\theta + R_1\theta \log \frac{\rho_0 V}{\xi} - R_1\theta - V_1\right) + \delta\eta\left(A_2\theta + R_2\theta \log \frac{\rho_0' V}{\eta} - R_2\theta - V_2\right);$$

but $\delta\eta = \delta\xi$, and when the system is in a steady state the variation of $\overline{L}$ vanishes, so that we have

$$\{A_1 - A_2 + (R_1 - R_2)\}\theta + R_1\theta \log \frac{\rho_0 V}{\xi} - R_2\theta \log \frac{\rho_0' V}{\eta} - (V_1 - V_2) = 0. \qquad (28)$$

If two of the molecules of the gas in the state B make up one of those in the state A, that is, if the gas on dissociation splits up into two components, we have

$$R_2 = 2R_1.$$

Making this substitution, equation (28) becomes

$$R_1 \log \frac{\eta^2\alpha}{\xi V} + A_1 - A_2 + R_1 = \frac{V_1 - V_2}{\theta}, \qquad (29)$$

where α is a constant.

If the density of the mixture of normal and dissociated gas be δ, and the density of the normal gas at the same density and pressure d, then we have

$$\frac{\delta}{d} = \frac{\xi + \eta}{\xi + 2\eta},$$

so that

$$\frac{2\delta - d}{d - \delta} = \frac{\xi}{\eta},$$

and

$$\frac{2\delta - d}{\delta} = \frac{\xi}{\xi + \eta};$$

again

$$V\delta = \xi + \eta;$$

so that

$$\xi = V(2\delta - d),$$

$$\eta = V(d - \delta).$$

Substituting these values for ξ and η in equation (29), we get

$$R_1 \log \frac{\alpha(d-\delta)^2}{2\delta - d} + A_1 - A_2 + R_1 = \frac{V_1 - V_2}{\theta}. \qquad (30)$$

This formula agrees substantially in form with one given by Professor WILLARD GIBBS in his paper on the "Equilibrium of Heterogeneous Substances" ('Transactions of the Connecticut Academy of Arts and Sciences,' vol. 3, 1874–8, p. 239). In his paper on the vapour-densities of nitrogen tetroxide, formic acid, acetic acid, and perchloride of phosphorus ('American Journal of Science and Arts,' vol. 18, 1879, p. 277), Professor GIBBS compares the density given by his formula with those found by various experimenters for the substances mentioned in the title of his paper, and he finds that the two sets of values agree very closely.

Liquefaction and Solution.

§ 9. We can apply the Hamiltonian principle to cases when a solid and liquid are in equilibrium in presence of each other, as, for example, when we have a mixture of ice and water, or a salt in a saturated solution of a liquid in which it can dissolve.

Let us first consider the case of liquefaction and take the case of the melting of ice as the typical one. We must first find the value of $\overline{L}$ for a mixture of ice and water.

The positional part of $\overline{T}$ for a solid or a liquid contains the term

$$-\theta \int_{v_0}^{v} \frac{\partial p}{\partial \theta}\, dv,$$

where v is the volume of the solid or liquid, v_0 is a constant, and $\partial p/\partial\theta$ is obtained on the supposition that v is constant. Thus the positional part of $\overline{L}$ for the mixture of ice and water equals

$$-\left(\theta\int_{v_0}^{v}\frac{\partial p}{\partial\theta}\,dv\right)_{\text{ice}}-\left(\theta\int_{v_0'}^{v'}\frac{\partial p}{\partial\theta}\,dv\right)_{\text{wa}}-\overline{\mathrm{V}}, \quad \ldots \ldots \quad (31)$$

where $\overline{\mathrm{V}}$ is the mean potential energy of the mixture. By HAMILTON'S principle this expression must be stationary when there is equilibrium, the temperature remaining constant. Let us suppose unit mass of ice to melt: the change in the first terms is

$$\theta\int_{v_1'}^{v_1}\frac{\partial p}{\partial\theta}\,dv,$$

where v_1' is the volume of unit mass of water, and v_1 the volume of unit mass of ice. The increase in the potential energy is equal to the latent heat under the pressure p of water, so that HAMILTON'S principle gives

$$\theta\int_{v_1'}^{v_1}\frac{\partial p}{\partial\theta}\,dv=\lambda. \quad \ldots \ldots \ldots \quad (32)$$

Now PLANCK has shown * that

$$\int_{v_1'}^{v_1}\frac{\partial p}{\partial\theta}\,dv=\frac{d\mathrm{P}}{d\theta}(v_1-v_1'), \quad \ldots \ldots \ldots \quad (33)$$

where P, the pressure to which both the ice and water are subjected, is regarded as a function of the temperature alone.

Hence we have, by equation (32),

$$\theta\frac{d\mathrm{P}}{d\theta}(v_1-v_1')=\lambda, \quad \ldots \ldots \ldots \quad (34)$$

the well-known equation connecting the change in the melting point of ice with the change in the pressure to which it is subjected. In this equation λ is the increase in the potential energy when unit mass of the ice melts. It will depend to some extent upon external circumstances; thus, if the water is of such a shape that the area of its free surface changes when ice melts, then, on account of the energy due to surface-tension, λ will depend upon the change in the surface. As the volume diminishes as the ice melts, the surface will in general diminish, so that the energy due to this cause will be diminished by the liquefaction, and the effect of pressure upon the freezing-point increased. Since the volume changes, work is done by or against the external pressure. thus λ will be a function of the pressure. If λ_0 be the value of λ when the pressure is zero, λ the value when the pressure is p, we may easily prove that

$$\lambda=\lambda_0-p(v_1-v_1'),$$

so that λ diminishes as the pressure increases, and the effect on the freezing-point of a given increment of pressure will increase as the pressure increases.

* 'WIEDEMANN'S Annalen,' vol. 13, p. 541.

Solution.

§ 10. The next case we shall consider is that of a saturated salt solution.

Let w be the mean value of $\partial p/\partial\theta$ (v constant) at the temperature θ for the salt, w' the value of the corresponding quantity for the solution; let v be the volume of the salt, v' that of the solution, q the mass of the salt, q' that of the solution; V_1 the mean potential energy of unit mass of the salt, V_2 that of unit mass of the solution. Then, if there is no energy due to strain, electrification, &c., the positional part of $\overline{L}$ will be equal to

$$\theta wv + \theta w'v' - qV_1 - q'V_2. \qquad (35)$$

When there is equilibrium this must be stationary, so that, if we suppose a small quantity $\delta q'$ of the salt to melt, the value of $\overline{L}$ must remain unaltered.

If σ and ρ are the densities of the salt and the solution respectively, then when the mass $\delta q'$ of the salt melts the changes in the volume of the salt and the solution are given by

$$\left.\begin{aligned}\delta v &= -\delta q'/\sigma \\ \delta v' &= \delta q_1'/\rho + q'\frac{d}{dq_1}\left(\frac{1}{\rho}\right)\delta q'\end{aligned}\right\}. \qquad (36)$$

As the properties of the fluid may alter with the amount of salt dissolved, we must regard w' and V_2 as functions of q. Remembering this, we see that the change in the value of $\overline{L}$, when a mass $\delta q'$ of the salt dissolves, is

$$\delta q'\left\{-\theta\frac{w}{\sigma} + \theta\frac{w'}{\rho} + \theta w'q'\frac{d}{dq'}\left(\frac{1}{\rho}\right) + \theta v'\frac{dw'}{dq'} + V_1 - V_2 - q'\frac{dV_2}{dq'}\right\},$$

and this, by the Hamiltonian principle, must vanish. Hence, equating the quantity inside the brackets to zero, we have

$$\frac{w'}{\rho} + w'q'\frac{d}{dq'}\frac{1}{\rho} + v'\frac{dw'}{dq'} = \frac{w}{\sigma} + \frac{1}{\theta}\left(V_2 - V_1 + q'\frac{dV_2}{dq'}\right). \qquad (37)$$

This equation would determine ρ, the density of the saturated solution, if we knew how w', V_2, and ρ depended upon the amount of salt in unit volume of the solution; as, however, we have not this information, we cannot reduce this formula to numbers. We can, however, use it to calculate the effect upon the density of the solution of any change in the external circumstances. In the preceding investigation we have supposed that the energy possessed by the salt and the solution was all intrinsic as it were, and that none of it depended upon strain, electrification, magnetisation, and so on.

The methods of calculating the potential energy due to strain, &c., are, of course, well known. The corresponding terms in the kinetic energy can be calculated in the following way.

We saw, on p. 480, that if y is a coordinate of any type, and P a quantity such that, when y is increased by δy, the energy in the system is increased by $\mathrm{P}\delta y$, then

$$\frac{d\mathrm{T}}{dy} = -\theta\frac{d\mathrm{P}}{d\theta}.$$

P may be regarded as a force of the type y, so that, if this force depends upon the temperature, there will be a term in the mean kinetic energy equal to

$$-\theta\int\frac{d\mathrm{P}}{d\theta}dy,$$

so that the expression for the positional part of the kinetic energy equals

$$\theta\int\frac{dp}{d\theta}dv - \theta\Sigma\int\frac{d\mathrm{P}}{d\theta}dy;$$

the summation being extended over all the types of coordinates. The term in the potential energy corresponding to the coordinate of type y will be

$$\int\mathrm{P}_0\,dy,$$

where P_0 is the part of P which is independent of the temperature.

By the above equations we can calculate the term in the expressions for both the kinetic and potential energies involving the coordinates of any type. If, however, we only require to calculate the value of $\overline{\mathrm{L}}$, a much simpler process is applicable. For when the system is in a steady state

$$\overline{\mathrm{P}} = \frac{d\overline{\mathrm{L}}}{dy},$$

so that we have

$$\overline{\mathrm{L}} = \int\mathrm{P}\,dy. \qquad (38)$$

Let us now apply these equations to calculate the effect which any change in the external circumstances has upon the solubility of a salt Let us, for example, consider the effect of capillarity. Then, if the process of solution alters the volume of the mixture of salt and water, it will in general alter the surface, and so alter the energy due to the surface-tension. If it increases the energy, the surface-tension will tend to stop the solution; if it diminishes the energy, the surface-tension will facilitate the solution. The energy due to the surface-tension will change even though the volume remains unaltered, if the surface-tension depends upon the quantity of salt dissolved in the liquid; if the surface-tension diminishes as the salt dissolves, then the surface-tension will facilitate the solution; if it increases, it will tend to prevent the salt dissolving. This result, as well as the preceding, follows from equation (37). We see, by that equation, that anything which makes $d(q'\mathrm{V}_2)/dq'$ increase will increase

the difference between ω'/ρ for the solution, and the same quantity for the salt, and therefore probably diminish the quantity of salt in the solution; for it seems natural to expect that, the more salt we have dissolved in the solution, the more nearly will the properties of the solution approximate to those of the salt Any additional energy which increases as solution goes on makes $d(q'\mathrm{V}_2)/dq'$ positive, and therefore tends to stop the solution

To calculate the magnitude of the effect due to capillarity, let us suppose that the mixture of salt and solution is in the form of a spray of spherical drops of radius a. The additional term in $q'\mathrm{V}_2$ due to the surface-tension will be

$$4\pi a^2\mathrm{T},$$

where T is the surface-tension. We have then, from (37),

$$\theta\delta\left(\frac{w'}{\rho}+w'q'\frac{d}{dq'}\frac{1}{\rho}+v'\frac{dw'}{dq'}\right)=8\pi a\mathrm{T}\frac{da}{dq'}+4\pi a^2\frac{d\mathrm{T}}{dq'};$$

but

$$\tfrac{4}{3}\pi a^3=\frac{q}{\sigma}+\frac{q'}{\rho},$$

so that

$$4\pi a^2\frac{da}{dq'}=\frac{1}{\rho}-\frac{1}{\sigma}+q'\frac{d}{dq'}\left(\frac{1}{\rho}\right),$$

and therefore

$$\theta\delta\left(\frac{w'}{\rho}+w'q'\frac{d}{dq'}\frac{1}{\rho}+v'\frac{dw'}{dq'}\right)=\frac{2}{a}\left(\frac{1}{\rho}-\frac{1}{\sigma}+q'\frac{d}{dq'}\frac{1}{\rho}\right)\mathrm{T}+\frac{3}{a}\left(\frac{q}{\sigma}+\frac{q'}{\rho}\right)\frac{d\mathrm{T}}{dq'}. \quad (39)$$

We can determine experimentally the value of all the quantities on the right-hand side of this equation, and, if we know the quantity in brackets on the left-hand side as a function of ρ, we can at once determine the change in the density of the solution. We can express this in a way which more readily admits of comparison with experiment Suppose that the existence of the surface-tension causes as much salt to be absorbed at θ as would be absorbed if there were no surface-tension at $\theta+\delta\theta$; then, by comparing equations (37) and (39), we see that

$$\frac{\delta\theta}{\theta}=-\frac{\left\{\frac{2}{a}\left(\frac{1}{\rho}-\frac{1}{\sigma}+q'\frac{d}{dq'}\frac{1}{\rho}\right)\mathrm{T}+\frac{3}{a}\left(\frac{q}{\sigma}+\frac{q'}{\rho}\right)\frac{d\mathrm{T}}{dq'}\right\}}{\mathrm{V}_2-\mathrm{V}_1+q'\frac{d\mathrm{V}_2}{dq'}}, \quad (40)$$

but the denominator is the increase of potential energy when unit mass of the salt dissolves. We can measure this by the cooling. Let it be denoted by λ; then

$$\lambda\frac{\delta\theta}{\theta}=-\left\{\frac{2}{a}\left(\frac{1}{\rho}-\frac{1}{\sigma}+q'\frac{d}{dq'}\frac{1}{\rho}\right)\mathrm{T}+\frac{3}{a}\left(\frac{q}{\sigma}+\frac{q'}{\rho}\right)\frac{d\mathrm{T}}{dq'}\right\}. \quad (41)$$

Hence we see that the density of the saturated solution depends upon the extent of surface of the liquid, so that, if we alter the surface, we may either deposit or absorb salt. Effects of this kind have, I believe, been observed by chemists.

Let us now proceed to investigate the effects of pressure on the solubility. Let us suppose that the mixture of salt and solution is under a pressure p per unit area. The effect of the pressure will be due to two causes. The first is the change of volume which accompanies solution, and which causes the process of solution to be accompanied by an expenditure or absorption of work; the second cause is that the energy due to the strain in unit mass of the salt is not, in general, the same as that of an equal mass of the solution, so that there will be a change in the potential energy when the salt dissolves. If κ be the bulk modulus for the salt, κ' for the solution, the rest of the notation being the same as before, then the change in the potential energy due to strain is, when unit mass of salt dissolves,

$$\tfrac{1}{2}p^2\left\{\frac{1}{\kappa'\rho}+v'\frac{d}{dq'}\frac{1}{\kappa'}+\frac{v'}{\kappa'}\frac{d}{dq'}\frac{1}{\rho}-\frac{1}{\sigma\kappa}\right\};$$

the increase in the potential energy due to the change in volume is

$$p\frac{d}{dq'}(v+v'),$$

so that we have, by equation (37),

$$\theta\delta\left(\frac{w}{\rho}+w'q'\frac{d}{dq'}\frac{1}{\rho}+v'\frac{dw'}{dq'}\right)=p\frac{d}{dq'}(v+v')+\tfrac{1}{2}p^2\left(\frac{1}{\kappa'\rho}+v'\frac{d}{dq'}\frac{1}{\kappa'}+\frac{v'}{\kappa'}\frac{d}{dq'}\frac{1}{\rho}-\frac{1}{\sigma\kappa}\right); \quad (42)$$

or, if the change due to pressure be the same as the change due to an increase $\delta\theta$ in the temperature, we have

$$\lambda\frac{\delta\theta}{\theta}=-p\frac{d}{dq'}(v+v')-\tfrac{1}{2}p^2\left(\frac{1}{\kappa'\rho}+v'\frac{d}{dq'}\frac{1}{\kappa'}+\frac{v'}{\kappa'}\frac{d}{dq'}\frac{1}{\rho}-\frac{1}{\sigma\kappa}\right). \quad (43)$$

Thus we see that the effect of pressure consists of two parts, one of which is proportional to the pressure, and the other to the square of it. It will be well to try to gain some idea of the relative magnitude of these effects. Let us suppose that the pressure is n atmospheres, that is, in C.G.S. units, $n\times 10^6$. We do not know accurately the value of the second term on the left-hand side of equation, but, since κ for water is about 2×10^{10}, the multiplier of p^2 will be of about the order 10^{-11}, so that the ratio of the second term to the first will be of the order

$$\frac{n\times 10^{-5}}{\frac{d}{dq'}(v+v')}.$$

If the alteration in the volume of the saturated solution amounts to a cubic millimetre per gramme of salt dissolved, $d(v+v')/dq'$ is 10^{-3}, so that

$$\frac{n \times 10^{-5}}{\frac{d}{dq'}(v+v')} = n/100,$$

so that with this amount of alteration in volume the terms proportional to the pressure will be the most important for pressures up to 100 atmospheres, while for pressures greater than this the term proportional to the square of the pressures will be the most important. In the case of a solution of sal-ammoniac the increase in volume is enormously greater than that stated, so that for such a salt the effect of pressure on the solubility ought to be very nearly proportional to the pressure. This agrees with SORBY's experiments on the solubility of certain salts.*

In this case the effect of pressure is expressed by the equation

$$\lambda \frac{\partial\theta}{\theta} = -p \frac{d(v+v')}{dq'}. \quad \quad \quad (44)$$

We can test the formula by comparing it with the result of SORBY's experiments on the effect of pressure upon the solubility. The two salts we shall take are potassium sulphate and sodium chloride, as we can get approximate values for these salts for all the quantities involved in equation (44).

According to J. THOMSEN, the heat absorbed when 174 grammes of K_2SO_4 dissolve in water is 6380 gramme-degree units, so that the heat absorbed for one grain of K_2SO_4 will be about 36 gramme-degrees or $36 \times 4{\cdot}2 \times 10^7$ C.G.S. units. We may take this as an approximation to the value of λ, though it must be remembered that λ is the heat absorbed when one gramme of salt is dissolved in a nearly saturated solution.

According to SORBY, when K_2SO_4 crystallises out of a saturated solution, the volume increases from 100 to 134, so that, when a cubic centimetre of salt dissolves, the volume diminishes by about ·25 c c. The specific gravity of K_2SO_4 is about 2 5, so that, when one gramme of salt dissolves, the volume diminishes by ·1 c.c., and $d(v+v')\,dq' = -10^{-1}$, so that for 100 atmospheres the change in temperature required to produce the same change in solubility is at the temperature 15° C. given by the equation

$$\delta\theta = \frac{288 \times 10^8 \times 10^{-1}}{36 \times 4{\cdot}2 \times 10^7}$$
$$= 2^\circ.$$

SORBY found that the K_2SO_4 dissolved increased by about 3 per cent. According to KOPP, 100 parts of water dissolve 8·36 parts of K_2SO_4 at 0°, and ·1741 part

* SORBY, 'Roy. Soc Proc,' vol. 12, 1863, p 538

for every degree above 0°, so that at 15° C an increase of 2 per cent would cause an increase of 348 part in 10·9, which is about 3·2 per cent., so that in this case the agreement is almost closer than we could have expected

In the case of NaCl we have, getting the data from the same sources as before—

$$\lambda = 20 \times 4{\cdot}2 \times 10^7,$$
$$d(v + v')/dq' = 05\,;$$

hence the change in temperature required to produce the same change as 100 atmospheres at 15°

$$= \frac{288 \times 10^8 \times 5 \times 10^{-2}}{20 \times 42 \times 10^7}$$
$$= 1{\cdot}6^\circ \text{ approximately.}$$

Now, according to the curve of solubility of NaCl given in OSTWALD's 'Lehrbuch der Allgemeinen Chemie,' vol. 1, p 380, about 34 parts dissolve at 15°, and the increase is about $\frac{1}{12}$ of a part per degree Centigrade, so that for 1·6° the increase would be about 13 in 34 · this is 40 per cent. The value found by SORBY was ·419 per cent, so that the agreement is again very close.

Chemical Combination.

§ 11. We can apply HAMILTON's principle to the case of chemical combination. Let us in the first place take cases of the type studied by GULDBERG and WAAGE in their theory of chemical combination ('Études sur les Affinités Chimiques').

In these cases there is equilibrium between various chemical actions which tend to reverse each other : a good example of such cases is that of a mixture of dilute solutions of sulphuric and nitric acids, sodium nitrate, and sodium sulphate. When the sulphuric acid acts on the sodium nitrate it produces nitric acid and sodium sulphate, while nitric acid by its action on sodium sulphate produces sulphuric acid and sodium nitrate. The problem is, given four substances of this kind, to find the quantity of each when there is equilibrium. Let us begin with the case of four gases, which we will call A, B, C, D, such that A by its action on B produces C and D, while C by its action on D produces A and B.

Let $p\xi$, $q\eta$, $r\zeta$, $s\epsilon$, be the number of molecules of A, B, C, D, respectively, when p, q, r, s, are the numbers of molecules in equivalent molecules. By equivalent molecules we mean molecules, or groups of molecules, such that, $\{A\}$ being the equivalent molecule of the gas A, with a corresponding notation for the others, the chemical action which goes on may be expressed by the equation

$$\{A\} + \{B\} = \{C\} + \{D\}.$$

Let us take as an example the case mentioned above, where the chemical action is expressed by the equation

$$H_2SO_4 + 2NaNO_3 = 2HNO_3 + Na_2SO_4;$$

here the equivalent molecules are H_2SO_4, $2NaNO_3$, $2HNO_3$, and Na_2SO_4, and, if A, B, C, D, denote sulphuric acid, sodium nitrate, nitric acid, and sodium sulphate respectively, $p = 1$, $q = 2$, $r = 2$, $s = 1$

Let m_1, m_2, m_3, m_4, denote the masses of the molecules of A, B, C, D, respectively. Then, if these are gases, by what we have proved before (p 486), the value of $\overline{L}$ equals

$$\theta\left[m_1p\xi\left(c_1 + R_1\log\frac{\rho_0 Q}{m_1p\xi}\right) + m_2q\eta\left(c_2 + R_2\log\frac{\rho_0' Q}{m_2q\eta}\right)\right.$$
$$\left. + m_3r\zeta\left(c_3 + R_3\log\frac{\rho_0'' Q}{m_3r\zeta}\right) + m_4s\epsilon\left(c_4 + R_4\log\frac{\rho_0''' Q}{m_4s\epsilon}\right)\right] - w, \quad . \quad . \quad (45)$$

where w is the mean potential energy of the four gases, and the remaining notation is the same as that on p. 486

The quantities ξ, η, ζ, ϵ, are not independent of each other; in fact, there are three relations between them. Thus suppose, for example, that $\{A\}$ consists of the two components α, β; $\{B\}$ of γ, δ, $\{C\}$ of α and γ; $\{D\}$ of β and δ, then the chemical reaction is expressed by the equation

$$(\alpha\beta) + (\gamma\delta) = (\alpha\gamma) + (\beta\delta),$$

so that we have evidently

$$\xi + \zeta = \text{a constant},$$
$$\xi + \epsilon = \quad . \; . \; . \; . \; . \quad ,$$
$$\eta + \zeta = \quad . \; . \; . \; . \; . \; ,$$

so that

$$d\xi = d\eta = - d\zeta = - d\epsilon. \quad . \quad . \quad . \qquad . \quad . \quad (46)$$

Thus, if ξ be increased by $d\xi$, the change in $\overline{L}$ equals

$$d\xi\left[\theta\left\{m_1p\left(c_1 + R_1\log\frac{\rho_0 Q}{m_1p\xi} - R_1\right) + m_2q\left(c_2 + R_2\log\frac{\rho_0' Q}{m_2q\eta} - R_2\right)\right.\right.$$
$$\left.\left. - m_3r\left(c_3 + R_3\log\frac{\rho_0'' Q}{m_3r\epsilon} - R_3\right) - m_4s\left(c_4 + R_4\log\frac{\rho_0''' Q}{m_4s\epsilon} - R_4\right)\right\} - \frac{dw}{d\xi}\right];$$

and by HAMILTON'S principle the quantity in square brackets must vanish when there is equilibrium.

For perfect gases (*i.e.*, gases which obey BOYLE'S Law)

$$R_1m_1 = R_2m_2 = R_3m_3 = R_4m_4. \quad . \quad . \quad . \quad . \quad (47)$$

Let each of these quantities equal K; then the equation of equilibrium becomes

$$\theta\left\{m_1p_1c_1 + m_2qc_2 - m_3rc_3 - m_4sc_4 + \mathrm{K}\left(p\log\frac{\rho_0\mathrm{Q}}{m_1p\xi} + q\log\frac{\rho_0'\mathrm{Q}}{m_2q\eta} - r\log\frac{\rho_0''\mathrm{Q}}{m_3r\zeta}\right.\right.$$
$$\left.\left. - s\log\frac{\rho_0'''\mathrm{Q}}{m_4s\epsilon}\right) - (p+q-r-s)\right\} = \frac{dw}{d\xi}. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

We see from this equation that anything which increases $d\omega/d\xi$ will increase $\log \rho_0\mathrm{Q}/m_1p\xi +$ . ., and so will diminish ξ, so that, if there is any kind of potential energy which increases as ξ increases, the value of ξ when there is equilibrium will be smaller than it would have been if this energy had not been present, or, in more general terms, any circumstance which causes the potential energy to increase as chemical action goes on tends to stop the action, while, if it causes the energy to diminish, it will facilitate the action.

Equation (47) may be written

$$\frac{\zeta^r\epsilon^s}{\xi^p\eta^q} = \mathrm{C}\,e^{\frac{dw}{d\xi}\frac{1}{\mathrm{K}\theta}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

where C is independent of $\xi\eta$, $\zeta\epsilon$.

In the case of gases combining in a vessel of constant volume, and when there is no action on the sides of the vessel, $d\omega/d\xi$, being the increase in the potential energy when one equivalent of C acts on one of D, to produce one each of A and B, may be measured by the heat developed in the reverse process, that is, when an equivalent of A acts on one of B to produce one each of C and D. Let us call this quantity of heat (measured in mechanical units) H; then

$$\frac{\zeta^r\epsilon^s}{\xi^p\eta^q} = \mathrm{C}\,e^{\mathrm{H}/\mathrm{K}\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

If H be positive, and θ zero, then either ξ or η must be zero, that is, the chemical action which is attended by the production of heat will go on as far as possible. This is BERTHELOT's Law of Maximum Work, and we see from the above expression that it holds at the zero of absolute temperature, but only then. Equation (49) also shows that the tendency of any chemical reaction to take place is greater, the larger the amount of heat developed by it.

If all the equivalents contain the same number of molecules, we may put $p = 1$, $q = 1$, $r = 1$, $s = 1$, and equation (50) takes the simple form

$$\frac{\zeta\epsilon}{\xi\eta} = \mathrm{C}\,e^{\mathrm{H}/\mathrm{K}\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

GULDBERG and WAAGE put $\zeta\epsilon = \kappa\xi\eta$, where κ is a constant. This agrees with equation (50) if the temperature remains constant. Equation (50) shows how the

equilibrium varies with the temperature, and shows that, the lower the temperature, the further that action which is attended by the evolution of heat goes on, and that the equilibrium will vary more quickly with the temperature when the heat developed by the reaction is great than when it is small. By determining the state of equilibrium at two different temperatures, we could determine H.

Since, by the kinetic theory of gases,

$$K\theta = R_1 m_1 \theta = p/N,$$

where p is the pressure and N the number of molecules, we see that $K\theta$ is one-third of the mean energy of the molecules at the temperature θ. In many cases of chemical combination the heat developed by the combination of the gases is enormously greater than that required to raise their temperature through 300° or 400°, and in these cases $H/K\theta$ will be very large, so that the combinations will nearly obey BERTHELOT's Law at moderate temperatures But this law will not nearly hold when only a small quantity of heat is developed in the reaction.

Equation (49) only agrees with that given by GULDBERG and WAAGE when the number of molecules in the equivalent is the same in each of the gases; and, if we look at the subject from another point of view, we shall also see reasons for supposing that GULDBERG's and WAAGE's equation is not likely to hold when the equivalents contain different numbers of molecules. Let us take first the case where the molecule and the equivalent molecule are identical. Then, calling the four substances A, B, C, D, as before, combinations will take place by a molecule of A pairing with one of B. The number of collisions in unit time between the A and B molecules is proportional to $\xi\eta$, and if combination takes place in a certain fraction of the number of cases of collision the number of A and B molecules which disappear in unit time through this combination, or, what is the same thing, the number of C and D molecules produced, is

$$p\xi\eta,$$

where p is a constant.

In a similar way we may show that the number of C and D molecules disappearing by their combination to form A and B molecules is

$$q\zeta\epsilon,$$

where q is again a constant quantity.

When there is equilibrium the number of A and B molecules which disappear must equal the number which appear, so that in this case

$$p\xi\eta = q\zeta\epsilon,$$

which, so long as the temperature is constant, agrees with equation (49) and with GULDBERG's and WAAGE's equation.

Let us now suppose that the equivalents of B and D each contain two molecules.

In this case, for chemical combination to take place, one molecule of A must come into collision with two of B simultaneously The number of such collisions is proportional to $\xi\eta^2$, so that the number of equivalents of A and B destroyed and of C and D produced by this combination will be

$$p'\xi\eta^2,$$

where p' is a constant.

In a similar way we can show that the number of equivalents of C and D destroyed and of A and B produced in unit time by the combination of C and D is

$$q'\zeta\epsilon^2,$$

where q' is a constant

In the state of equilibrium these quantities must be equal, so that

$$p'\xi\eta^2 = q'\zeta\epsilon^2,$$

which, so long as the temperature remains constant, agrees with equation (49), but not with GULDBERG and WAAGE's equation.

In order to illustrate some important points connected with the theory of the combination of gases, we will consider the simpler case when two gases, A and B, combine to form a third, C, while C again splits up to form A and B. A particular case of this is dissociation, which we have already considered. Let ξ, η, ζ, be the number of equivalents of A, B, C, respectively, and let m_1, m_2, m_3, be the masses of the molecules of A, B, C.

Then, with the same notation as the last case, the value of $\overline{\mathrm{L}}$ is

$$\theta\left\{m_1 p\xi\left(c_1 + \mathrm{R}_1 \log\frac{\rho_0\mathrm{Q}}{m_1 p\xi}\right) + m_2 q\eta\left(c_2 + \mathrm{R}_2 \log\frac{\rho_0'\mathrm{Q}}{m_2 q\eta}\right)\right.$$
$$\left. + m_3 r\zeta\left(c_3 + \mathrm{R}_3 \log\frac{\rho_0''\mathrm{Q}}{m_3 r\zeta}\right)\right\} - w,$$

where w is the mean potential energy of the three gases. We have, as before,

$$d\xi = d\eta = -d\zeta;$$

so that the condition that the value of $\overline{\mathrm{L}}$ should be unaltered when ξ is increased by $d\xi$ gives

$$\theta\left\{m_1 p\,(c_1 - \mathrm{R}_1) + m_2 q\,(c_2 - \mathrm{R}_2) - m_3 r\,(c_3 - \mathrm{R}_3) + m_1 p\,\mathrm{R}_1 \log\frac{\rho_0\mathrm{Q}}{m_1 p\xi}\right.$$
$$\left. + m_2 q\,\mathrm{R}_2 \log\frac{\rho_0'\mathrm{Q}}{m_2 q\eta} - m_3 r\,\mathrm{R}_3 \log\frac{\rho_0''\mathrm{Q}}{m_3 r\zeta}\right\} = \frac{dw}{d\xi}. \quad \ldots \ldots \quad (52)$$

For perfect gases

$$\mathrm{R}_1 m_1 = \mathrm{R}_2 m_2 = \mathrm{R}_3 m_3 = \mathrm{K}.$$

Making this substitution, the equation of equilibrium becomes

$$\theta\Big\{m_1 p_1 c_1 + m_2 q c_2 - m_3 r c_3 + \mathrm{K}(p + q - r) + \mathrm{K}\left(p \log \frac{\rho_0 \mathrm{Q}}{m_1 p \xi} + q \log \frac{\rho_0' \mathrm{Q}}{m_2 q \eta} - r \log \frac{\rho_0'' \mathrm{Q}}{m_3 r \zeta}\right)\Big\} = \frac{dw}{d\xi}. \quad . \quad . \quad . \quad . \quad (53)$$

This equation may be written

$$\frac{\xi^p \eta^q}{\zeta^r} = \mathrm{CQ}^{p+q-r} e^{\frac{dw}{d\xi}\frac{1}{\mathrm{K}\theta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

If the combination takes place without alteration in the number of molecules,

$$p + q = r.$$

In this case the equilibrium state is independent of the volume of the vessel Q in which a given mass of gas is contained.

If $p + q > r$, that is, if the number of molecules after combination is less than that before, Q^{p+q-r} will increase with Q, so that $\xi^p \eta^q/\zeta^r$ will be larger, the greater the value of Q, so that for a given quantity of the gases there will not be so much combination in a large vessel when the pressure is small as in a smaller one when the pressure is large. If, on the other hand, $p + q < r$, then the amount of chemical combination will be greater at low than at high pressures.

We see from equation (54) that anything which affects the value of $d\omega/d\xi$ will affect the amount of the combination which takes place; anything which causes the potential energy to increase as chemical combination goes on, *i.e.*, which tends to make $d\omega/d\xi$ negative, increases, by equation (54), the value of $\xi^p \eta^q/\zeta^r$, that is, it increases the ratio of the number of uncombined atoms to the combined ones, and so tends to stop the combination; while, on the other hand, anything which makes the potential energy diminish as chemical combination goes on, since it tends to increase $d\omega/d\xi$, diminishes the ratio of the number of uncombined atoms to the number of combined ones, and so facilitates the combination.

If we define the coefficient of affinity, ω, of the gases A and B, to be the value of the steady state of

$$\frac{\zeta^r}{\xi^p \eta^q},$$

then

$$\omega = \mathrm{CQ}^{p+r-r} e^{-\frac{dw}{d\xi}\frac{1}{\mathrm{K}\theta}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

and if the potential energy be increased by $\delta\omega$, the corresponding increment $\delta\omega$ in the coefficient of affinity is given by the equation

$$\frac{\delta\omega}{\omega} = -\frac{1}{\mathrm{K}\theta}\frac{d\,\delta w}{d\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

an equation which connects the alteration in the coefficient of affinity with the alteration in the potential energy.

The layers of gas condensed on the surfaces of solids in contact with the gas may, perhaps, be looked upon as corresponding to the surface films of liquids, and as possessing energy different from that possessed by the same volume of gas when not attached to the sides of the vessel. In this case part of the energy of the gas would depend upon the surface of the solids in contact with it, just as in a liquid the existence of surface-tension makes part of the energy of a fluid proportional to its surface. It is, perhaps, worthy of notice that, according to the vortex ring theory of gases, part of the energy of a gas at a given pressure and volume depends upon the surface (J. J. THOMSON, 'Treatise on the Motion of Vortex Rings,' p. 112). If the layer of condensed gas were to contain an abnormal amount of energy, we could easily explain the influence exerted in some cases of chemical combination by the walls of the vessel in which the combination takes place (J H. VAN 'T HOFF, 'Études de Dynamique Chimique,' p. 58), and also the influence exerted by finely divided charcoal and platinum where a very large surface is exposed. For the explanation given above shows that, if the energy of unit area of the condensed gas varies as chemical combination goes on, the action of the surface layer will either promote or impede chemical combination. It will promote it if the energy per unit surface decreases as combination goes on; impede it, if this energy increases.

If the specific inductive capacity of the mixture of gases alters as the chemical combination goes on, their combination will be effected by placing them in an electric field. The chemical action will be checked if the specific inductive capacity increases as combination goes on; promoted, if it diminishes.

§ 12. The equations contained in the preceding investigation express the result of actions which are usually termed by the chemists "mass actions" These, however, have been chiefly studied in the case of very dilute solutions, so that it is important to endeavour to apply our results to this case.

If we regard a solution of one substance A in another B as equivalent to a distribution of the molecules of A through the volume occupied by B, then these molecules will behave with respect to each other very much like the molecules of a gas, and we may suppose that the value of $\overline{L}$ is expressed by an equation of the same form. This view would have to be modified if the nature of the solvent should be found to influence the equilibrium of a mixture of various reagents, and we should have to apply the more general method which we shall discuss later on. In those cases, however, where the solvent is without influence the above assumption that the solvent only separates the molecules of the substances dissolved seems legitimate.

If this be so, the investigation by the Hamiltonian principle of the case, when dilute solutions of four reagents, A, B, C, D, act upon each other, is the same in form as that investigated on p. 503, where four gases, A, B, C, D, act upon each other. Let us suppose that when A acts on B it produces C and D, and when C acts on D it pro-

duces A and B. Then, if ξ, η, ζ, ϵ, be the number of equivalents of A, B, C, D, respectively, p, q, r, s, the number of molecules in these equivalents, w the mean potential energy of the mixture of the reagents, θ the absolute temperature, we have

$$\frac{\zeta^r\epsilon^s}{\xi^p\eta^q} = CQ^{r+s-p-q}e^{-\frac{dw}{d\xi}\frac{1}{K\theta}}, \quad \ldots \ldots \ldots \quad (57)$$

where Q is the volume of the solvent, and C and K are constants; we must not, however, assume without proof that the value of K is the same as that of the quantity denoted by the same letter for gases.

The same conclusions as to the coincidence of this law with that of BERTHELOT at the zero of absolute temperature, and the close approximation between the two at ordinary temperatures in those cases where a very large amount of heat is developed in the reaction, hold in this case as well as in that of the gases.

We will now apply this formula to some cases which have been experimentally investigated. The one to which most attention has been directed is that of a mixture of dilute solutions of nitric and sulphuric acids, sodium nitrate, and sodium sulphate. Here the reaction is represented by the equation

$$H_2SO_4 + 2NaNO_3 = 2HNO_3 + Na_2SO_4,$$

and equation (57) becomes

$$\gamma\zeta^r\epsilon^s = \xi^p\eta^q,$$

where γ depends upon the temperature, but not on ξ, η, ζ, or ϵ. Here ξ, η, ζ, ϵ, are respectively the number of equivalents of sulphuric acid, sodium nitrate, nitric acid, and sodium sulphate. In order to fix the values of p, q, r, s, we must know whether the molecules of sodium nitrate and nitric acid in the solution are to be represented by $Na_2N_2O_6$, $H_2N_2O_6$, or by $NaNO_3$ and HNO_3; if the first supposition is correct, then $p = q = r = s = 1$; if the latter, $p = 1$, $q = 2$, $r = 2$, $s = 1$. In the first case the equation is

$$\gamma\zeta\epsilon = \xi\eta;$$

in the second,

$$\gamma^1\zeta^2\epsilon = \xi\eta^2.$$

THOMSEN* has determined the state of equilibrium when solutions of nitric acid and sodium sulphate are mixed together in varying proportions. I have calculated from his results the corresponding value of γ and γ'. In the following Table n is the ratio of the number of equivalents of sodium sulphate to the number of equivalents of nitric acid before chemical combination commences.

* THOMSEN, 'Thermochemische Untersuchungen,' I, 112.

n	γ	γ'
8	1·93	28·1
4	2 89	28·9
2	2 90	13 05
1	3 47	6·8
$\frac{1}{2}$	4 1	3·2
$\frac{1}{4}$	4·1	1 0

It is evident from the above Table that, except when the amount of nitric acid originally present is very small, the second equation, which requires γ' to be constant, does not agree with the experiments, while the first, which requires γ to be constant, does agree fairly well, except in those cases where the quantity of nitric acid present is originally small.

Hence we conclude that, except in these cases, the molecule and the equivalent coincide, that is, the molecule of nitric acid in the solution is represented by $H_2N_2O_6$; a similar conclusion applies to the molecule of sodium nitrate, the molecule of which in the solution has to be represented by $Na_2N_2O_6$. When the quantity of nitric acid initially present is very small, the second equation seems to agree with the experiments better than the first, so that it might seem as if, when the quantity of nitric acid was very small, the molecule was HNO_3 and not $H_2N_2O_6$, but, as in this case a very small error in the experiments would make a large error in γ or γ', too much weight must not be attached to it. It seems quite possible that there are molecules of both types, and that in concentrated solutions those of the type $H_2N_2O_6$ are by far the most numerous.

Cases when one or more of the Reagents are insoluble.

§ 13. Let us suppose that A is insoluble; then for A the positional part of the kinetic energy equals

$$\theta \overline{\frac{\partial p}{\partial \theta}} v,$$

where $\overline{\partial p/\partial \theta}$ is the mean value of $\partial p/\partial \theta$ (v constant), and v is the volume. If σ be the density of A, we may write this

$$\theta \overline{\frac{\partial p}{\partial \theta}} \cdot \frac{m_1 p \xi}{\sigma}.$$

Hence $\overline{\mathrm{L}}$ for the system equals

$$\theta \left\{ \overline{\frac{\partial p}{\partial \theta}} \frac{m_1 p \xi}{\sigma} + m_2 q c_2 \eta + \mathrm{K} q \eta \log \frac{\rho_0' \mathrm{Q}}{m_2 q \eta} + m_3 r c_3 \zeta + \mathrm{K} r \zeta \log \frac{\rho_0'' \mathrm{Q}}{m_3 r \zeta} + m_4 s c_4 \epsilon + \mathrm{K} s \epsilon \log \frac{\rho_0''' \mathrm{Q}}{m_4 r \epsilon} \right\} - w.$$

The condition that $\overline{\mathrm{L}}$ should be stationary is therefore

$$\theta\left\{\frac{\overline{\partial p}}{\partial\theta}\frac{m_1 p}{\sigma} + m_2 q c_2 - m_3 r c_3 - m_4 s c_4 + \mathrm{K}(q-r-s) + \kappa\left(q\log\frac{\rho_0 \mathrm{Q}}{m_2 q\eta} - r\log\frac{\rho_0' \mathrm{Q}}{m_3 r\zeta} - s\log\frac{\rho_0'' \mathrm{Q}}{m_4 s\epsilon}\right)\right\} = \frac{dw}{d\xi}. \quad (58)$$

This equation may be written as

$$\frac{\zeta^r\epsilon^s}{\eta^q} = \mathrm{Q}^{r+s-q}\,\mathrm{C}\,e^{\frac{dw}{d\xi}\frac{1}{\mathrm{K}\theta}} \quad (59)$$

If two of the constituents, say A and C, are insoluble, then we can easily prove in a similar way that

$$\frac{\epsilon^s}{\eta^q} = \mathrm{Q}^{s-q}\,\mathrm{C}'e^{\frac{dw}{d\xi}\frac{1}{\mathrm{K}\theta}} \quad (60)$$

The Effect of Temperature on the Equilibrium.

§ 14. If we define the coefficient of the reaction to be the value of $\zeta^r\epsilon^s/\xi^p\eta^q$ when there is equilibrium, and denote it by the symbol ω, then we have

$$\omega = \mathrm{Q}^{r+s-p-q}\,\mathrm{C}\,e^{\frac{dw}{d\xi}\frac{1}{\mathrm{K}\theta}}; \quad (61)$$

so that, if $\delta\omega$ be the alteration in the value of ω when the temperature is increased by $\delta\theta$, we have approximately, if $dw/d\xi$ does not change with the temperature, and if its ratio to $\mathrm{K}\theta$ is large, so that the term in ω which varies most rapidly with the temperature is $e^{\frac{dw}{d\xi}\frac{1}{\mathrm{K}\theta}}$,

$$\frac{\delta\omega}{\omega} = -\frac{1}{\mathrm{K}\theta}\frac{dw}{d\xi}\frac{\delta\theta}{\theta}, \quad (62)$$

so that the percentage change in ω for a given change in θ varies inversely as the square of the absolute temperature, and directly as $dw/d\xi$, which, when the system is free from strain, electrification, &c., is the amount of heat given out when one equivalent of A combines with one of B to form one each of C and D.

The greater this amount of heat, the more quickly will the coefficient of the reaction vary with the temperature.

We shall now proceed to reduce this expression to numbers, assuming, as an approximate value of K, that it is the same as for gases. In this case

$$K\theta = \frac{pm}{\rho},$$

where p is the pressure and ρ the density.

Let H′ be the amount of heat produced when c grammes of A combine, c being the combining weight of A, then

$$H' = Nc\frac{dw}{d\xi},$$

where N is the number of equivalents in a gramme of A. In order to simplify the reasoning, let us suppose that we are considering a case where the molecule and the equivalent are identical,

$$\frac{1}{K\theta}\frac{dw}{d\xi} = \frac{H'}{\frac{Nm\,pc}{\rho}}$$

$$= \frac{H'}{pc/\rho},$$

since

$$Nm = 1.$$

Since pc/ρ is the same for all gases, and since for Hydrogen it is about $10^{11}/4{\cdot}5$, we have

$$\frac{1}{\kappa\theta}\frac{dw}{d\xi} = \frac{4{\cdot}5\,H'}{10^{11}},$$

and therefore, by equation (62),

$$\frac{\delta\omega}{\omega} = -\frac{4{\cdot}5\,H'}{10^{11}}\frac{\delta\theta}{\theta}.$$

In the case of hydrochloric acid, sulphuric acid, sodium sulphate, and sodium chloride THOMSEN * found $H' = 2 \times 10^3 \times 4{\cdot}2 \times 10^7$, so that in this case

$$\frac{\delta\omega}{\omega} = 3\frac{\delta\theta}{\theta}\text{ approximately};$$

so that a temperature of about 30° C, an alteration of 1° C., would make a change of about 1 per cent. in the value of ω.

§ 15. The formula

$$\frac{\eta^r\epsilon^s}{\xi^p\eta^q} = Q^{r+s-p-q}\,C\,e^{\frac{dw}{d\xi}\frac{1}{K\theta}}$$

enables us to calculate readily the disturbing effect produced by any slight alteration in the external circumstances.

* LOTHAR MEYER · 'Theorien der Chemie,' 4th edit., p. 478.

Putting, as before,

$$\frac{\zeta^r \epsilon^s}{\xi^p \eta^q} \equiv \omega,$$

we see that $\delta\omega$, the change in ω due to a change δw in the value of the potential energy, is given by the equation

$$1 + \frac{\delta\omega}{\omega} = e^{\frac{d}{d\xi}\frac{\delta w}{}\frac{1}{\kappa\theta}}, \qquad (63)$$

so that, if $d\,.\,\delta w/d\xi$ is positive, $\delta\omega$ is positive, that is, in the state of equilibrium. ξ and η are smaller than they would have been if δw had been zero; so that, if an increase in ξ increases the additional potential energy, the value of ξ in the state of equilibrium will be diminished. If, on the other hand, the additional potential energy diminishes as ξ increases, the value of ξ in the state of equilibrium will be increased.

Effect of Capillarity.

§ 16 Let us first consider the case when the additional potential energy is due to capillarity, then

$$\delta w = \mathrm{TS},$$

where S is the area of the surface, and T the surface-tension

Hence we have, from Equation (63),

$$1 + \frac{\delta\omega}{\omega} = \epsilon^{\frac{d(\mathrm{TS})}{d\xi}\frac{1}{\kappa\theta}}. \qquad (64)$$

Thus, if either the surface-tension or the area of the surface alters as chemical combination proceeds, the final state of equilibrium will depend upon the extent of surface. The state of equilibrium will be different when the solution is spread out over a large surface from that which exists when the solution exposes only a small free surface. Considerations of this kind would explain the experiments of Professor LIEBREICH on the precipitation of chloroform by the mixture of hydrate of chloral and an alkaline solution. ('Nature,' vol. 35, p. 264.) In these experiments it was observed that when the solutions were mixed in a test-tube the top layer remained clear, no precipitation taking place inside it, and the same phenomenon occurred in the capillary space between two plates, in short capillary tubes the reaction failed altogether. The equations obtained above apply to this case. Since the surface-tension of pure water is altered by the addition of other substances, the surface-tension of the mixture of chloral and alkaline solution will be altered by the withdrawal from the solution of chloral and alkali; let us suppose that it is increased; then the precipitation of the chloroform will increase the surface-tension, and therefore the potential energy due to it. Thus, if ξ in this case is the number of equivalents of chloroform, $d\,.\,\delta w/d\xi$ is positive, and

therefore the existence of surface-tension diminishes the value of ξ, that is, it diminishes the precipitation. In the case mentioned above it seems to have been able to stop the precipitation altogether. The effects of surface-tension will be most noticeable when the heat developed by the chemical action is small. Some other effects produced by surface-tension on chemical action are considered in a paper read before the Cambridge Philosophical Society, February 1887, by Professor LIVEING

Surface-tension will also affect the equilibrium when chemical combination produces a change in volume, even though the surface remains unaltered.

Effects of Pressure.

§ 17. If Q be the volume, p the pressure, κ the coefficient of compressibility of the solution, then, just as in the case of solution, we have

$$\frac{d}{d\xi}\delta\omega = p\frac{dQ}{d\xi} + \frac{1}{2}\frac{d}{d\xi}\left(\frac{Qp^2}{\kappa}\right). \qquad (65)$$

Thus pressure produces two effects, the one being proportional to the pressure, the other to the square of it.

If the volume increases as chemical combination goes on, external pressure will tend to stop the combination; and, *vice versâ*, the change in the coefficient of reaction due to this case is given by the equation

$$1 + \frac{\delta\omega}{\omega} = e^{\frac{p\,dQ}{\kappa\theta\,d\xi}},$$

or, approximately,

$$\frac{\delta\omega}{\omega} = \frac{1}{K\theta} p \frac{dQ}{d\xi}. \qquad (66)$$

If the coefficient of compressibility increases as chemical action goes on, external pressure will tend to facilitate the combination, and the change in the coefficient of the reaction due to this cause is

$$\frac{\delta\omega}{\omega} = \frac{1}{2}\frac{1}{K\theta}\frac{d}{d\xi}\left(\frac{Qp^2}{\kappa}\right). \qquad (67)$$

HORSTMANN'S *Experiments on the Division of Oxygen between Carbonic Oxide and Hydrogen.*

§ 18. We shall now proceed to apply the Hamiltonian principle to this case, which is more complicated than any of the preceding. In these experiments hydrogen, carbonic oxide, and oxygen were mixed together and exploded, and the proportion of hydrogen, oxygen, carbonic oxide, carbonic acid, and water in the mixture after the explosion determined.

Let ξ, η, ζ, ϵ, ω, be the number of molecules of hydrogen, oxygen, carbonic oxide, carbonic acid, and water respectively; then the value of $\overline{\mathrm{L}}$ will be

$$\theta \left\{ m_1 \xi \mathrm{R}_1 \log \frac{\rho_0 \mathrm{Q}}{m_1 \xi} + m_2 \eta \mathrm{R}_2 \log \frac{\rho_0' \mathrm{Q}}{m_2 \eta} + m_3 \zeta \mathrm{R}_3 \log \frac{\rho_0'' \mathrm{Q}}{m_3 \zeta} + m_4 \epsilon \mathrm{R}_4 \log \frac{\rho_0''' \mathrm{Q}}{m_4 \epsilon} + m_5 \omega \mathrm{R}_5 \log \frac{\rho_0'''' \mathrm{Q}}{m_5 \omega} \right\} - w,$$

where m_1, m_2, m_3, m_4, m_5 are respectively the masses of the hydrogen, oxygen, carbonic oxide, carbonic acid, and water molecules respectively; $\mathrm{R}_1, \mathrm{R}_2, \ldots \mathrm{R}_5$ the value of $p/\rho\theta$ for these gases; w the mean potential energy of the mixture of gases, and Q the volume of the vessel in which they are contained.

Since the quantity of hydrogen in the mixture remains constant,

$$\xi + \omega = \text{a constant},$$

and, since the quantity of oxygen remains constant,

$$\eta + \tfrac{1}{2}\zeta + \epsilon + \tfrac{1}{2}\omega = \text{a constant}; \qquad (68)$$

since the quantity of carbon remains constant,

$$\zeta + \epsilon = \text{a constant}.$$

Since there are three equations between these five quantities, we may take two of them as independent variables. Let us take ξ and ϵ as our independent variables; then we have from the above equations

$$\left.\begin{aligned} \frac{d\eta}{d\xi} &= \tfrac{1}{2}, & \frac{d\zeta}{d\xi} &= 0, & \frac{d\omega}{d\xi} &= -1, \\ \frac{d\eta}{d\epsilon} &= -\tfrac{1}{2}, & \frac{d\zeta}{d\epsilon} &= -1, & \frac{d\omega}{d\epsilon} &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (69)$$

By the Hamiltonian principle we have

$$\frac{d\overline{\mathrm{L}}}{d\xi} = 0,$$

$$\frac{d\overline{\mathrm{L}}}{d\epsilon} = 0,$$

so that

$$\theta \left\{ m_1 \mathrm{R}_1 \log \frac{\rho_0 \mathrm{Q}}{m_1 \xi} - m_1 \mathrm{R}_1 + \tfrac{1}{2} \left(m_2 \mathrm{R}_2 \log \frac{\rho_0' \mathrm{Q}}{m_2 \eta} - m_2 \mathrm{R}_2 \right) - \left(m_5 \mathrm{R}_5 \log \frac{\rho_0'' \mathrm{Q}}{m_5 \omega} - m_5 \mathrm{R}_5 \right) \right\} = \left(\frac{dw}{d\xi} \right)_{\epsilon \text{ constant.}} \quad . \quad (70)$$

Since $m_1 \mathrm{R}_1 = m_2 \mathrm{R}_2 = m_3 \mathrm{R}_3 = m_4 \mathrm{R}_4 = m_5 \mathrm{R}_5 = \kappa$, we may write this equation as

$$\tfrac{1}{2}\kappa\,\theta\left\{\log\frac{\rho_0^{2}\,\rho_0'\,Q\,\omega^2\,m_5^{2}}{m_1^{2}\,m_2\,\rho_0''^{2}\,\xi^2\eta}-\tfrac{1}{2}\right\}=\left(\frac{dw}{d\xi}\right)_{\epsilon\ \text{constant}},$$

or

$$\frac{\omega^2}{\xi^2\eta}=\frac{C_1}{Q}\,e^{\frac{2}{\kappa\theta}\frac{dw}{d\xi}}, \qquad (71)$$

where C_1 is a constant

From the equation

$$\frac{d\bar{L}}{d\epsilon}=0$$

we get, in a similar way,

$$\frac{\zeta^2\eta}{\epsilon^2}=Q\,C_2\,\epsilon^{\frac{2}{\kappa\theta}\left(\frac{dw}{d\epsilon}\right)}. \qquad (72)$$

Equations (71) and (72), along with the three equations (68), are sufficient to determine the five quantities ξ, η, ζ, ϵ, ω.

If we multiply equations (71) and (72) together, we get

$$\frac{\omega^2\zeta^2}{\xi^2\epsilon^2}=C_1\,C_2\,\epsilon^{\frac{2}{\kappa\theta}\left\{\left(\frac{dw}{d\xi}\right)+\frac{dw}{d\epsilon}\right\}}; \qquad (73)$$

so that, as long as the temperature remains constant, $\omega\zeta/\xi\epsilon$ is constant; or, in words, the ratio of the quantities of water and carbonic acid formed by the explosion always bears a constant-ratio to the ratio of the quantity of hydrogen left free to the quantity of free carbonic-acid.

Since equations (71) and (72) involve Q, the amount of combination which goes on when a given quantity of the gases are exploded will depend upon the volume in which they are confined. The equations show that the amount of combination will increase as the volume diminishes. In a paper on the "Chemical Combination of Gases"* I arrived at similar results (except with regard to the effect of temperature, which I did not investigate) by a purely kinematical method.

Method applicable to Solutions of any Strength.

§ 19. When the solutions are too concentrated to permit us to assume that the energy possessed by the molecules of the salt is the same as the energy possessed by the same number of molecules in the gaseous state, we must use the more general expression given on p. 487 for the positional part of $\bar{L}$, viz.,

$$\theta\int_{v_0}^{v}\frac{\partial p}{\partial\epsilon}\,dv-w,$$

* 'Phil. Mag.,' vol. 19, 1884

where, in finding the value of $\partial p/\partial\epsilon$, we suppose v to be constant, and w is the mean potential energy of the system.

If $\xi, \eta, \zeta \ldots$ be the number of equivalent molecules of the substances A, B, C... in the solution, then, since $\overline{\mathrm{L}}$ must be stationary when the system is in equilibrium, we have

$$\theta \frac{d}{d\xi}\int_{v_0}^{v} \frac{\partial p}{\partial} dv - \frac{dw}{d\xi} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

with as many equations of a similar type as there are independent variables. We can experimentally determine the way in which $\partial p/\partial\epsilon$ and ω vary with the quantities of the various substances in the solution; and, if we have sufficient data to express these quantities as functions of $\xi, \eta, \ldots$, the equations of the type (74) will enable us to find the values of $\xi, \eta, \ldots$ when there is equilibrium The part of $dw/d\xi$ which depends upon the chemical affinity of the substances for each other can be measured by the heat developed when the chemical action which causes the diminution of the number of equivalents of the substance A by one goes on. But, just as in the case of the more dilute solutions, part of w may be the energy due to surface-tension, electrification, compression, &c, and the presence of this energy will affect the state of equilibrium

The Velocity of Chemical Change.

§ 20. It is much easier in many cases to measure the rate at which chemical change takes place than to determine the final state of equilibrium. It seems desirable, therefore, in order to facilitate the comparison of theory with experiment, to endeavour to deduce some expression for the velocity of chemical change.

Let us suppose that, as before, we have a number of substances which can act chemically on each other. Let $\xi, \eta, \zeta \ldots$ be the number of equivalents of these substances: these will be connected by various equations; let us choose $\xi, \eta \ldots$ as independent variables. Then, since when there is equilibrium $d\overline{\mathrm{L}}/d\xi = 0, d\overline{\mathrm{L}}/d\eta = 0$, we conclude that these quantities have something to do with the velocity with which chemical change goes on. Now the approach of a mixture of various reagents to its state of equilibrium is not like the approach of a vibrating body resisted by a frictional force to its position of equilibrium, for after the mixture has got to its position of equilibrium it stays there instead of vibrating about it like the ordinary dynamical system. Thus the mixture behaves like a system in which inertia is absent or comparatively unimportant; we may, therefore, suppose that the accelerations ξ, η, &c., are absent from the equations of motion of the mixture, and we may assume, at any rate when $d\overline{\mathrm{L}}/d\xi$, $d\overline{\mathrm{L}}/d\eta$, are small, that

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \mathrm{A}\frac{d\overline{\mathrm{L}}}{d\xi} \\ \frac{d\eta}{dt} &= \mathrm{B}\frac{d\overline{\mathrm{L}}}{d\eta} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

where A and B are either constants or functions of ξ, η, &c., whose values we do not know. From the indeterminateness of A and B, we cannot use these equations to determine the absolute velocity of chemical change. We can, however, use them to determine the effect on this velocity of any alteration in the external circumstances. Thus, suppose that the circumstances are changed so that $\bar{L}$ is increased by δL; then, if $\delta\dot{\xi}$ denotes the change in the velocity of ξ, we have

$$\dot{\xi} + \delta\dot{\xi} = A\left\{\frac{d\bar{L}}{d\xi} + \frac{d\delta L}{d\xi}\right\},$$

so that

$$\frac{\delta\dot{\xi}}{\dot{\xi}} = \frac{\dfrac{d\delta L}{d\xi}}{\dfrac{d\bar{L}}{d\xi}}; \quad \ldots \ldots \ldots \quad (76)$$

and we can measure all the quantities occurring in this equation, and so compare it with experiment.

If the change in the energy be due to surface-tension, then, if S be the area of the surface, T the surface-tension, we have

$$\frac{\delta\dot{\xi}}{\dot{\xi}} = -\frac{\dfrac{d}{d\xi}(ST)}{\dfrac{d\bar{L}}{d\xi}}. \quad \ldots \ldots \ldots \quad (77)$$

If the change in the circumstances be due to the action of the pressure p, then

$$\frac{\delta\dot{\xi}}{\dot{\xi}} = -\frac{p\dfrac{dv}{d\xi} - \frac{1}{2}p^2\dfrac{d\frac{1}{\kappa}}{d\xi}}{\dfrac{d\bar{L}}{d\xi}}, \quad \ldots \ldots \ldots \quad (78)$$

where V is the volume of the mixture and κ its coefficient of compressibility. The value of $d\bar{L}/d\xi$ for gases is given by equation (14); for solids and liquids by equation (74).

Irreversible Effects.

§ 21. So far we have only considered those cases which involve nothing but reversible processes, and have left out of consideration the effect of such things as friction, electrical resistance, and so on, which destroy the reversibility of any process in which they play a part. If, however, as we have done in this paper, we take the view that the properties of matter in motion, as considered in abstract dynamics—when all the

processes are reversible—are sufficient to account for any physical phenomenon, then we must show how to explain irreversible processes as the effect of changes all of which are reversible It would not be sufficient to explain these irreversible effects by means of ordinary dynamical systems with friction, as friction itself ought on this view to be explained by means of the action of frictionless systems.

If every physical phenomenon can be explained by means of frictionless dynamical systems, each of which is reversible, then it follows that, if we could only control the phenomenon in all its details, it would be reversible, so that the irreversibility of any system is due to the limitation of our powers of manipulation. It is because we only possess the power of dealing with the molecules *en masse* and not individually, while the reversal of these processes would require us to be able to reverse the motion of each individual molecule This was pointed out by MAXWELL, who showed that an army of his "demons" would be able to prevent the dissipation of energy.

Our want of power of dealing with very minute portions of matter imposes one kind of limitation on our control of physical processes; another limitation to our power of interpreting them is caused by the time which our sensations last, causing any phenomenon which consists of events following one another with great rapidity to present a blurred appearance, so that what we perceive at any moment is not what is happening at that moment, but merely an average effect, which may be quite unlike the effect at any particular instant In consequence of the finiteness of the time taken by our senses to act we are incapable of separating two events which happen within a very short interval of each other, just as the finiteness of the wave-length of light prevents us separating two points which are very close together. Thus, if we observe any effect, we cannot tell by our senses whether it represents a steady state of things or a state which is rapidly changing, and whose mean is what we actually observe. Thus we are at liberty, if it is more convenient for the purposes of explanation, to look upon any effect as the average of a series of other rapidly changing effects.

Let us consider the case of a system whose motion is such that, in order to represent it, "frictional terms" have to be introduced Let us first assume that the motion is represented at each instant by the equations with these terms in, and that these equations are not equations which are only true on the average. Let us assume that any phenomenon is capable of explanation by the principle of abstract dynamics. Then, from our point of view, we shall have explained the phenomenon when we have found a frictionless system whose motion would produce the phenomenon.

It might at first sight appear as if we could explain the frictional terms in the equations of motion as arising from the connection of other subsidiary systems with the original system, just as in the first part of the Paper ('Phil. Trans.,' 1885, p. 311) we explained the "positional" forces as due to changes in the motion of a system connected with the original system. Let us suppose for a moment that this is possible. Then, if T be the kinetic energy of the original system, and T′ that of the subsidiary

system whose motion is to explain the frictional forces, then we have, by LAGRANGE'S equation,

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\dot{x}} - \frac{d\mathrm{T}}{dx} - \frac{d}{dt}\frac{d\mathrm{T}'}{d\dot{x}} - \frac{d\mathrm{T}'}{dx} + \frac{d\mathrm{V}}{dx} = \text{external force of type } x\,;$$

thus the term

$$-\frac{d}{dt}\frac{d\mathrm{T}'}{d\dot{x}} - \frac{d\mathrm{T}'}{dx}$$

must be equal to the "frictional term" which is proportional to $\dot{x}$. For this to be the case, it is evident that T′ must involve $\dot{x}$. The momentum of the system is, however, $d(\mathrm{T} + \mathrm{T}')/d\dot{x}$. This momentum must, however, be the same as that given by the ordinary expression in Rigid Dynamics, viz, $d\mathrm{T}/d\dot{x}$. If these two expressions, however, are identical, $d\mathrm{T}'/d\dot{x}$ must vanish for all values of $\dot{x}$, that is, T′ cannot involve x', which is inconsistent with the condition necessary in order that the motion of the subsidiary system should give rise to the "frictional" terms. Hence we conclude that the frictional terms cannot be explained by supposing that any subsidiary system with a finite number of degrees of freedom is in connection with the original system.

If we investigate the case of a vibrating piston in connection with an unlimited volume of air, we shall find that the waves starting from the piston dissipate its energy just as if it were resisted by a frictional force proportional to its velocity; this, however, is only the case when the medium surrounding the piston is unlimited; when it is bounded by fixed obstacles the waves originated by the piston get reflected from the boundary, and thus the energy which went from the piston to the air gets back again from the air to the piston. Thus the frictional terms cannot be explained by the dissipation of the energy by waves starting from the system and propagated through a medium surrounding it, for in this case it would be possible for energy to flow from the subsidiary into the original system, while, if the frictional terms are to be explained by a subsidiary system in connection with the original one, the connection must be such that energy can flow from the original into the subsidiary system, but not from the subsidiary into the original.

Hence we conclude that the equations of motion, with frictional terms in, represent the average motion of the system, but not the motion at any particular instant.

Thus, to take an example, let us suppose that we have a body moving rapidly through a gas; then, since the body loses by its impacts with the molecules of a gas more momentum than it gains from them, it will be constantly losing momentum, and this might on the average be represented by the introduction of a term expressing a resistance varying as some power of the velocity; but the equations of motion, with this term in, would not be true at any instant, neither when the body was striking against a molecule of the gas, nor when it was moving freely and not in collision with any of the molecules. Again, if we take the resistance to motion in a gas which arises from its own viscosity, the kinetic theory of gases shows that the equations of motion

of the gas, with a term included expressing a resistance proportional to the velocity, are not true at any particular instant, but only when the average is taken over a time which is large compared with the time a molecule takes to traverse its own free path.

The irreversible effects which have the closest connection with the phenomena we have been considering in this paper are those of electrical resistance, and we shall now go on to consider the application of dynamical principles to phenomena of this type. In accordance with what we have already stated, we regard the ordinary electrical equations containing the terms which express the effects of the resistance as equations which only apply to the average state of the system, the average being taken over a time which is too small to allow us to perceive the changes taking place inside it; about these changes the ordinary equations give us no information. It is evident from this point of view that we cannot hope to deduce directly the ordinary electrical equations from these dynamical equations, which are always true, and which, if we could solve them, would describe the whole history of the electrical configuration. We should expect the electrical equations to be obtained from the dynamical ones by some process of averaging.

If this view is right, the passage of a "steady" current is not, strictly speaking, a steady phenomenon; but only one in which the average effect, taken over some very small time, is steady. We must therefore take a view of the electric current somewhat different from that usually taken. In order to explain this view, let us begin by considering a case which is plainly discontinuous, but which, when the changes succeed each other sufficiently rapidly, will produce the same effect as a steady current. The case is that of the passage of electricity through a tube containing gas at a low pressure. In this case the electric force inside the tube increases until it gets too great for the electric strength of the gas, the field then breaks down, and for a moment the electric force either vanishes or is very much diminished, or, what is the same thing, a quantity of electricity passes from the one terminal to the other; after this the force increases until it gets great enough to again overcome the electric strength of the gas, when discharge again takes place. The constant succession of such discharges produces the same effect as a current flowing through a metallic conductor. In the case of metallic conductors we may suppose that very much the same kind of thing goes on, only that now the electric field is dissipated by the breaking up of molecular aggregations which split up independently of the electric field. Let us imagine a conductor placed in the electric field, and suppose that at first induction occurs in it as well as in the surrounding dielectric: then in each unit of volume of the conductor there is a certain amount of energy. If the molecules or the atoms in the molecules can move so that this energy diminishes, they will do so; in general, however, we should expect the forces existing between the atoms in the molecule to be so large that no very extensive re-arrangement of the atoms or molecules in the way suggested by the electric forces would take place unless the electric field were excessively strong. If, however, the molecules or the aggregations of molecules were to

break up independently of the electric field, then these inter-atomic or inter-molecular forces would be absent, and the atoms or molecules would be free to arrange themselves so as to diminish the potential energy due to the electric field. This diminution in the electrical energy would be equivalent to a discharge of the electric field, partial or total, according as the energy is only partially or totally exhausted. According to this view, the electric current is a discontinuous phenomenon, though there need not be anything corresponding to a definite period, as the field may not be simultaneously discharged at all points. We may suppose that much the same kind of thing occurs in electrolytes, and in this case the view has much in common with the WILLIAMSON-CLAUSIUS hypothesis. According to this view the electrolyte is not decomposed by the electric field, the function of the electric forces being merely to direct the motion of the components of the molecules dissociated by other means According to our view it is the re-arrangement of the components of dissociated molecules or groups of molecules which produces the current. This view is in accordance with FARADAY'S remark that induction always precedes conduction. If, as in the case of the electric discharge through permanent gases, the electric field were strong enough to separate the molecules without any independent dissociation, we should expect the law connecting the current with the electromotive force to be different from the law connecting the same quantities when the electromotive force is too weak to decompose the molecule. There seems to be evidence for such a difference, for QUINCKE* has shown that when the E.M.F. is very large the current through badly conducting liquids, such as olive oil or benzene, does not obey OHM'S Law, while the experiments of Mr. NEWALL and myself have shown that when the E.M.F. is small, not more than a few hundred volts per centimetre, the current does obey OHM'S Law.

We shall now proceed to endeavour to represent the theory symbolically. To fix our ideas, let us consider the case of an air-condenser whose armatures are connected with the poles of a battery whose E.M.F. is greater than the air-space can stand: then, as soon as the armatures are connected to the poles of the battery, there is an electric displacement across the air-space in the direction of the E.M.F.; then the air breaks down, and there is the passage of a quantity of electricity—equal per unit area to the displacement across the unit area—from one armature to the other, and then the disappearance of the displacement. Now, whether we take the ordinary two-fluid theory, or MAXWELL'S displacement theory, the result is the same; let us take the two-fluid theory first—then the displacements count for nothing, and we have only to consider the passage of the electricity across the air-space. Let us next take MAXWELL'S theory—the effect of the disappearance of the displacement is equal and opposite to that of the passage of the electricity across the air-space—so that we are left with the effect of the establishment of the displacement, which is the same as that of the passage of the electricity, and the two theories lead to identical results; and we may conclude that when a field, such that the displacement across unit area

* 'WIEDEMANN'S Annalen,' vol. 28, p. 529.

is η, is established, and then breaks down by reason of the re-arrangement of the dielectric, the whole effect is the same as if a quantity of electricity equal to η passed through each unit area of the dielectric.

Let us suppose that in the expression for the electrical part of $\mathrm{T}-\mathrm{V}$ for unit volume of a conductor through which the electrical displacements are f, g, h, in addition to the usual term

$$\tfrac{1}{2}\left\{\mathrm{L}\frac{df}{dt}+\mathrm{M}\frac{dg}{dt}+\mathrm{N}\frac{dh}{dt}\right\},$$

where L, M, N, are the components of the electrical momentum parallel to the axes of x, y, z, respectively, there is the term

$$-\tfrac{1}{2}\mathrm{Q}\left(f^2+g^2+h^2\right);$$

then LAGRANGE's equations give

$$\begin{aligned}\frac{d\mathrm{L}}{dt}+\mathrm{Q}f &= \text{external electromotive force tending to increase } f,\\ &= \mathrm{X} \text{ say.}\end{aligned}$$

Now, in the case of a conductor, f is changing very rapidly, while the other terms in this equation only change gradually. Let us then take the mean over unit time of each side of the equation, then we have

$$\overline{\frac{d\mathrm{L}}{dt}}+\mathrm{Q}\int_0^1 f\,dt=\overline{\mathrm{X}}, \qquad \text{. . . . . (79)}$$

where a bar placed over a symbol denotes the mean value of that symbol, and we suppose that Q does not change quickly with the time.

Now let us suppose, as before, that the polarisation is continually being broken down and renewed; let it break down n times a second, and let the mean value of f over one of these intervals of $1/n$ of a second be $\bar{f}$, and the maximum value be $\alpha\bar{f}$. the breaking down of the field each time is equivalent to the passage of $\alpha\bar{f}$ units of electricity across unit area at right angles to the axis of x. Since the field breaks down n times a second, it is equivalent to the passage of $n\alpha\bar{f}$ across this unit section in unit time, or, if u be the component of currents parallel to the axis of x,

$$u=n\alpha\bar{f}.$$

Now

$$\int_0^1 f\,dt=n\int_0^{\frac{1}{n}} f\,dt=n\cdot\frac{1}{n}\bar{f}=\frac{u}{\alpha n};$$

hence we have, from equation (79),

$$\frac{d\bar{L}}{dt} + \frac{Qu}{\alpha n} = \bar{\chi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

the ordinary electrical equation if the specific resistance of the substance equals $Q/\alpha n$. From this expression for the specific resistance we see that if Q remains constant, or nearly constant, the resistance will vary inversely as n, that is, the resistance will be inversely proportional to the number of discharges in unit time Now, since the discharges are caused by the breaking up of the molecular aggregations, the number of discharges in unit time will be greater, the greater the ease with which the molecular aggregations break up. The formation and breaking up of these molecular aggregations in a solid would play in it very much the same part as that played by the collision between the molecules in a gas. But, the greater the number of collisions in unit time, the greater the rapidity with which inequalities in the kinetic energy, momentum, &c, in the different parts of a gas disappear, so that we may conclude that, the greater the ease with which molecular aggregations are broken up in a solid, the more rapidly will the inequalities in the kinetic energy disappear, that is, the better conductor of heat the solid will be. Thus the expression for the resistance of a solid conductor contains one term which is inversely proportional to the conductivity for heat of the substance Hence we can understand why those metals which are good conductors of electricity are also good conductors of heat.

The term

$$-\tfrac{1}{2}Q(f^2 + g^2 + h^2),$$

which we have introduced into the expression for T − V of unit volume of the conductor, corresponds to the term

$$\frac{2\pi}{K}(f^2 + g^2 + h^2)$$

in the expression for the potential energy of unit volume of a dielectric in which the electric displacement is f, g, h. Thus we may look on Q as equal to $4\pi/K$, where K is the specific inductive capacity of the conductor. The rapid way the resistance increases with the temperature for metals shows that of the term $-\tfrac{1}{2}Q(f^2 + g^2 + h^2)$ the great part must in this case be due to the kinetic energy. Let us suppose that in the kinetic energy we have the term

$$-\tfrac{1}{2}a\theta(f^2 + g^2 + h^2),$$

where θ is proportional to the absolute temperature, and in the potential energy the term

$$\tfrac{1}{2}b(f^2 + g^2 + h^2);$$

then

$$Q = a\theta + b,$$

and the specific resistance σ equals

$$\frac{a\theta + b}{na}.$$

Now n will probably depend on the temperature, and will probably in most cases increase as it increases, as it seems likely that the molecular aggregations will split up more easily at high than at low temperatures. Thus there are two influences which, when the temperature increases, oppose each other in the effect they produce upon the resistance. On the one hand there is a tendency for the resistance to increase in consequence of the greater change in the value of T − V produced by a given electric displacement, and on the other hand there is a tendency for the conductivity to increase in consequence of the molecular aggregations breaking up with greater ease, and so increasing the number of discharges which take place in unit time. In the case of metals the first effect seems to be the most important, as the resistance of these substances increases with the temperature. In the case of electrolytes the second seems the most important, as the conductivity of these substances increases with the temperature. Anything which increases the complexity of the molecular aggregations will increase the importance of the second effect relatively to the first, hence we can understand why it is that alloys have so much smaller temperature coefficients than pure metals, as we should expect the molecular aggregations to be more complex in alloys than in pure metals, and therefore the second effect, which tends to make the resistance diminish as the temperature increases, is relatively more important.

It is evident that, if the view which we have taken of the electric discharge be correct, the specific inductive capacity of metals must alter more quickly with the temperature than the specific inductive capacities of dielectrics, as experiment has shown that these vary only slowly with the temperature.

The consideration of the term

$$-\tfrac{1}{2}Q(f^2 + g^2 + h^2),$$

which occurs in T − V, leads to many interesting results, of which we shall proceed to give one or two examples.

If we take the axis of x parallel to the electric displacement, and write σna for Q, where σ is the specific resistance, we may write this term as

$$-\tfrac{1}{2}\sigma naf^2.$$

Now the experiments of Sir W. THOMSON and Mr. TOMLINSON show that the resistance of metals is affected by strain, so that σna must be a function of the strain.

Let us suppose that e is a strain coordinate; then it follows, by LAGRANGE'S equations, that there is a force due to this term, tending to increase e, equal to

$$-\tfrac{1}{2}\frac{d}{de}(\sigma n\alpha)f^2.$$

Since f, according to our view, changes very rapidly, we must find the mean value of this term, that is, we must find the value of

$$\int_0^1 f^2 dt.$$

If the electric field breaks down and gets established n times in a second, this will equal

$$n\int_0^{\frac{1}{n}} f^2 dt.$$

Now

$$\int_0^{\frac{1}{n}} f dt = \bar{f}$$

Let

$$\int_0^{\frac{1}{n}} f^2 dt = \beta\,.\,(\alpha\bar{f})^2,$$

where, as before, $\alpha\bar{f}$ is the maximum value of f, and β is a quantity which will depend upon the way f reaches its maximum; then

$$\int_0^1 f^2 dt = n\beta(\alpha\bar{f})^2 = \frac{\beta}{n}u^2,$$

where u is the current through unit area of the section of the conductor. Then the average force tending to increase e equals

$$-\tfrac{1}{2}\frac{d}{de}(\sigma n\alpha)\frac{\beta}{n}u^2. \qquad (81)$$

If neither n nor α depend upon e, then this equals

$$-\tfrac{1}{2}\alpha\beta\frac{d\sigma}{de}u^2. \qquad (82)$$

Thus, if the specific resistance increases with the strain, there will be a tendency to diminish that strain when a current traverses the conductor.

We can apply precisely the same method to any other physical effect which influences the resistance. Suppose that an alteration in the configuration δp alters the resistance; then, when the current u passes through the conductor, there will be a force, tending to increase p, equal to

$$-\tfrac{1}{2}\frac{d}{dp}(\sigma n\alpha)\frac{\beta}{n}u^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Let us now consider the effect which this term in the expression for T − V will produce on the chemical phenomena which we discussed in the earlier part of this paper. The alteration in $\overline{\mathrm{T}} - \overline{\mathrm{V}}$ due to this term is

$$-\tfrac{1}{2}\sigma n\alpha\frac{\beta}{n}u^2$$

or

$$-\tfrac{1}{2}\sigma\alpha\beta u^2.$$

Now, if the specific resistance depends, as we know it does to a very large extent in the case of liquids, on the substances dissolved in them, the chemical equilibrium will be affected by the passage of an electric current through them.

Let us first take the case of solution; then, when we were considering that subject, we showed that, if $\overline{\mathrm{T}} - \overline{\mathrm{V}}$ were increased by $-\delta w$, the change $\delta\theta$ in the temperature, which would produce the same effect as that produced by this change in $\overline{\mathrm{T}} - \overline{\mathrm{V}}$, is given by the equation

$$\frac{\delta\theta}{\theta} = -\frac{\dfrac{dw}{dq'}}{\lambda},$$

where q' is the mass of the solution, and λ the amount of heat absorbed when one gramme of the salt dissolves at constant temperature. Applying this formula to our case, we have

$$\frac{\delta\theta}{\theta} = -\frac{\tfrac{1}{2}\alpha\beta u^2\dfrac{d\sigma}{dq'}}{\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

so that, if the specific resistance of the solution depends upon the amount of salt dissolved, then, if we make a current pass through a saturated solution, it will either absorb or deposit salt; if the specific resistance increases as the salt dissolves, it will deposit salt; if the resistance diminishes, it will absorb it. In trying the experiment, it would be well to use alternating currents to avoid any difficulty which might arise from the substances set free by electrolysis.

Again, if we have dilute solutions of any number of substances, and if the resistance of the solution depends upon the relative proportion of these substances, then the

proportion of these substances when there is equilibrium will be altered by passing an electric current through the solution. Thus the coefficient of reaction ω will be altered, and we see from equation (56) that the alteration $\delta\omega$ is given by the equation

$$\frac{\delta\omega}{\omega} = \frac{1}{\kappa\theta} \frac{d}{d\xi} \tfrac{1}{2}\sigma\alpha\beta u^2$$

$$= \frac{1}{2\kappa\theta} \alpha\beta u^2 \frac{d\sigma}{d\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (85)$$

where the notation is the same as that used on page 509.

Thus, if the specific resistance increases as any reaction goes on, the passage of an electric current through the solution will tend to stop that reaction.

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PHILOSOPHICAL TRANSACTIONS (A)

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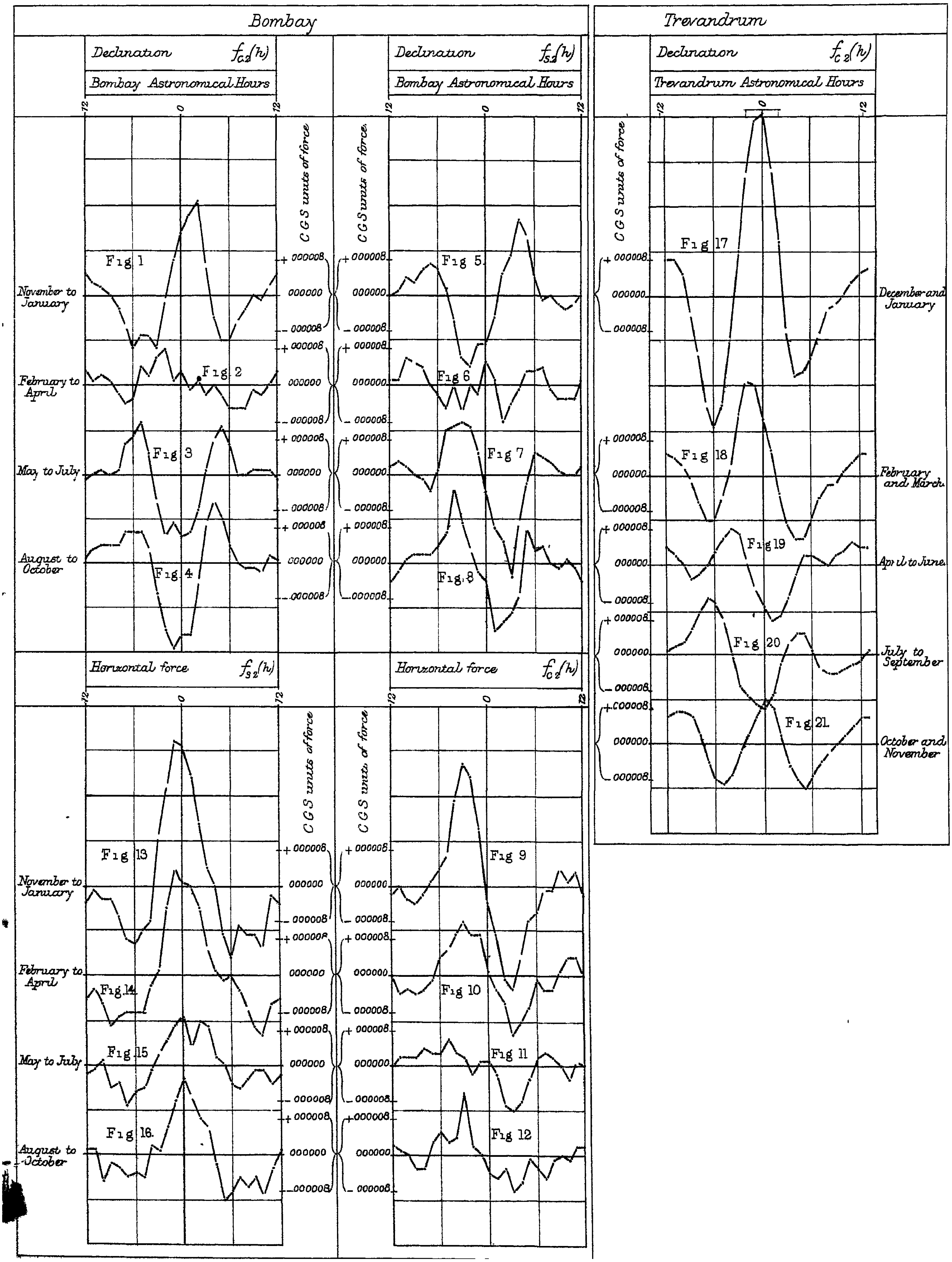
HARRISON AND SONS, PRINTERS IN ORDINARY TO HER MAJESTY, ST. MARTIN'S LANE.

Colaba Observatory, Bombay

Typical variations of declination and horizontal force at Bombay and of declination at Trevandrum



West, Newman & Co lith

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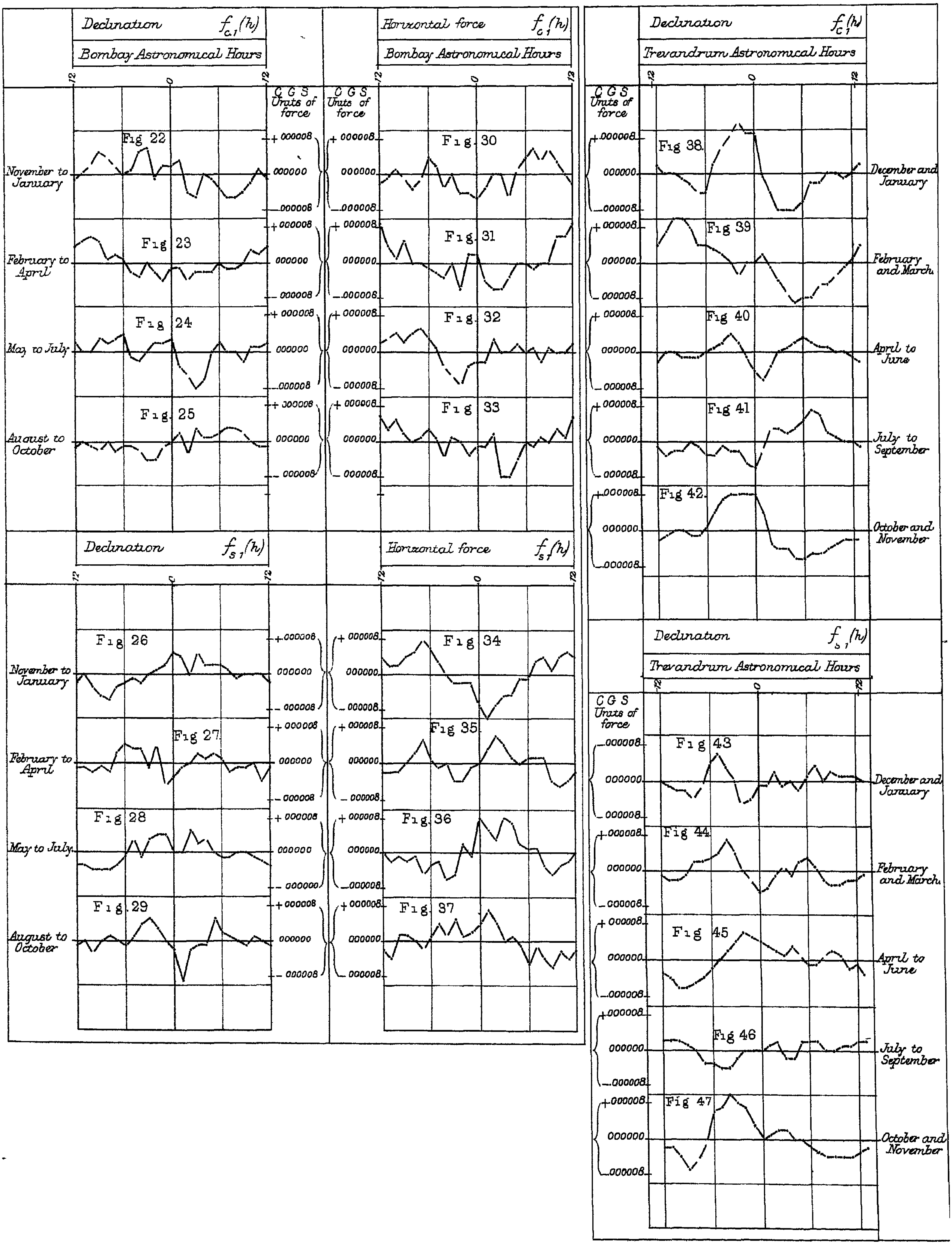
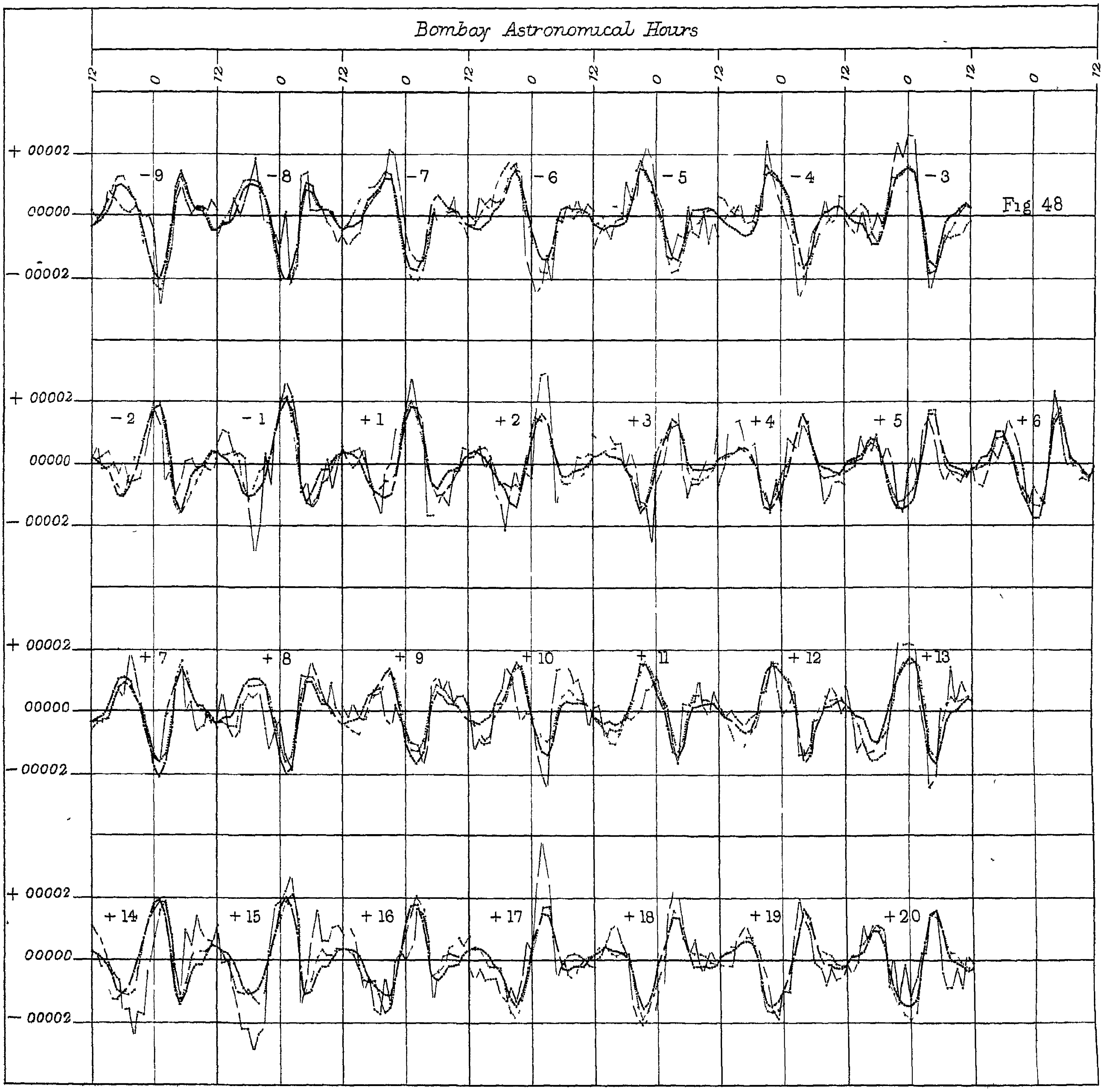
Typical variations of declination and horizontal force at Bombay

Typical variations of declination at Trevandrum



Colaba Observatory, Bombay

Curves representing, for a full lunation and for the quarter November to January, that part of the solar diurnal variation of declination which runs through a cycle of change in a lunation



Dotted curves - - - ——— *calculated by the formula*

$$f_{c1}(h)\cos\frac{2\pi}{P}t+f_{s1}(h)\sin\frac{2\pi}{P}t+f_{c2}(h)\cos 2\frac{2\pi}{P}t+f_{s2}(h)\sin 2\frac{2\pi}{P}t$$

h being the solar hour, t the age of the moon, and P the mean period of a lunation, and $f_{c1}(h)$, $f_{s1}(h)$, &c each being a constant solar diurnal variation

Thick-line curves ——— *calculated by the last two terms only of the same formula*

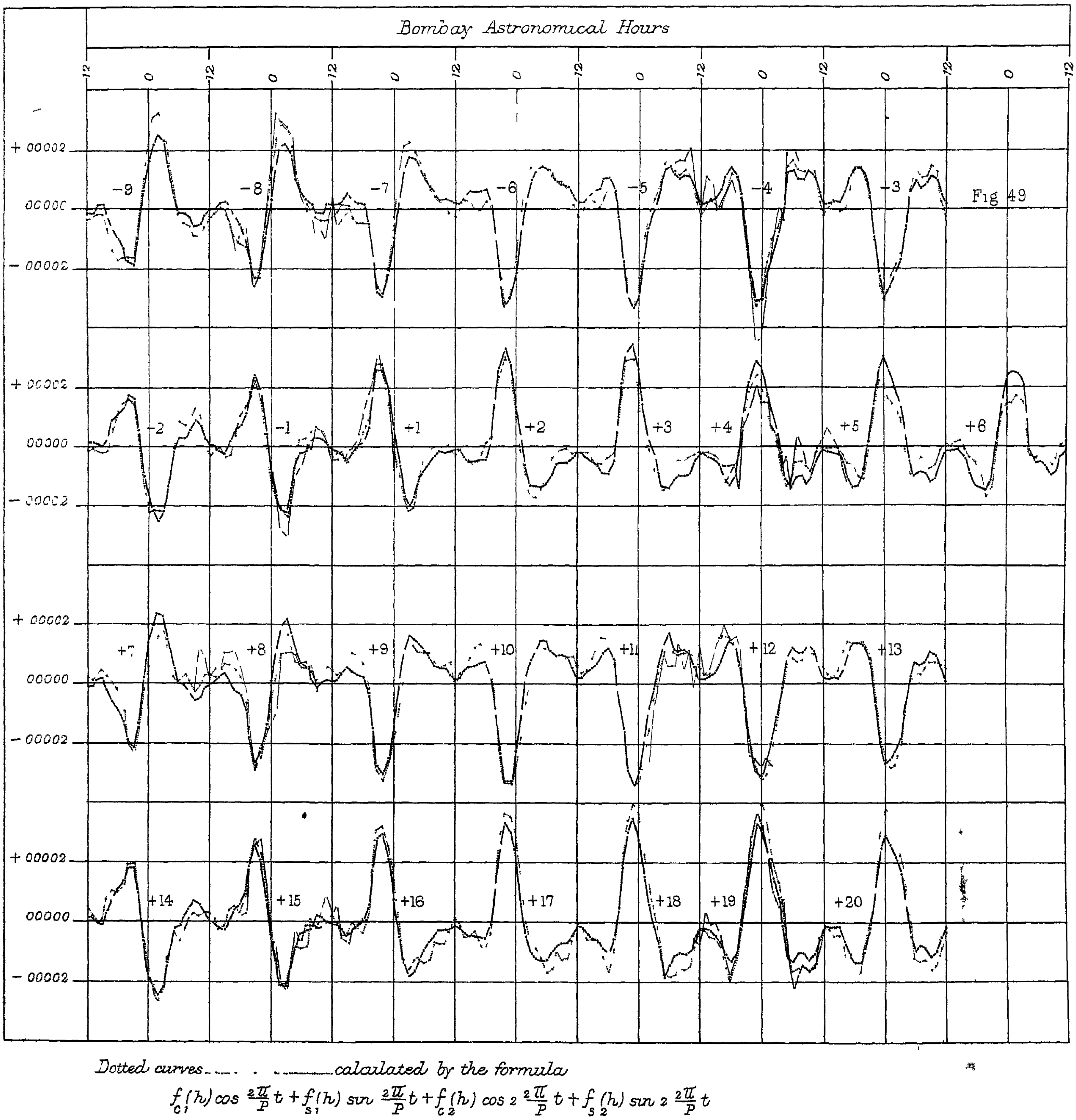
Thin line curves ——— *variations given by observation*

An increase (upward movement) of one inch in the ordinates indicates an increas of the easterly magnetic force of 000048 C G S unit

The numbers with which the curves are marked represent the days before (−) or after (+) New Moon

Colaba Observatory, Bombay

Curves representing, for a full lunation and for the quarter November to January, that part of the Solar diurnal variation of the horizontal force which runs through a cycle of change in a lunation



Dotted curves .... calculated by the formula

$$f_{c1}(h)\cos\frac{2\pi}{P}t + f_{s1}(h)\sin\frac{2\pi}{P}t + f_{c2}(h)\cos 2\frac{2\pi}{P}t + f_{s2}(h)\sin 2\frac{2\pi}{P}t$$

h being the solar hour, t the age of the moon, and P the mean period of a lunation, and $f_{c1}(h)$, $f_{s1}(h)$, &c each being a constant solar diurnal variation

Thick line curves ———— —— calculated by the last two terms only of the same formula

Thin line curves —— —— variations given by observation

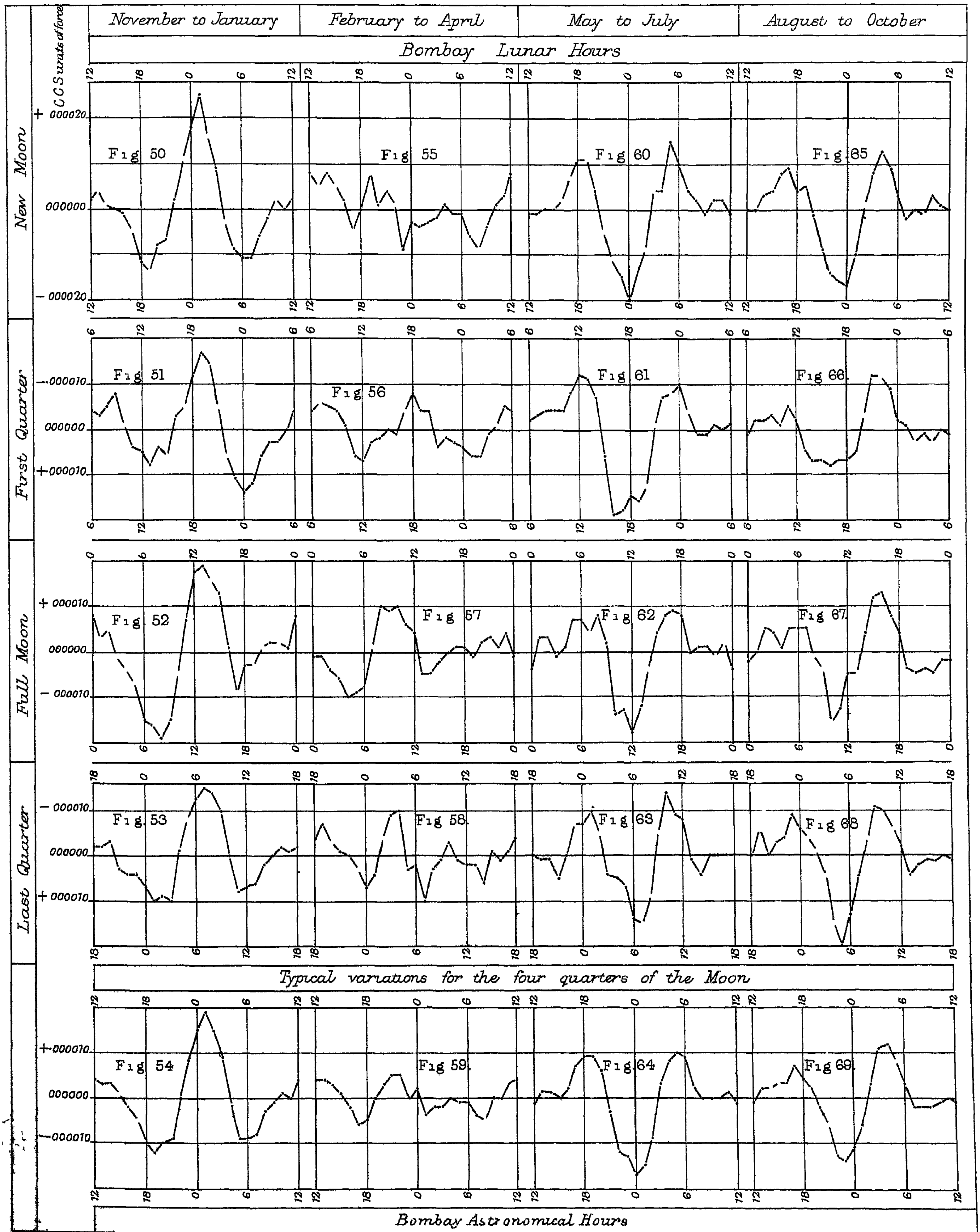
An increase (upward movement) of one inch in the ordinates indicates an increase of the northerly magnetic force of 000048 C G S unit

The numbers with which the curves are marked represent the days before (−) or after (+) New Moon

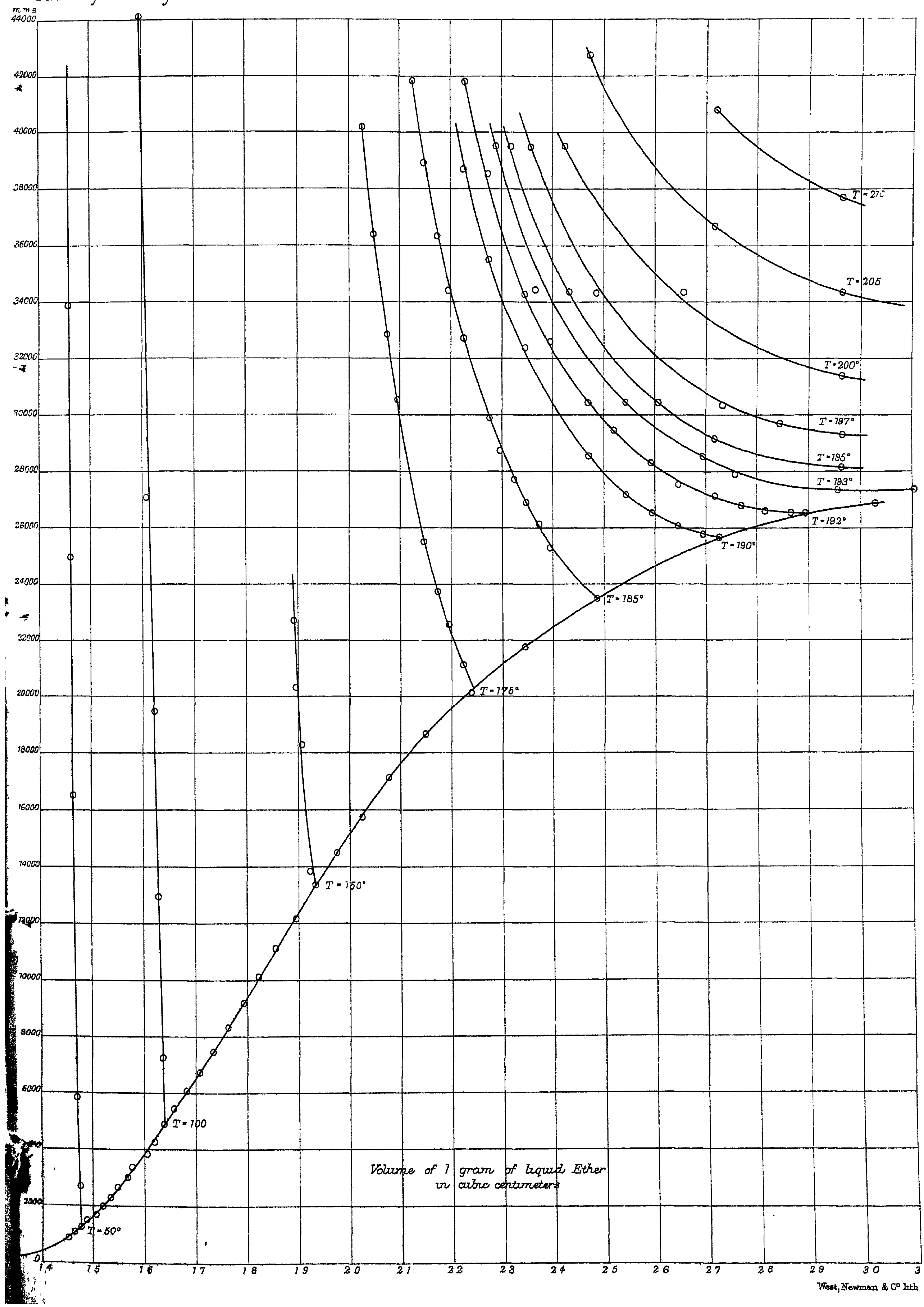
West, Newman & Co lith

Colaba Observatory, Bombay

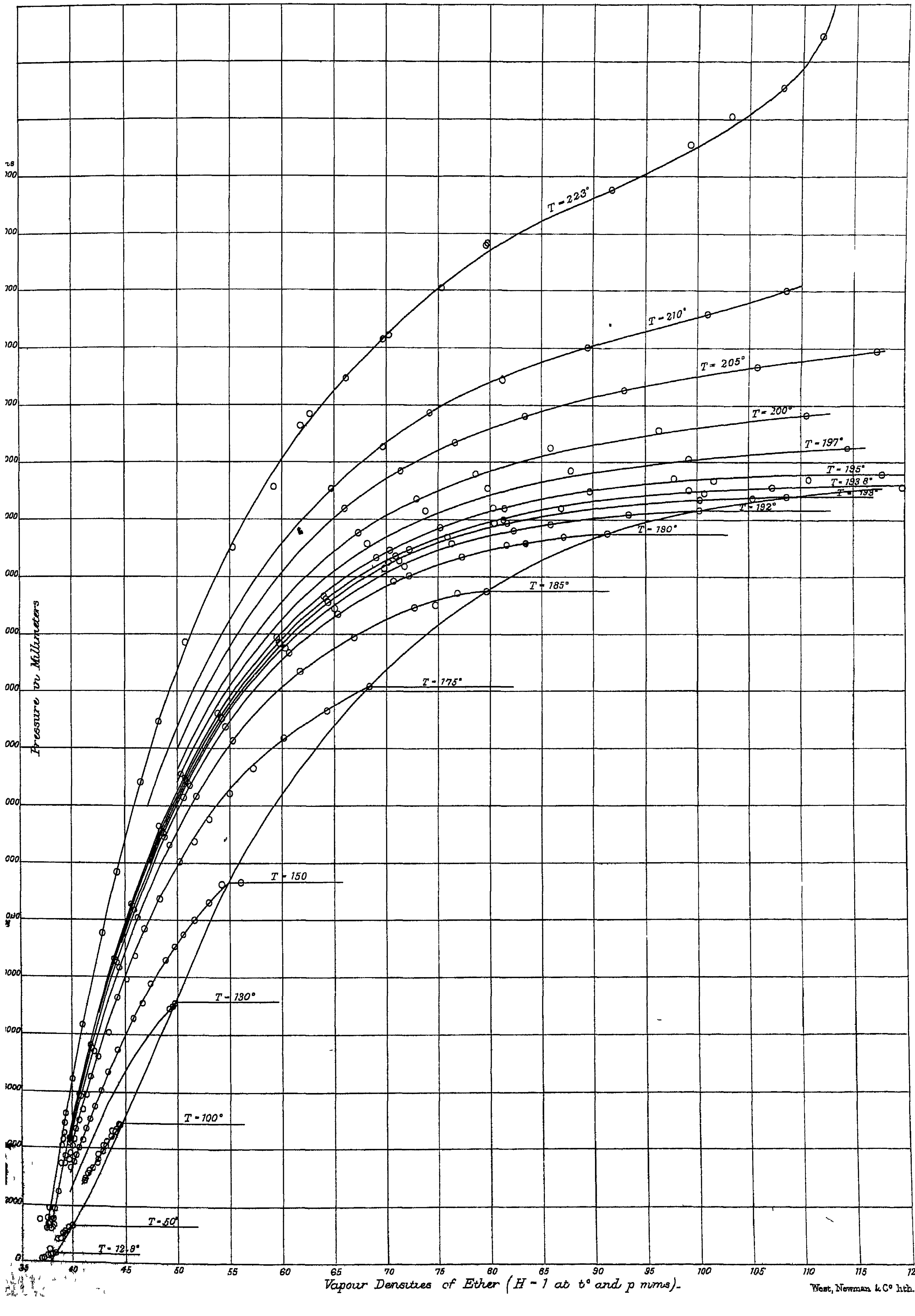
Curves representing the lunar diurnal variations of declination, for the four quarters of the Moon in each of four seasons of the year, and the typical variation for each season



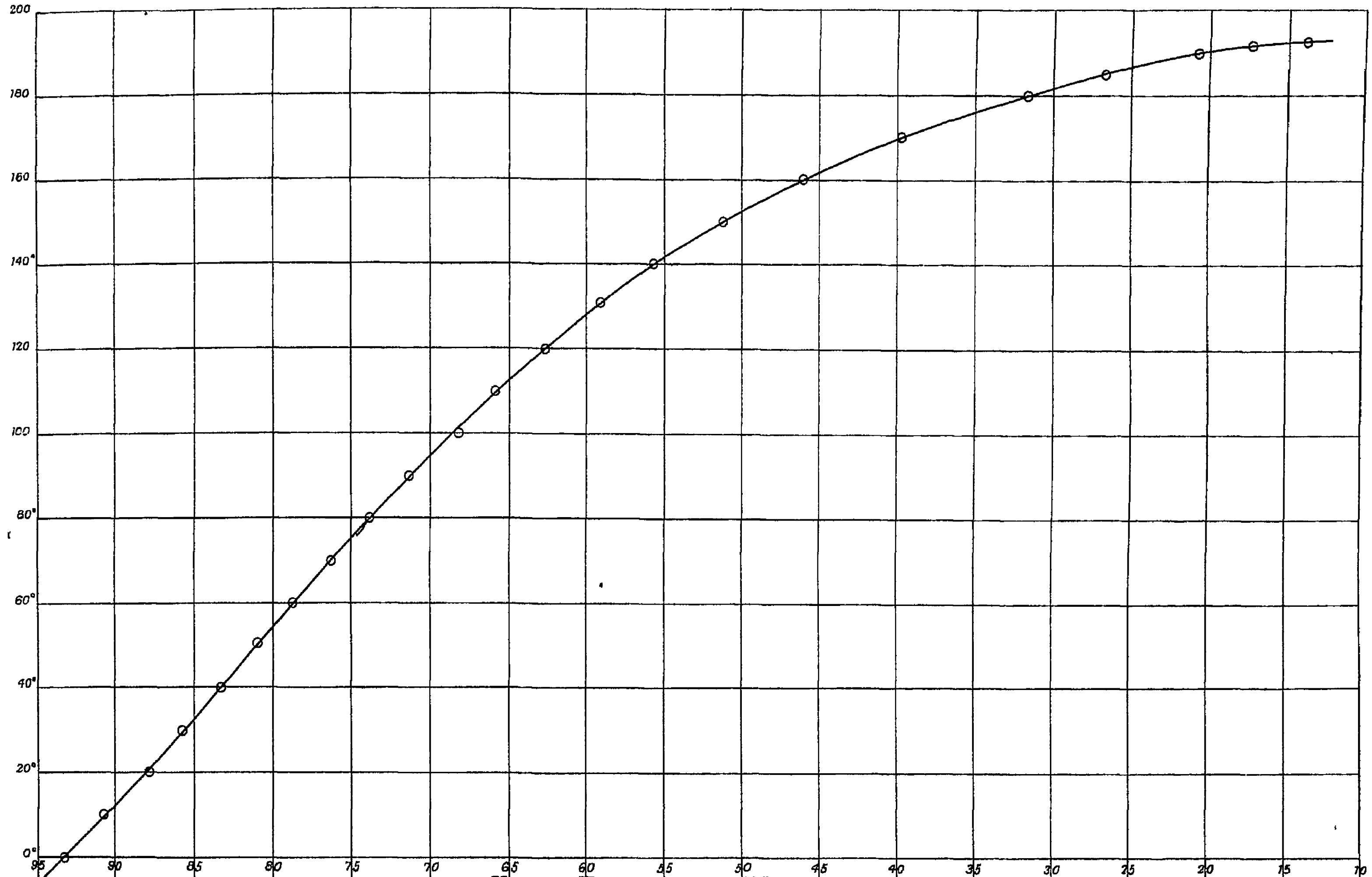
West Newman & Co lith.



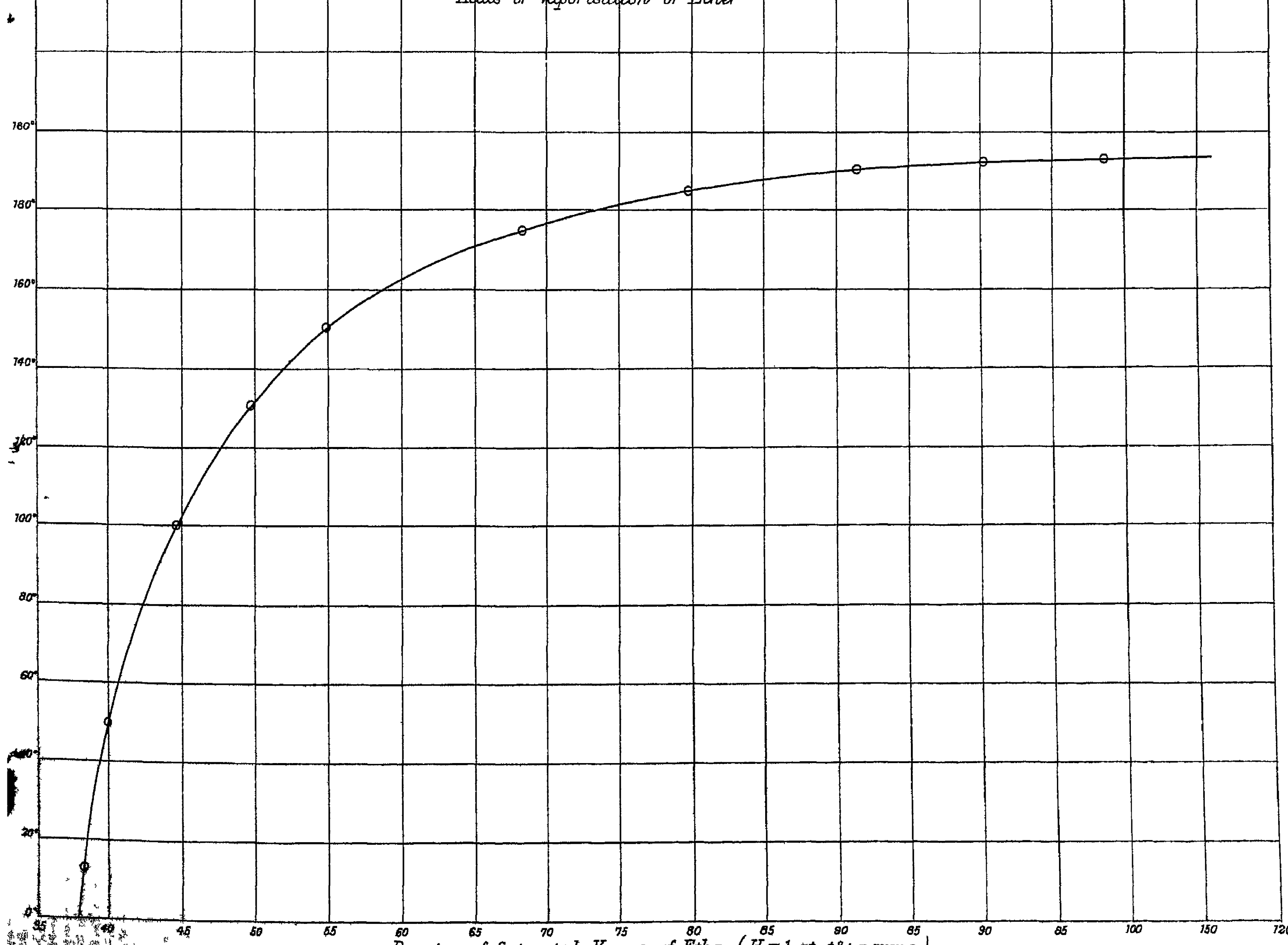
West, Newman & Co lith



West, Newman & Co lith.

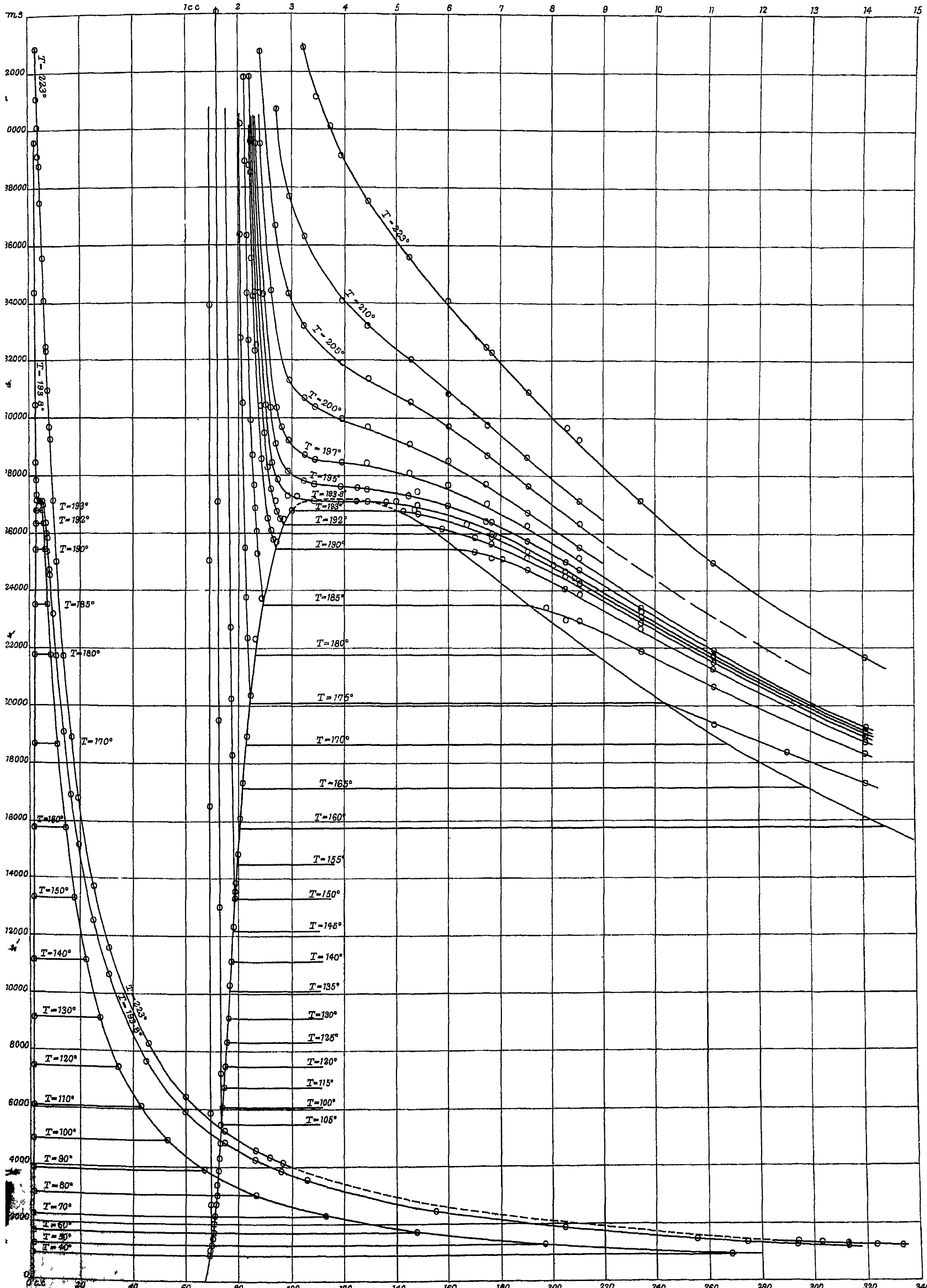


Heats of Vaporisation of Ether

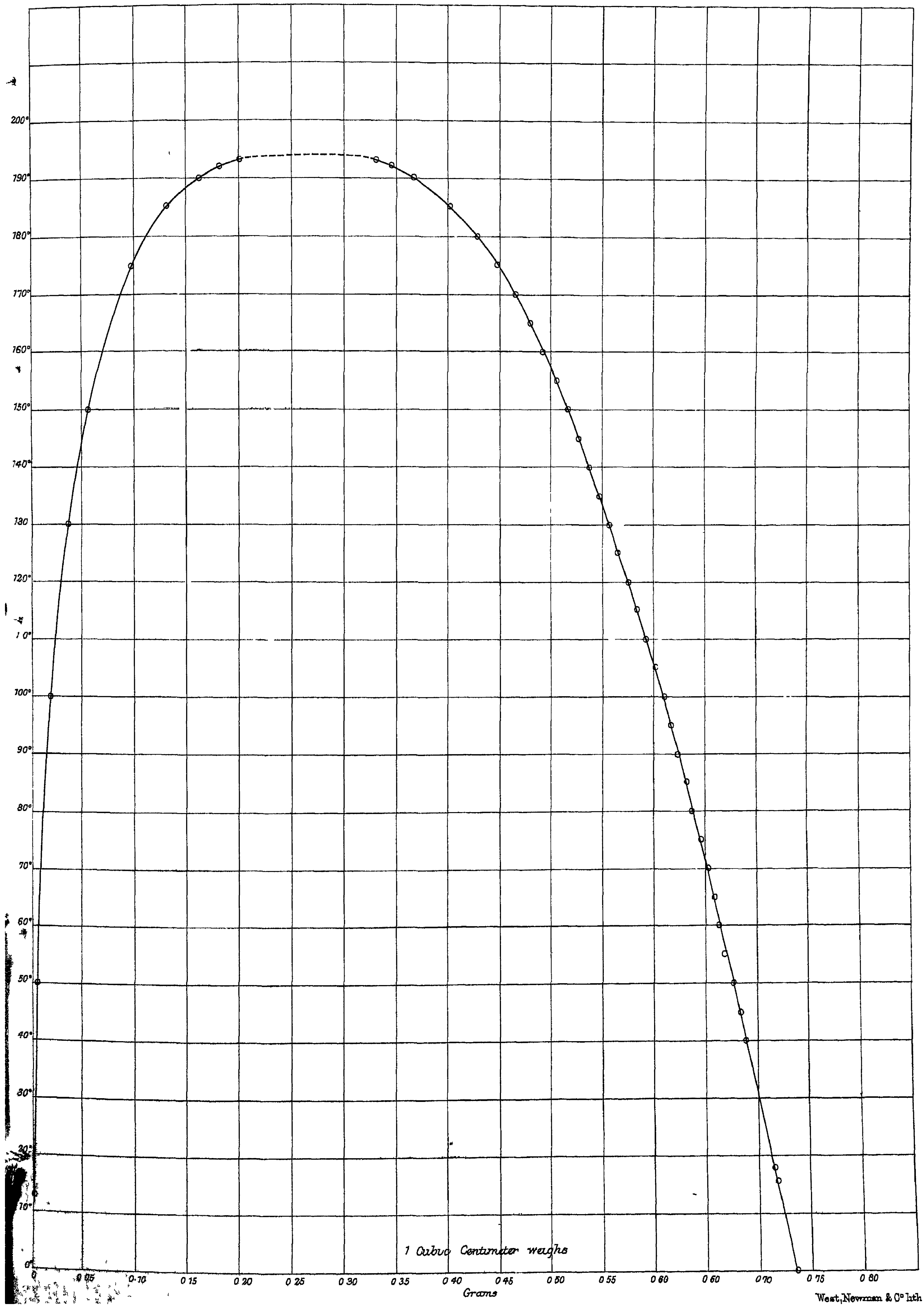


Densities of Saturated Vapour of Ether (H = 1 at t° + p mms).

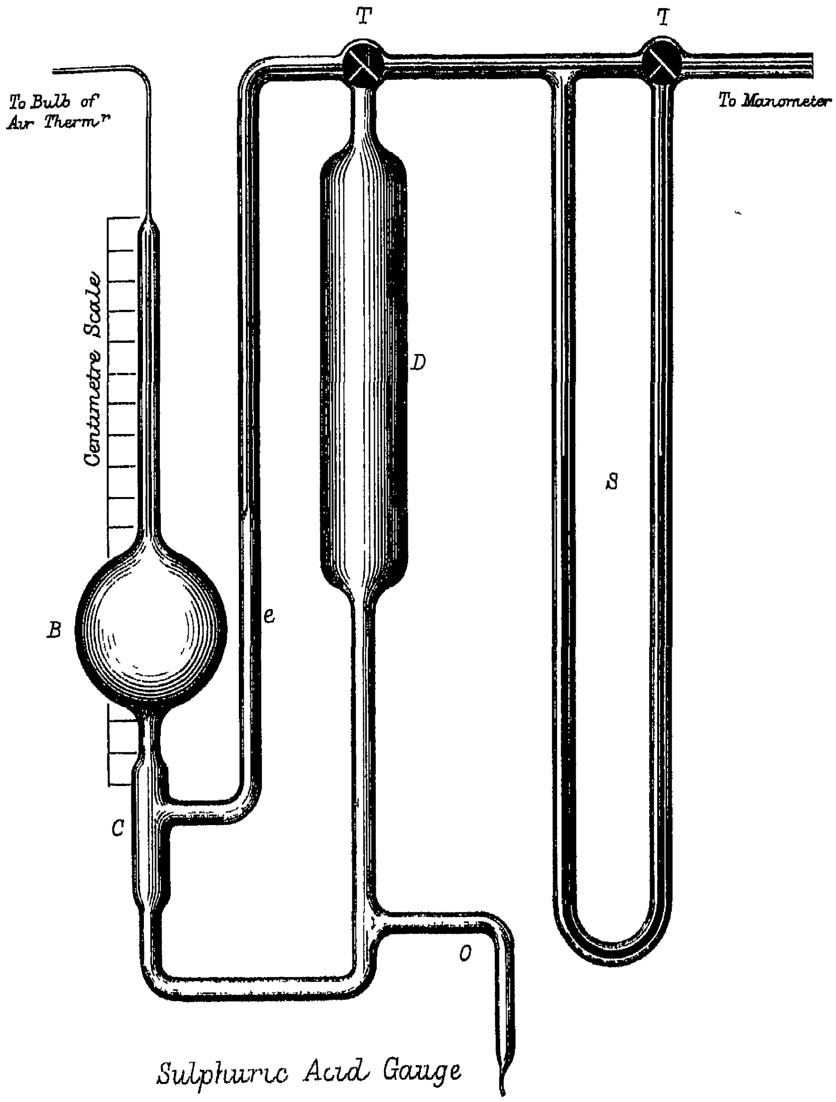
West, Newman & C° lith



West, Newman & Co lith.



West, Newman & C° lith



Sulphuric Acid Gauge

Fig 2

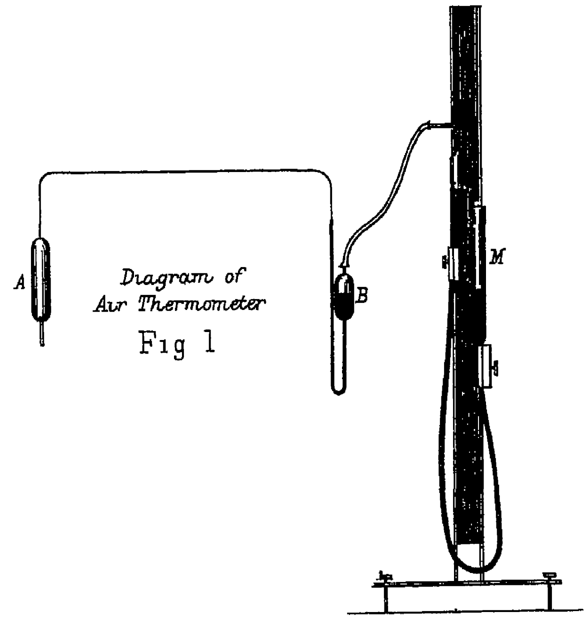


Diagram of Air Thermometer

Fig 1

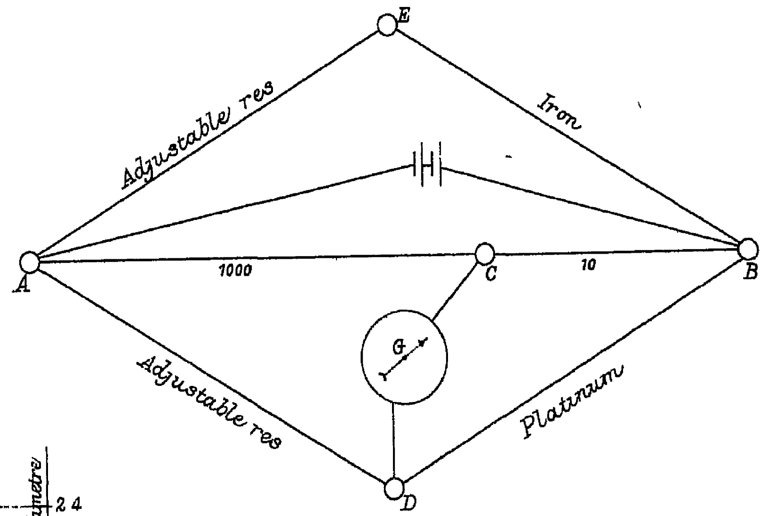


Diagram of Connections Fig 3

CORRECTION DIAGRAM

Air Thermometer N° 2

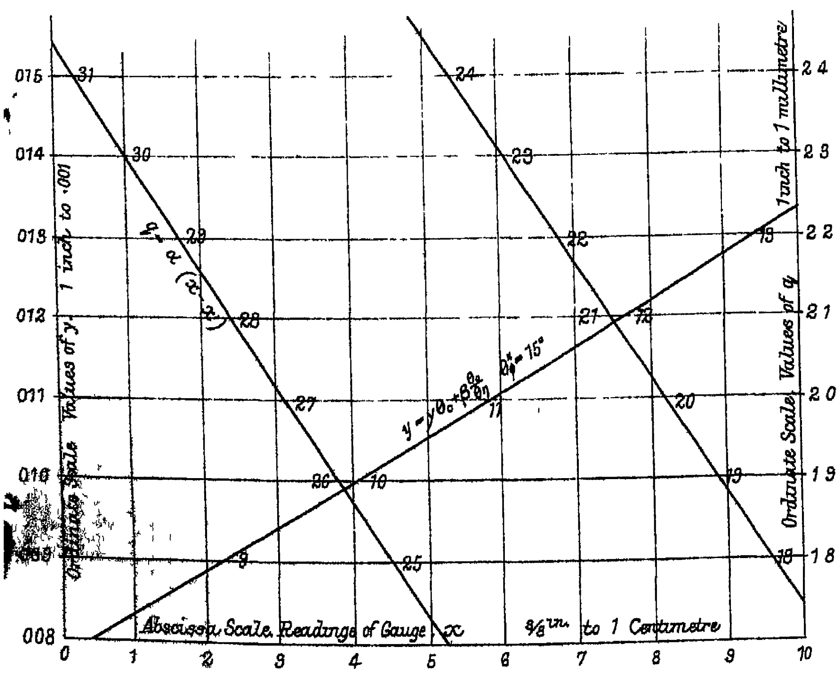
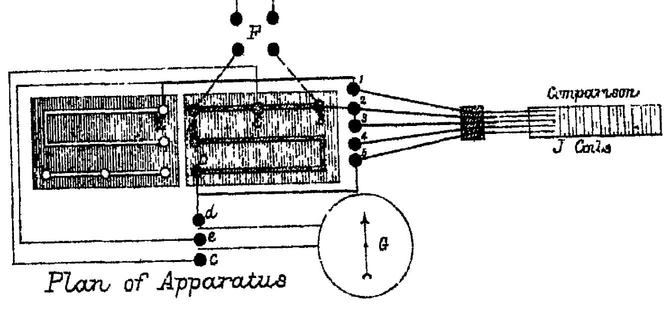


Fig 5

Fig 4



Plan of Apparatus

DIAGRAM OF SULPHURIC ACID GAUGE

Fig 6

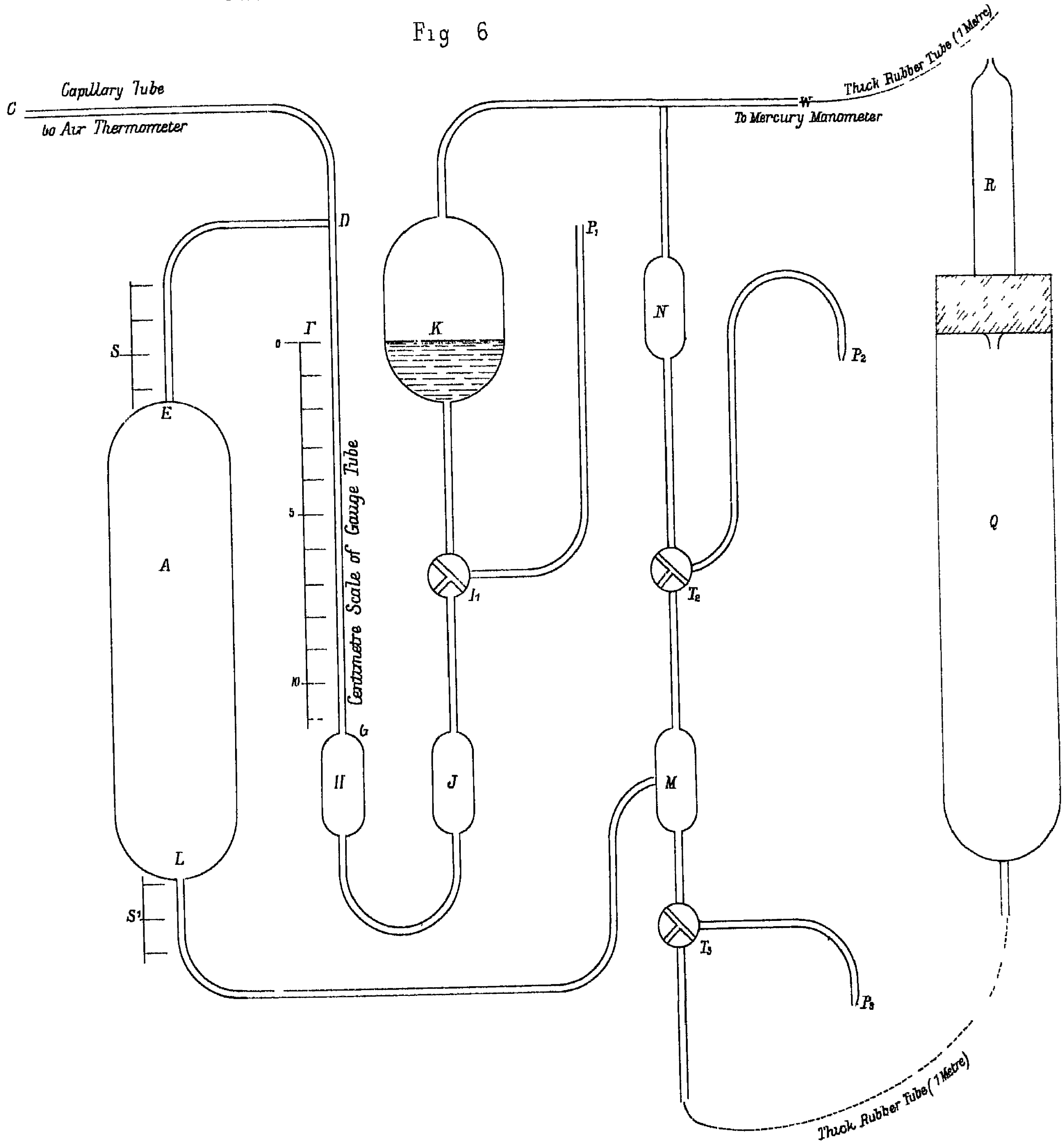


Fig 7

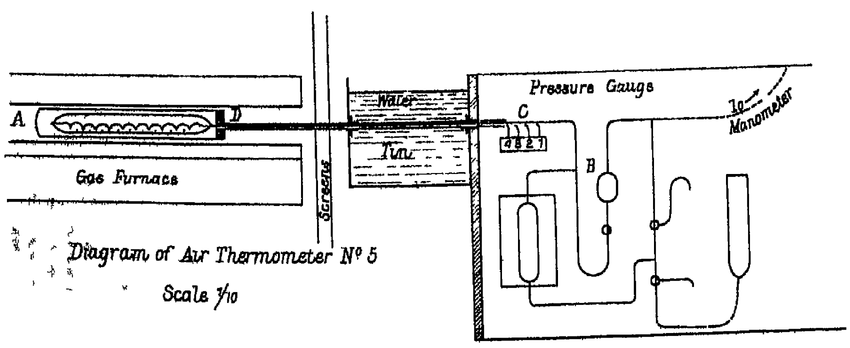
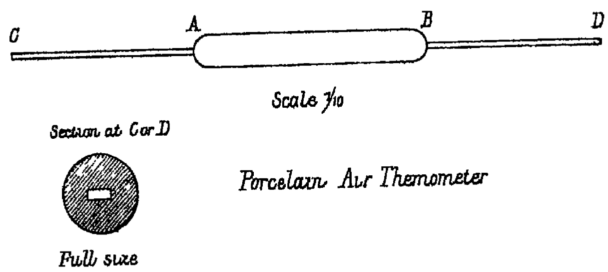


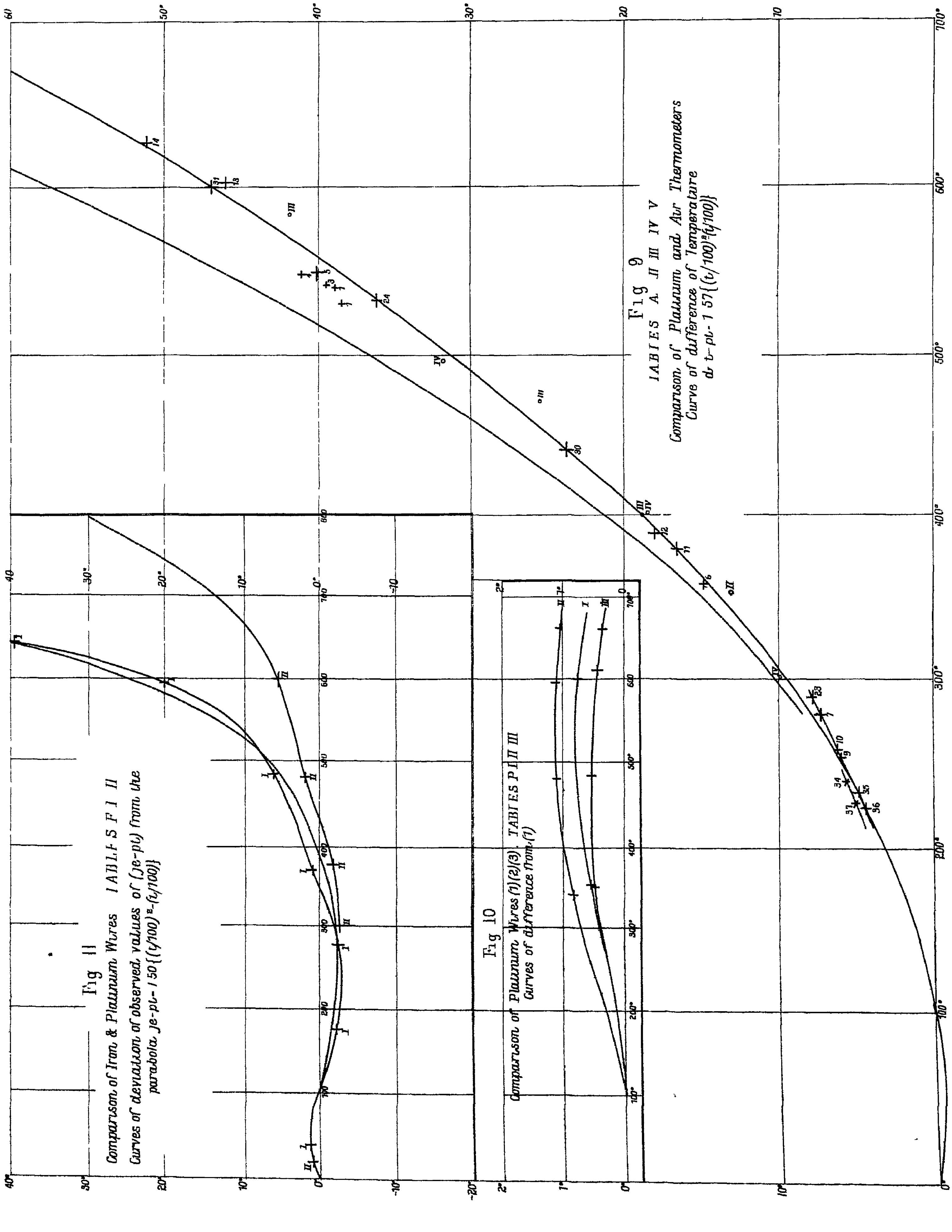
Diagram of Air Thermometer No 5

Scale 1/10

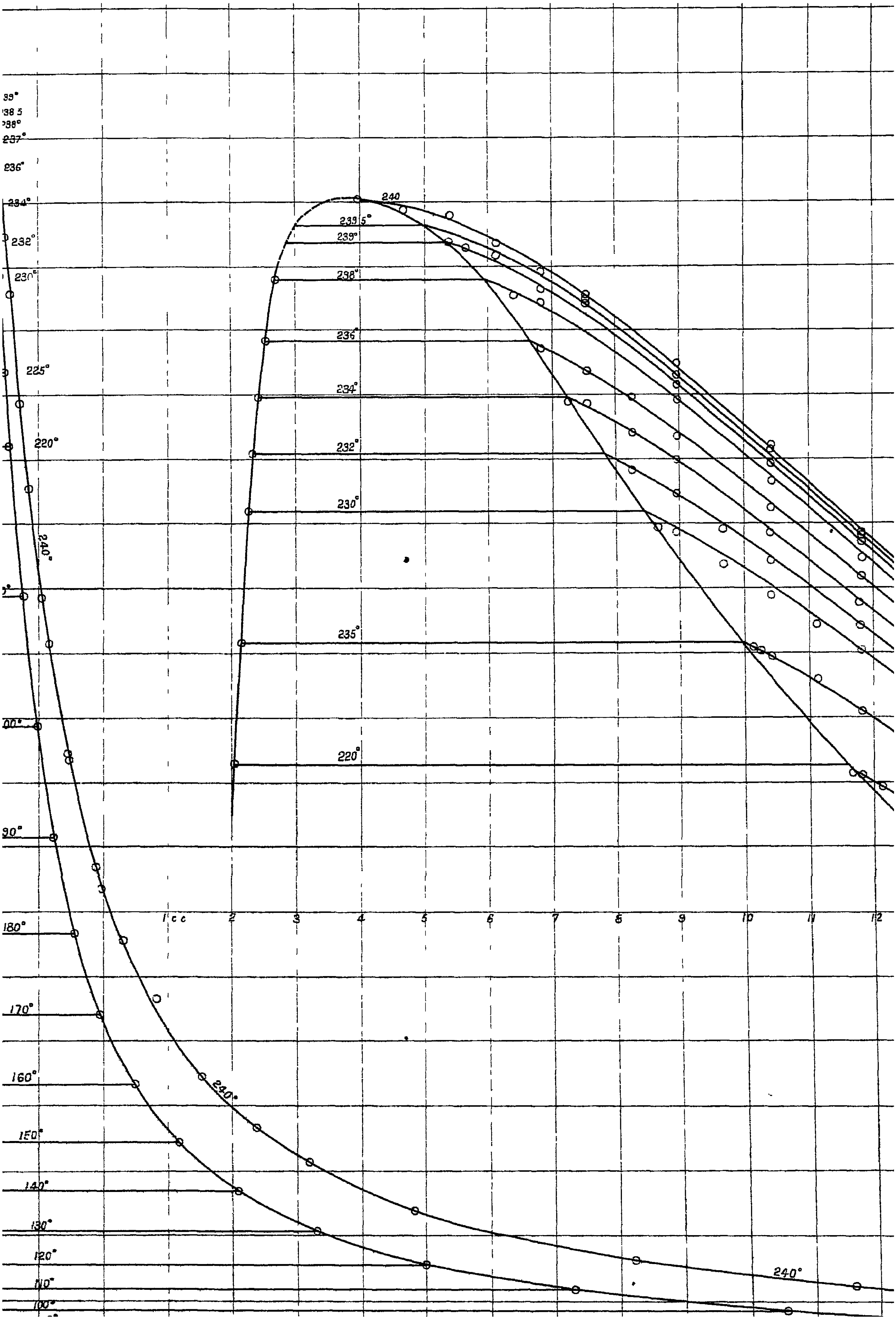
Fig 8



Porcelain Air Thermometer



West, Newman & Co. lith.



239°
238.5
238°
237°
236°
234°
232°
230°
225°
220°
240°
200°
190°
180°
170°
160°
150°
140°
130°
120°
110°
100°
240
239.5°
239°
238°
236°
234°
232°
230°
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220°
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2
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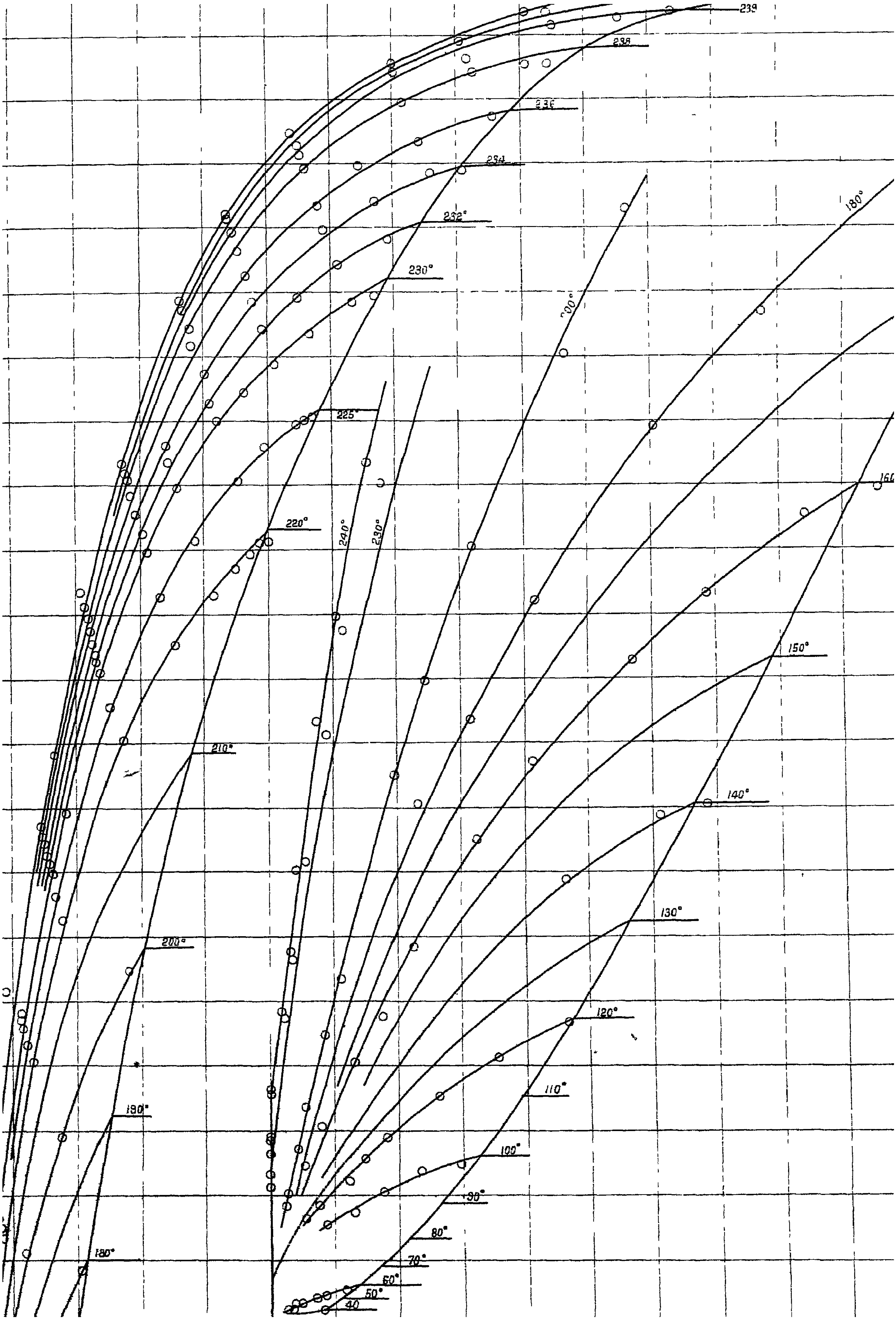
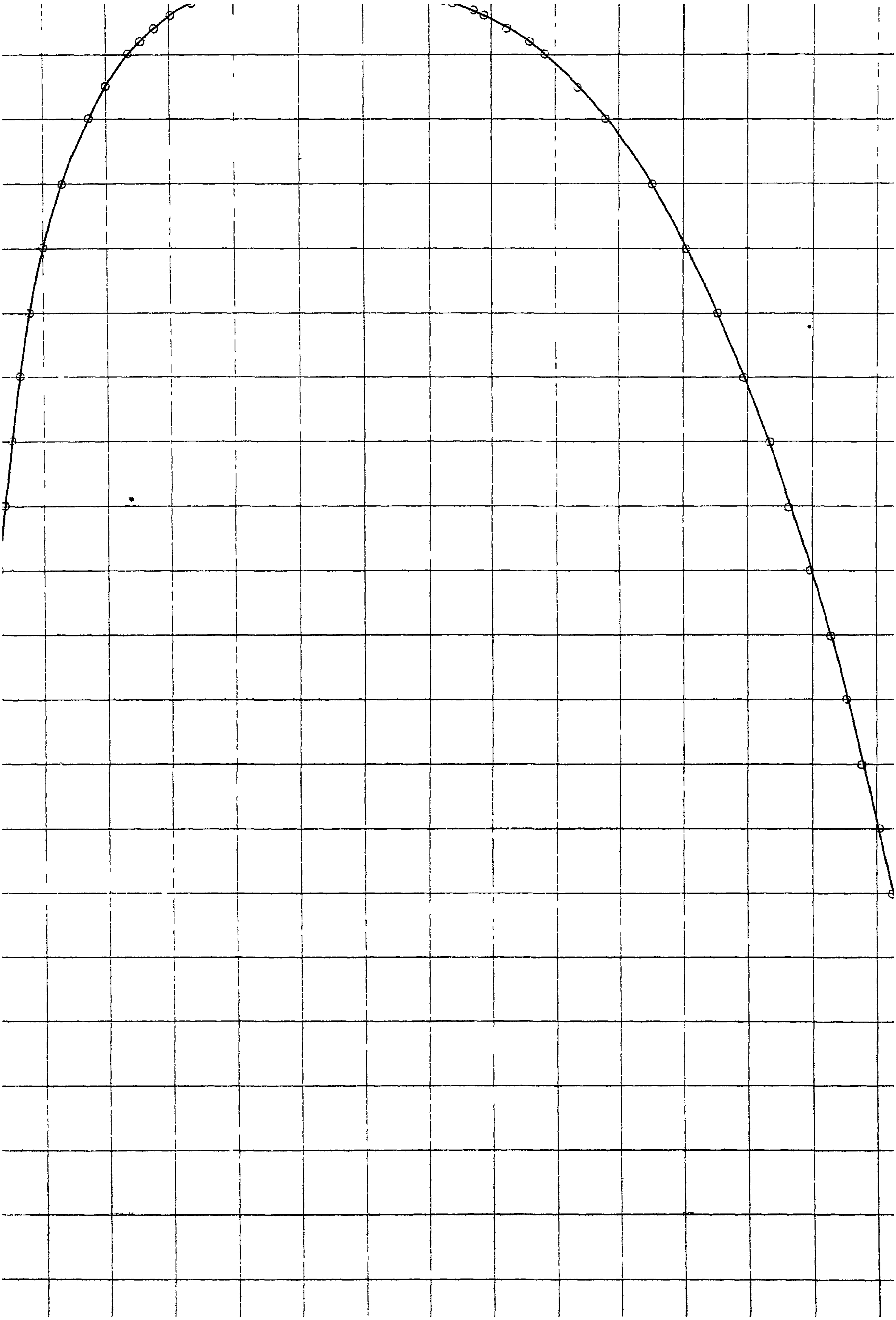
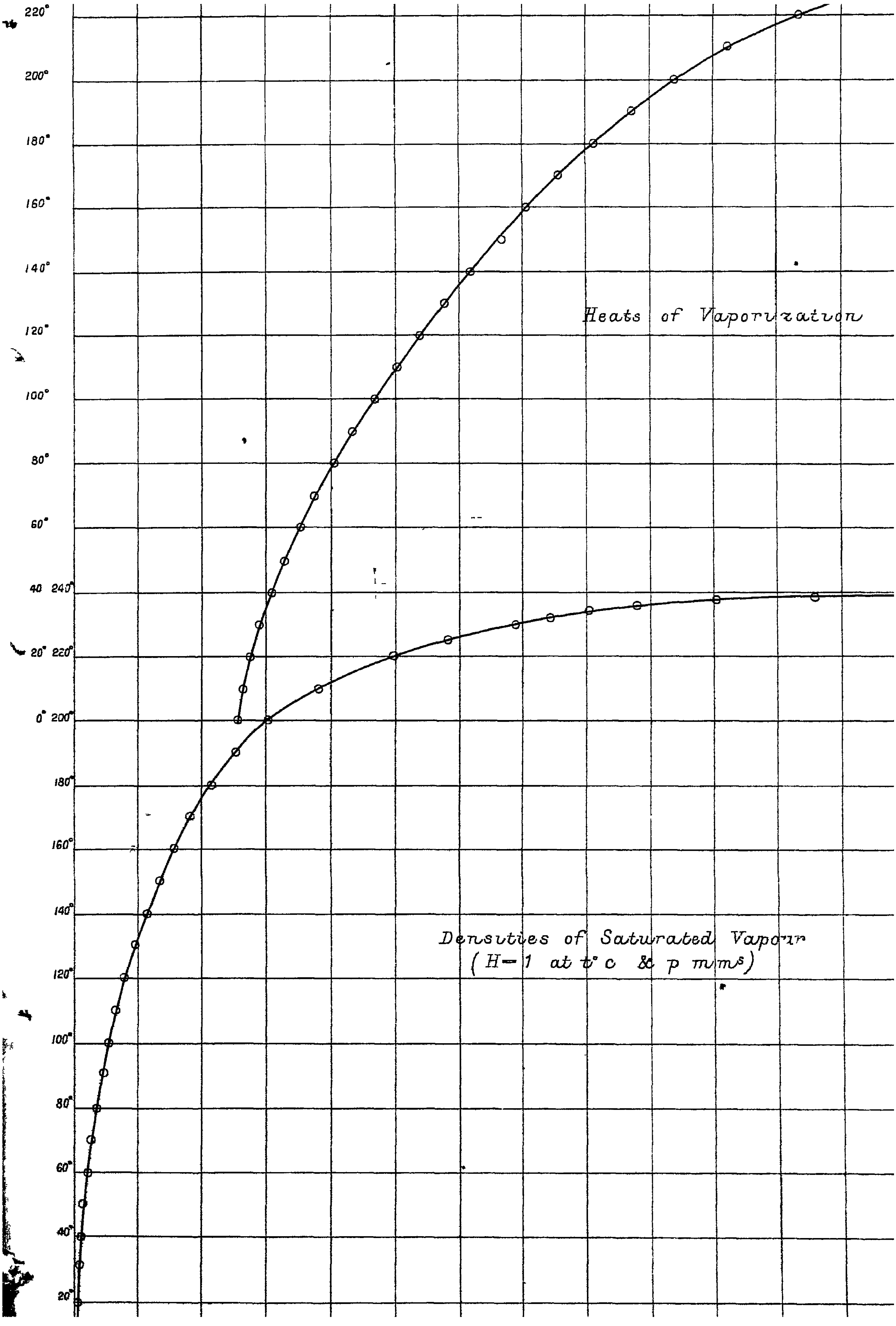
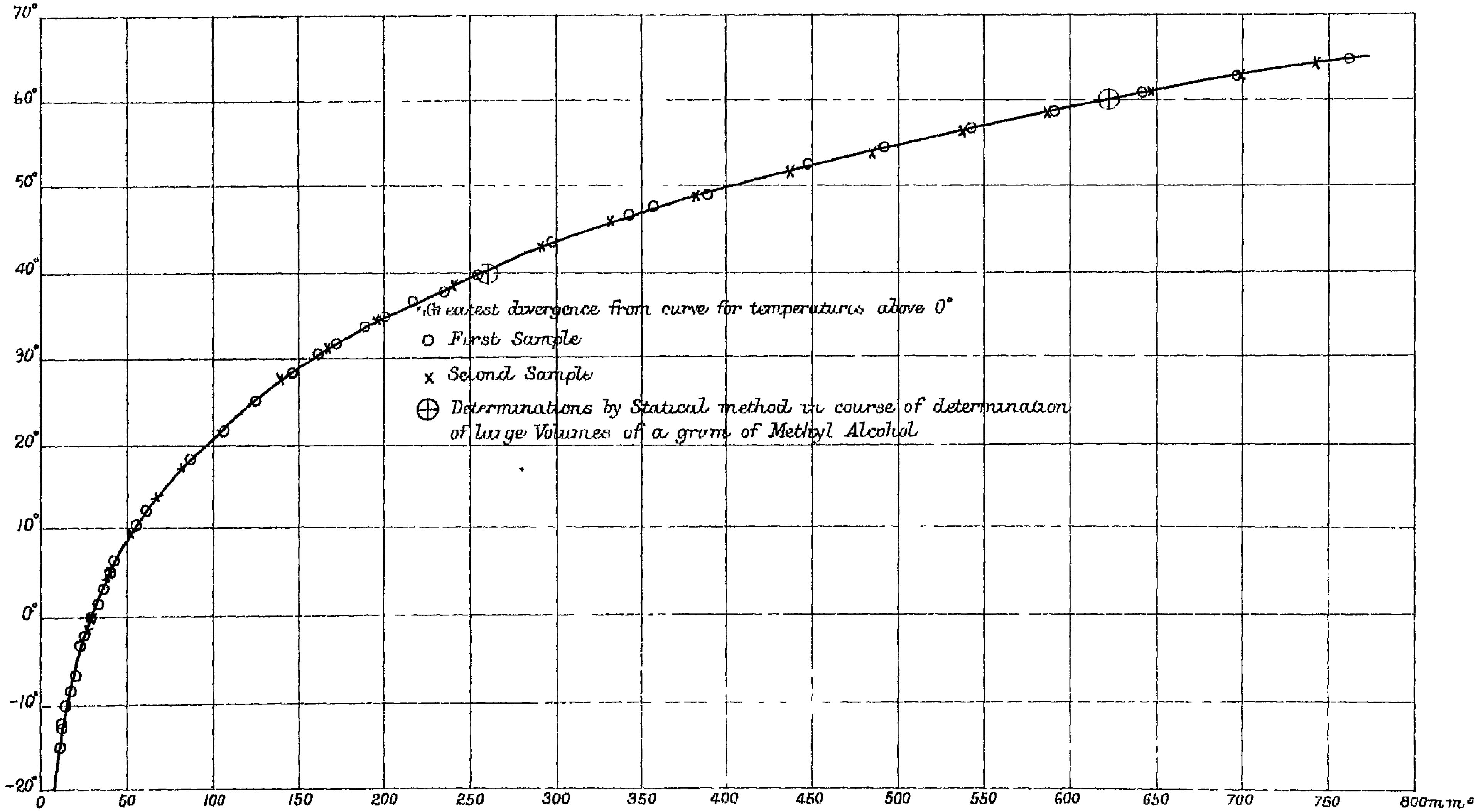
5
6
7
8
9
10
11
12
240°
240°


239
238
236
234
232°
230°
225°
220°
210°
200°
190°
180°
240°
230°
200°
180°
160
150°
140°
130°
120°
110°
100°
90°
80°
70°
60°
50°
40

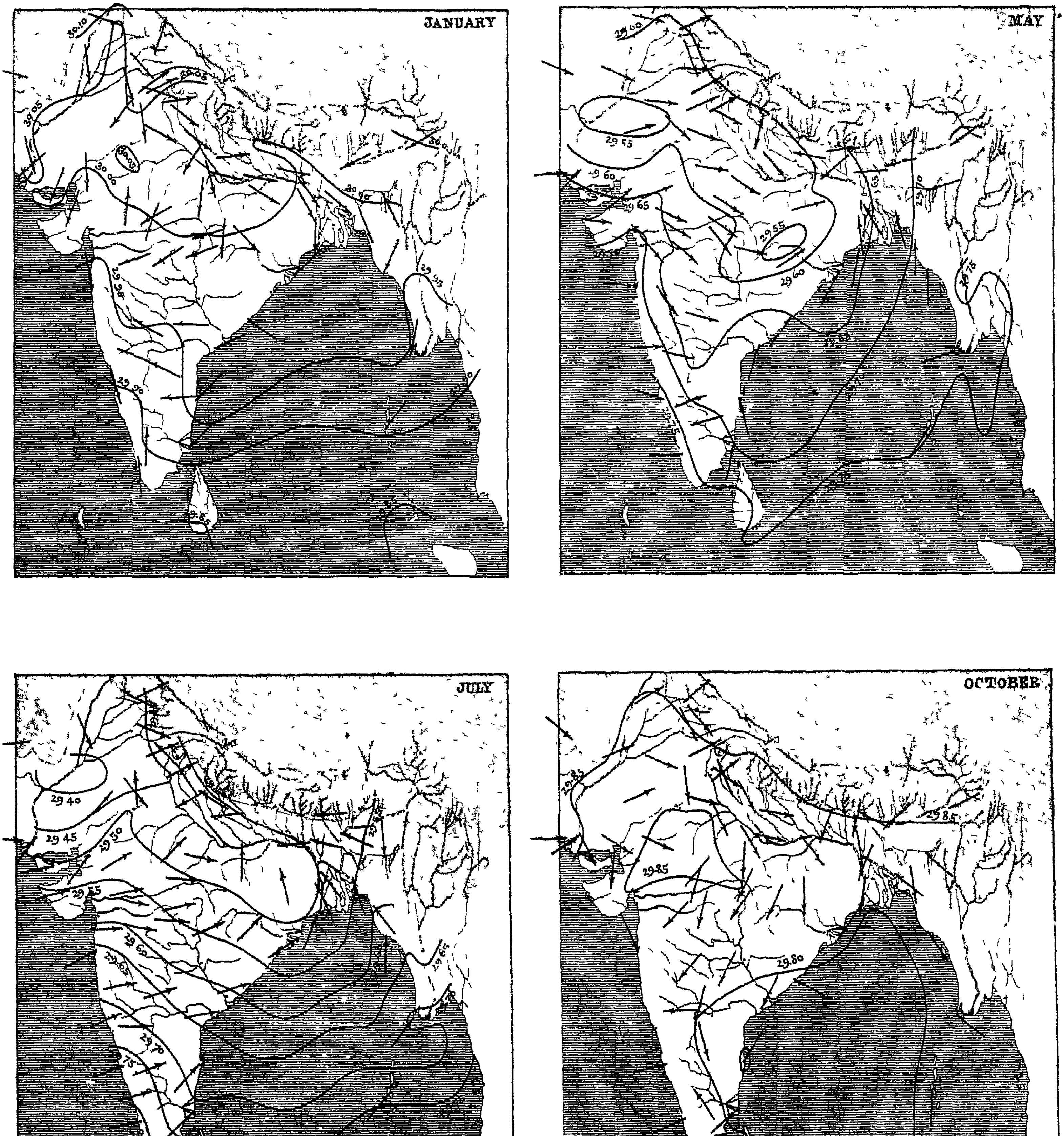




Heats of Vaporization
Densities of Saturated Vapour
(H = 1 at t° c & p m/m^s)
220°
200°
180°
160°
140°
120°
100°
80°
60°
40 240°
20° 220°
0° 200°
180°
160°
140°
120°
100°
80°
60°
40°
20°

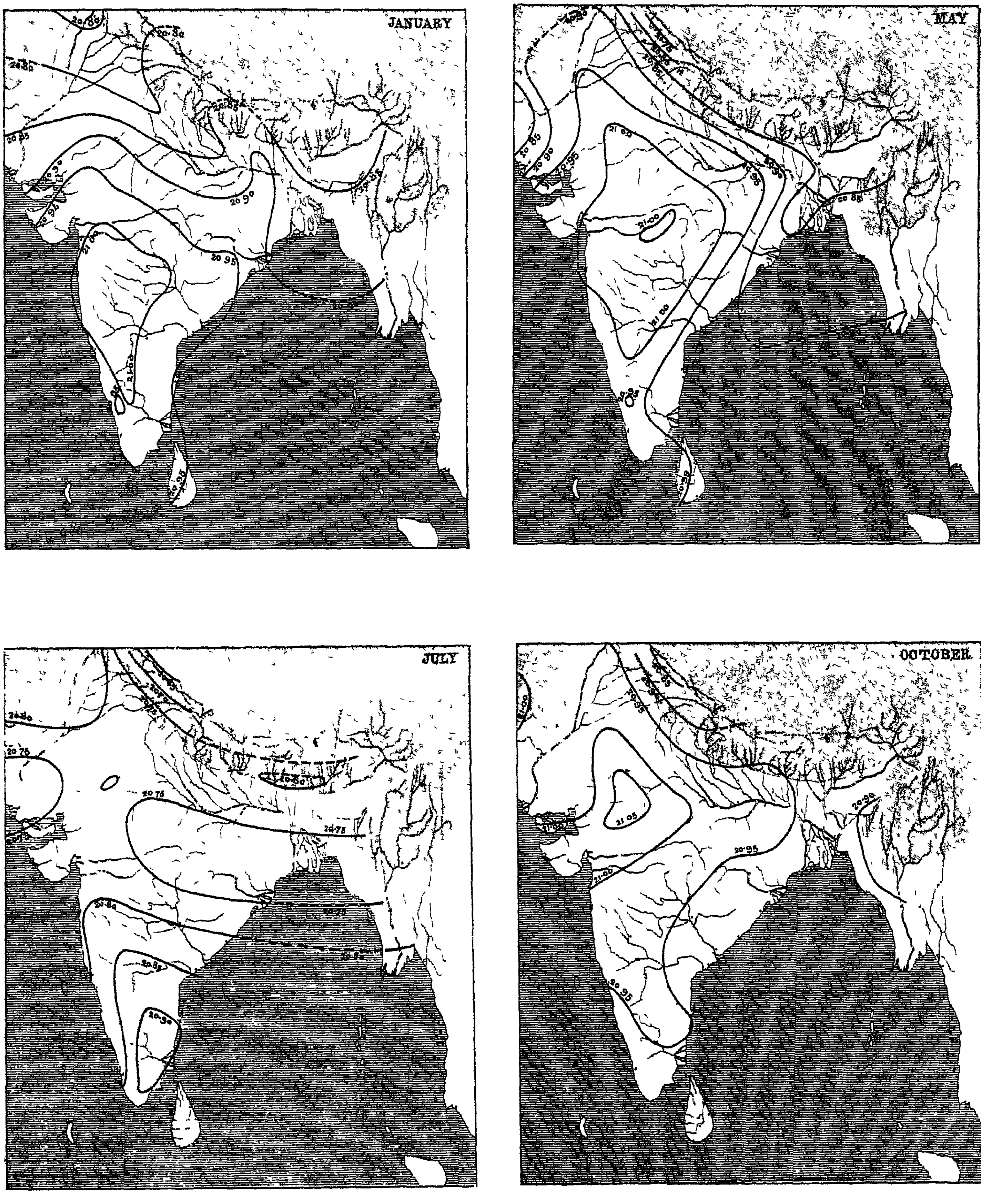

70°
60°
50°
40°
30°
20°
10°
0°
-10°
-20°
0
50
100
150
200
250
300
350
400
450
500
550
600
650
700
750
800mm.s
Greatest divergence from curve for temperatures above 0°
First Sample
Second Sample
Determinations by Statical method in course of determination of large Volumes of a gram of Methyl Alcohol

MEAN PRESSURES AT SEA-LEVEL

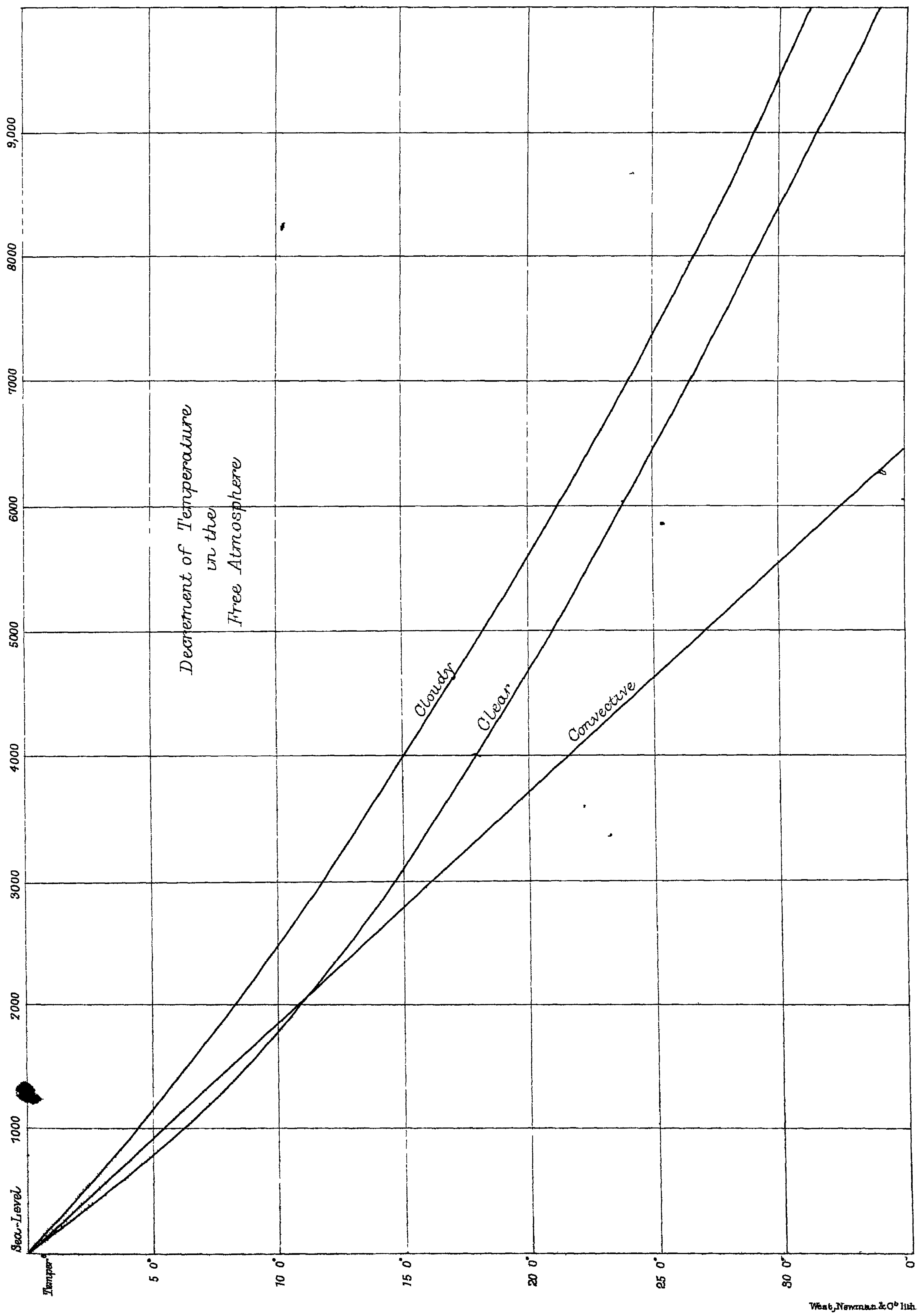


West Newman & C° lith

MID-DAY PRESSURES AT 10,000 FEET



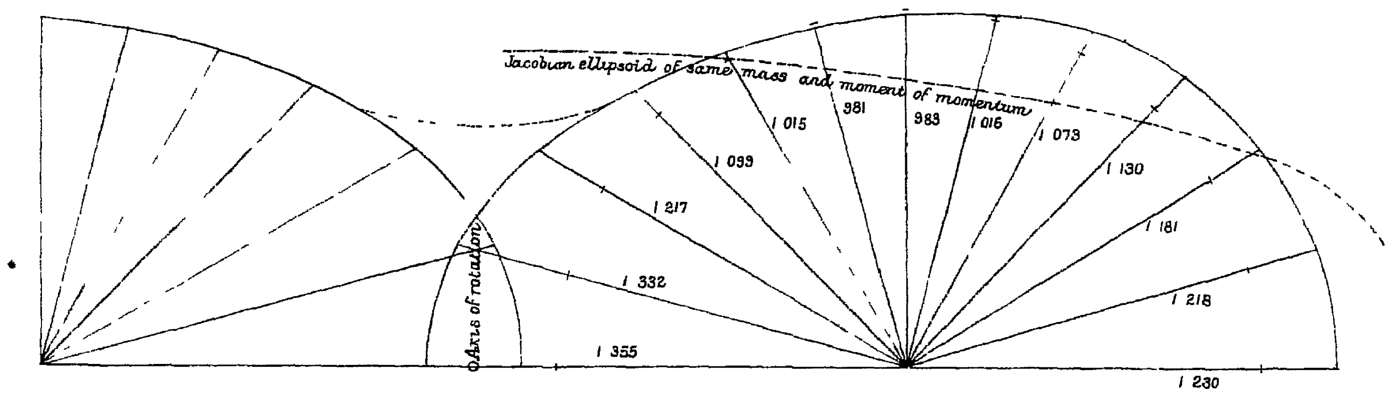
West, Newman & Co lith.



West, Newman & C^o lith.

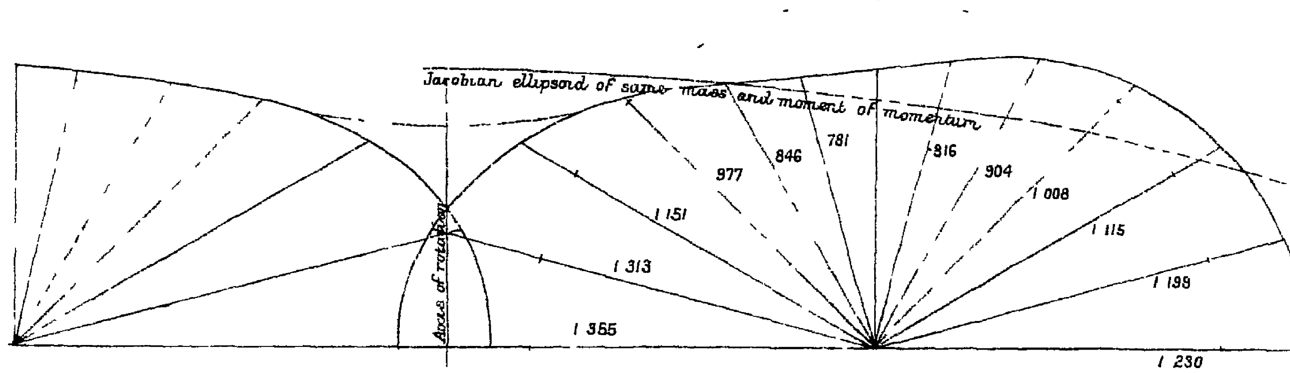
Dumb-bell figure of equilibrium
Section perpendicular to axis of rotation
[$A = a$, $\frac{c}{a} = 2\ 449$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{6}$, $\frac{\omega^2}{4\pi} = 0494$, momentum $= (\frac{4}{3}\pi)^{\frac{3}{2}} b^5 \times 482$]

Fig 2



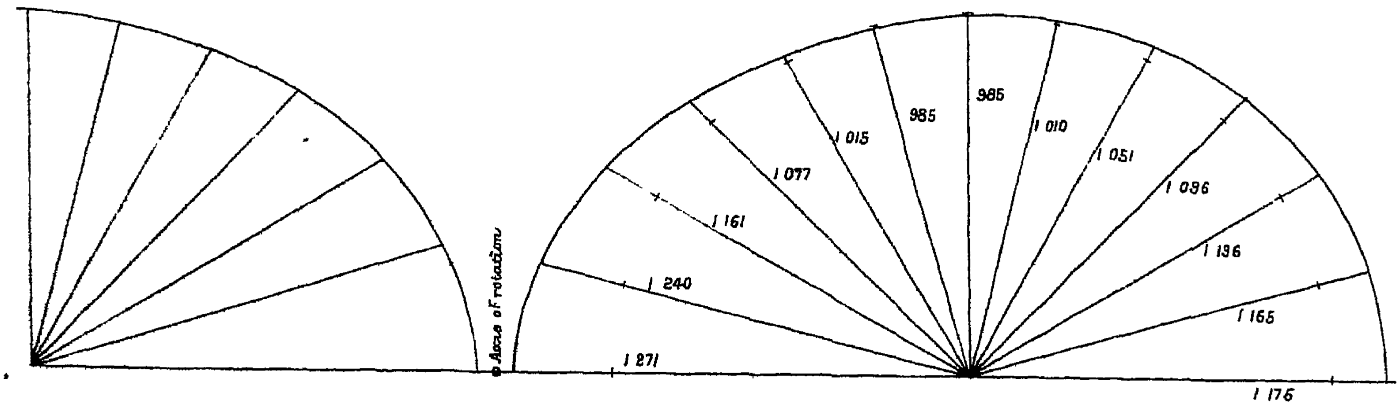
Dumb-bell figure of equilibrium
Section through axis of rotation
[$A = a$, $\frac{c}{a} = 2\ 449$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{6}$, $\frac{\omega^2}{4\pi} = 0494$, momentum $= (\frac{4}{3}\pi)^{\frac{3}{2}} b^5 \times 482$]

Fig 3



Equal masses nearly in contact
Section perpendicular to axis of rotation
[$A = a$, $\frac{c}{a} = 2646$, $\beta = \frac{1}{6}$, $\gamma = \frac{1}{7}$, $\frac{\omega^2}{4\pi} = 038$, momentum $= (\frac{4}{3}\pi)^{\frac{3}{2}} b^5 \times 472$]

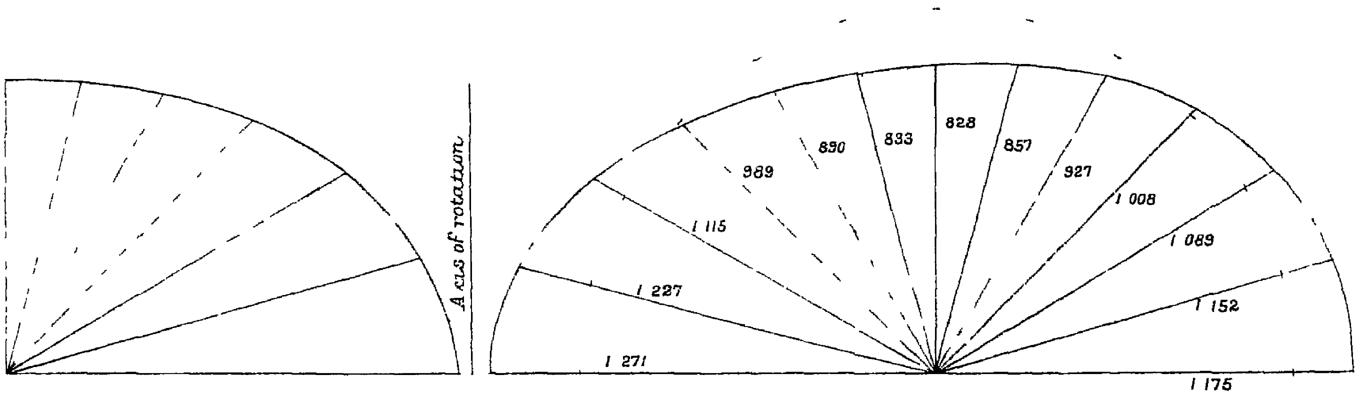
Fig 4



West, Newman & C° lith.

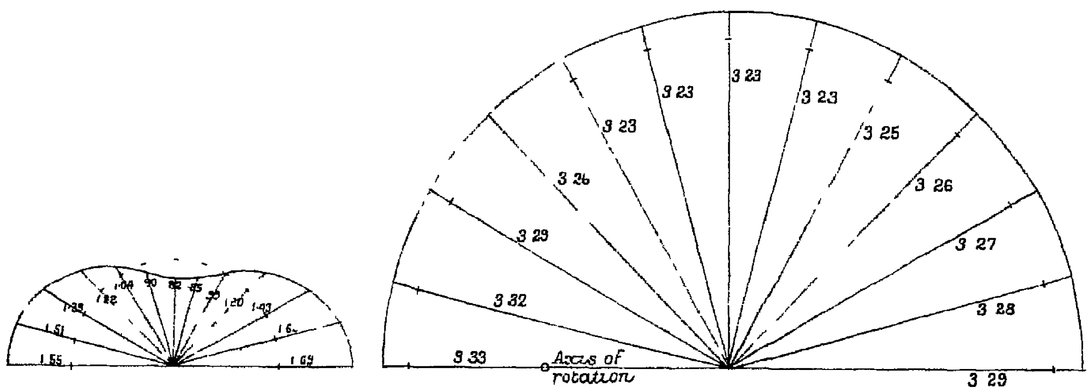
Equal masses nearly in contact
Section through axis of rotation
[$A = a$, $\frac{c}{a} = 2646$, $\beta = \frac{1}{6}$, $\gamma = \frac{1}{7}$, $\frac{\omega^2}{4\pi} = 038$, momentum $= (\frac{4}{3}\pi)^{\frac{1}{2}} b^5 \times 472$]

Fig 5



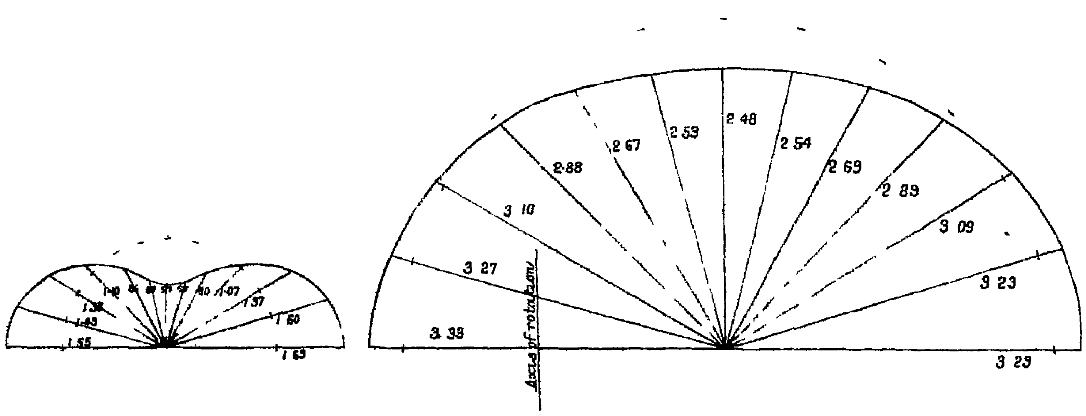
Unequal masses, section perpendicular to axis of rotation
[$\frac{A}{a} = 3$, $(\frac{A}{a})^3 = 27$, $\frac{a}{c} = 189$, $\frac{A}{c} = 566$, $\gamma = 036$, $\Gamma = 321$
$\frac{\omega^2}{4\pi} = 066$, momentum $= (\frac{4\pi}{3})^{\frac{1}{2}} b^5 \times 29$]

Fig 6



Unequal masses, section through axis of rotation
[$\frac{A}{a} = 3$, $(\frac{A}{a})^3 = 27$, $\frac{a}{c} = 186$, $\frac{A}{c} = 566$, $\gamma = 036$, $\Gamma = 321$
$\frac{\omega^2}{4\pi} = 066$, momentum $= (\frac{4\pi}{3})^{\frac{1}{2}} b^5 \times 29$]

Fig 7



West, Newman & Co lith.

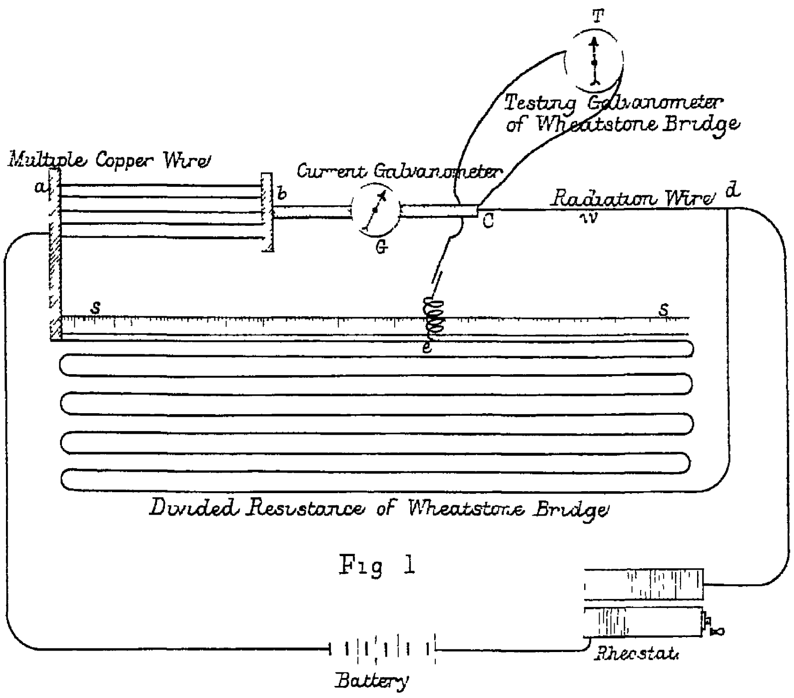


Fig 1

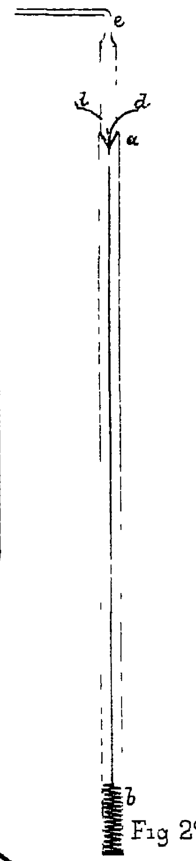


Fig 2^a

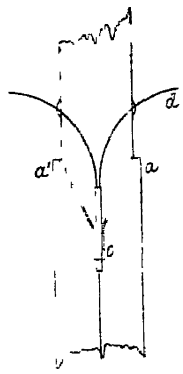


Fig 2^b

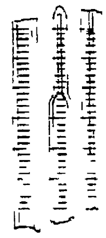


Fig 3^b

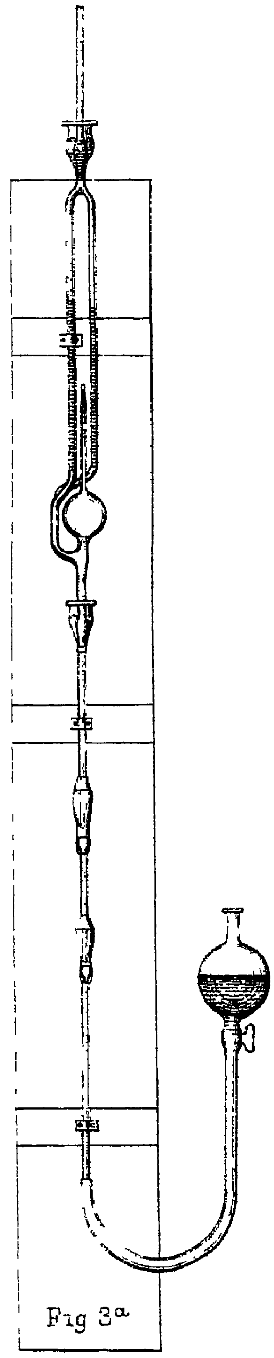


Fig 3^a

Mc Leod Gauge

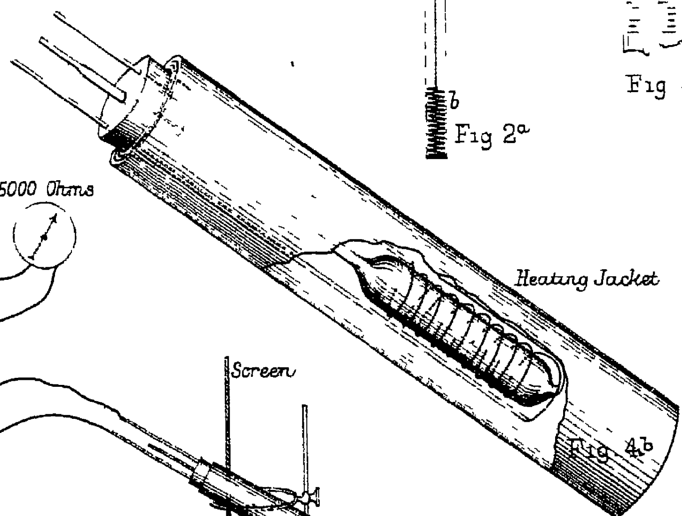


Fig 4^b

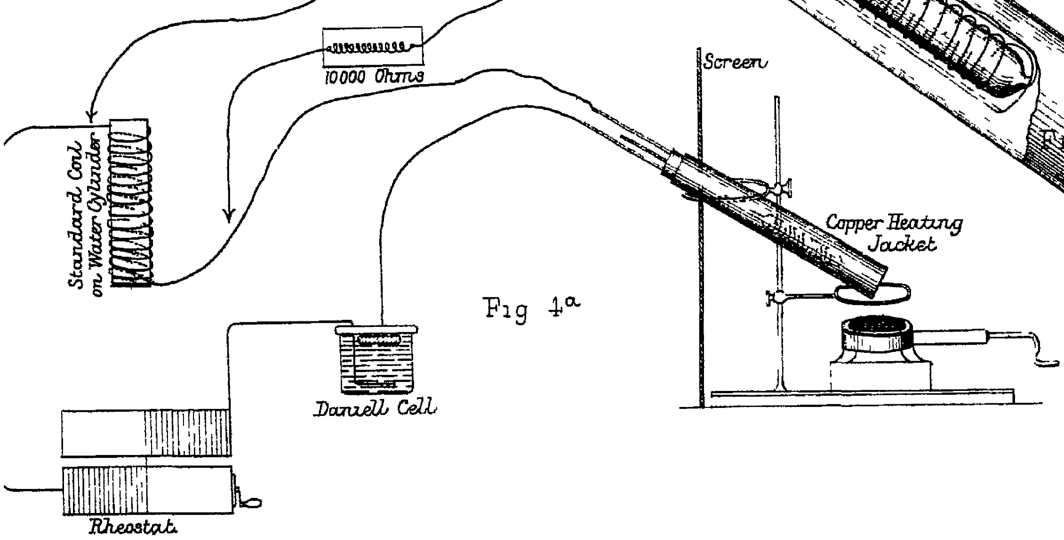


Fig 4^a

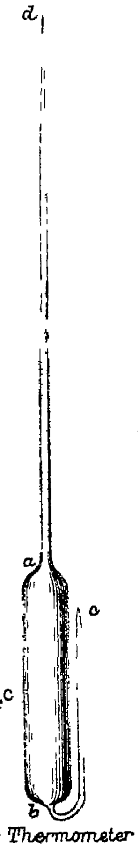


Fig 4^c

Air Thermometer

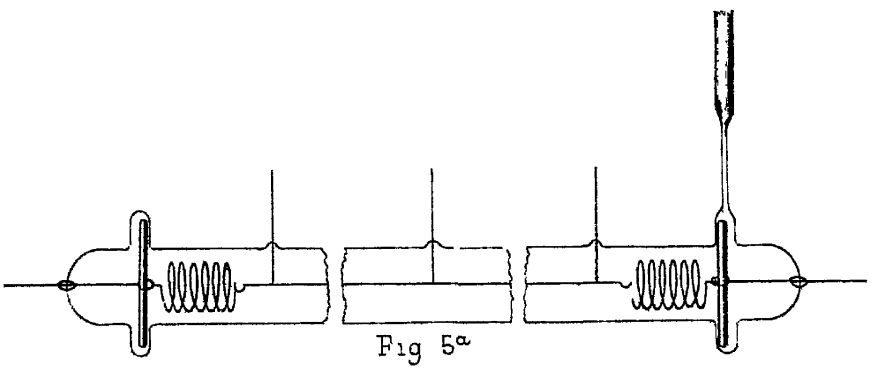


Fig 5^a

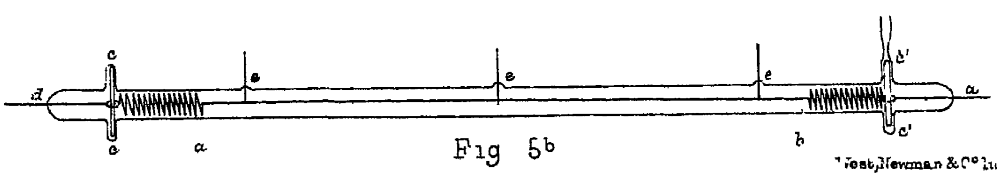
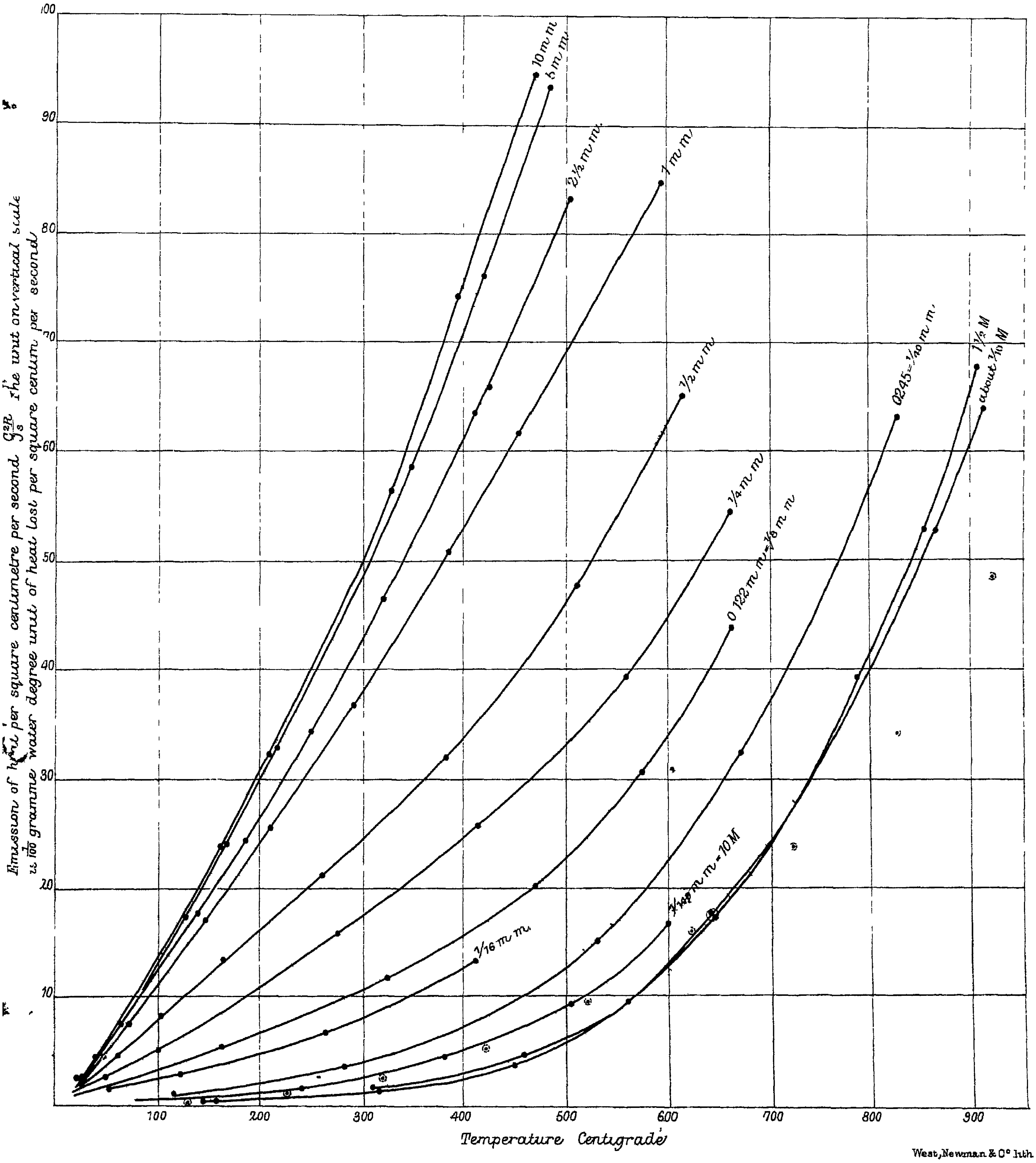


Fig 5^b

West, Newman & C° lith.

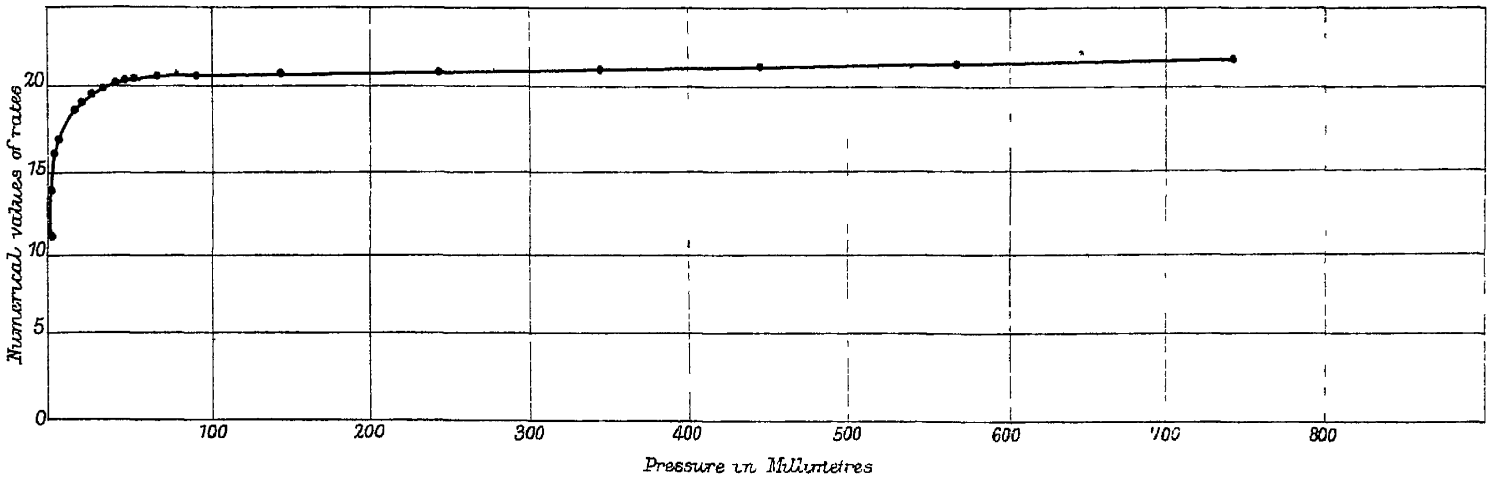
Diagram showing emission of heat at various pressures & temperatures. The Curves are drawn for the various pressures which are marked at their extremities.

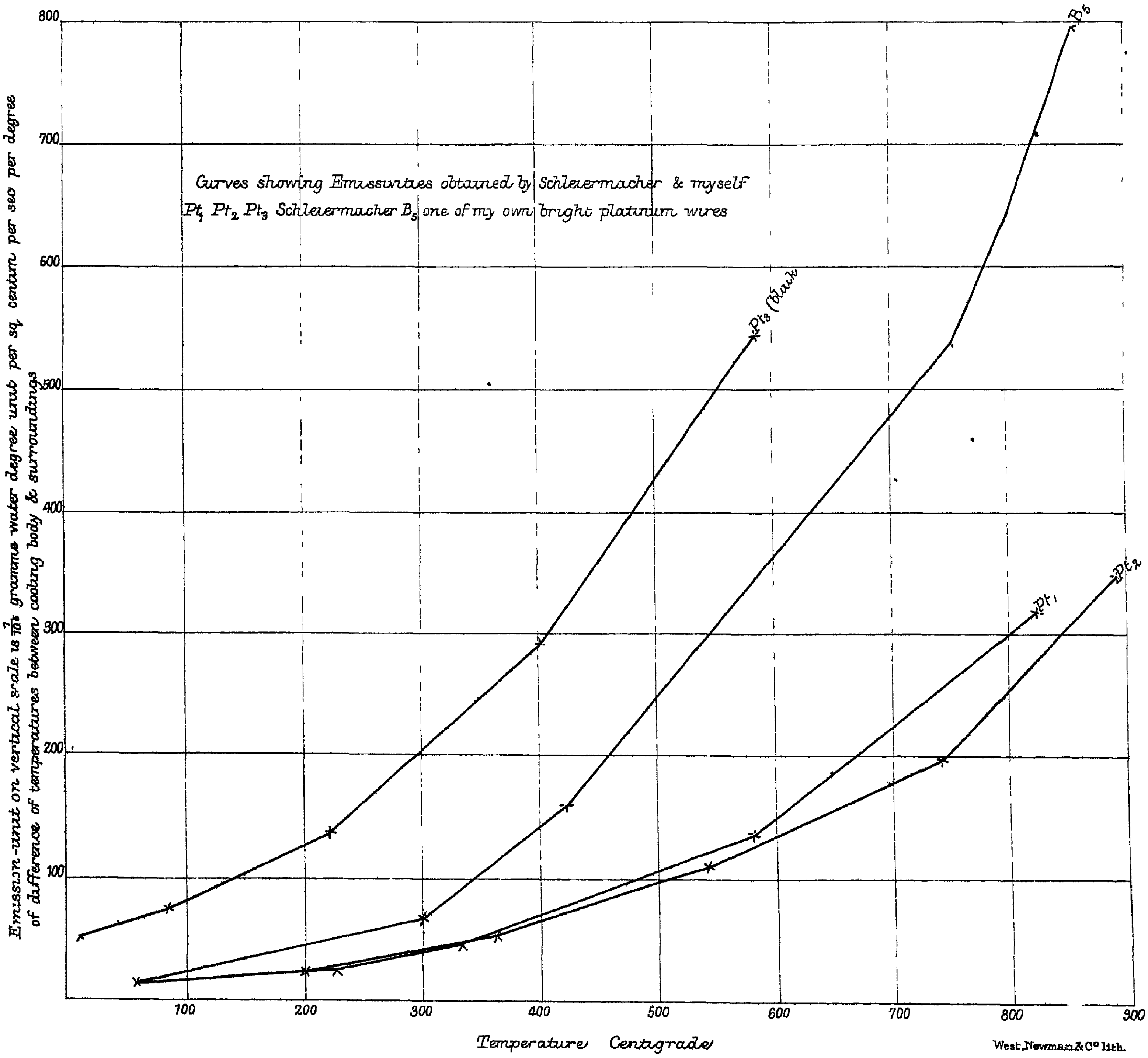
The points surrounded by dotted circles, are points on the Curve $y = a\,(T^4 - T_0^4)$ (Stefan's law), a being chosen to make $y = 0$ for $T = T' = 273 + 15$ & to make y correspond with a second well established experimental point on my emission curves at high vacuum.



West, Newman & Co lith.

Curve showing for Temperature 408°C the ratio between the rate of loss of heat with air present at various pressures & the rate of loss at the highest vacuum procurable with the Sprengel Pump the rate of loss at highest vacuum being taken as unity





West, Newman & Co. lith.

The following experiment, made upon a strip 45 millims. long and 5 millims broad, illustrates this point. Organ-current, + ·009 R.

Six Groves in Primary Circuit.

							Deflections (15″ readings).	
Secondary coil,	16	centims	(+)	$\dot{G}$	without	shunt	0	
,, ,,	15	,,	,,	,,	,,	,,	0	
,, ,,	14	,,	,,	,,	,,	,,	+ 300	15″ readings.
					,,	,,	+ 80	
					,,	,,	+ trace	
,, ,,	,,	,,	(−)		,,	,,	0	
,, ,,	12	,,	(+)		,,	,,	+ ∞	15″ readings
					,,	,,	+ 285	
					,,	,,	+ 160	
					,,	,,	+ 20	
,, ,,	,,	,,	(−)		,,	,,	+ 150	15″ readings.
					,,	,,	+ 10	
					,,	,,	0	

The effect is thus dependent upon the intensity of the current, which must vary with the dimensions of the preparation, so that an induction current which produces no effect upon a large thick block of tissue will produce an effect upon a long narrow strip cut from this block. Such a thin strip can be procured most readily from the organ of a small Torpedo, and it was with a preparation of this kind that the largest effects were observed. Thus a strip 15 millims. long, and only a little more than a millimetre wide, gave the following result:—

Six Groves in Primary Coil.

	Deflections.	
Secondary coil, 8 cm. (+) G. $\frac{1}{100}$	+ ∞	15″ readin
,,	+ 240	
,,	+ 100	
,,	+ 20	

The effect entirely disappears when the tissue is immersed for 15″ in hot water.

The effect being obviously an excitatory one, it is conceivable that it is nothing more than the response of the organ to the excitation of the nerve branches contained in the tissue. A strip of organ contains a very large number of nerve fibres, which may be traversed by the induction shock, and, if so, may be excited; on the other hand, the result may be due to excitation of the plates themselves. Any such confusion in the case of muscle and nerve is obviated by the use of curare, but this drug does not appear to be applicable to the Torpedo (40). In order, then, to determine whether or no the after-effect is a nerve-organ response, its character must be determined precisely as that of the true nerve-organ response has been determined —namely, by the rheotome method. The presence of the induction shock during the first three hundredths of a second is, however, a serious difficulty in the prosecution of such rheotome experiments. It is, however, quite easy to determine the relative values of the effect as disclosed by closures of from ·03″ to ·05″ and from ·05″ to ·07″; and it appears that the effect is always visible during these periods, but, though quite appreciable, it is not so strong as the nerve-organ response would be at the periods in question, especially when we take into consideration the magnitude of the total effect. Thus the largest nerve-organ response was imperceptible with the galvanometer ($\frac{1}{100}$ shunt) after an interval of one second. In the experiment to be next given the after-effect was so large that after an interval of one minute it was perceptible with the galvanometer $\frac{1}{100}$ shunt. There should, therefore, be a very large effect if the galvanometer circuit be closed for from ·03″–·05″ after the excitation. As a matter of fact, the effect is small under these conditions.

STRIP of Tissue from Small Torpedo. Six Groves in Circuit of Primary Coil.

Secondary coil, 9·5 cm (+) G. $\frac{1}{100}$	+ 420	15″ readings.
"	+ 190	
"	+ 70	
"	+ 30	

STRIP of Tissue from Large Torpedo. Five Groves in Circuit of Primary Coil.

Secondary coil, 0 cm. (+) G. $\frac{1}{10}$ . . .	. + 310	15″ readings
,, ...	+ 150	
,, .	+ 80	
,, .	+ 45	
,,	+ 25	
,, .	+ 8	

	Closure	
,, ,, ,, (+) G without shunt	·03″–·07″	+ 65
,, ,, ,, (+) ,, ,,	·03″–·05″	+ 30
,, ,, ,, (+) ,, ,,	·03″–·07″	+ 45

STRIP of Tissue from large vigorous Torpedo. Five Groves in Circuit of Primary Coil

	Closure	Alternate 15″ closure.
Secondary coil, 0 cm (+) G $\frac{1}{10}$	0·03″–15″	. . + ∞
,,	30″–45″	. . + 240
,, .	60″–75″	. + 112
,, .	90″–105″	. + 60
,, .	120″–135″	. . + 40

	Closure.	
,, ,, ,, (+) G. all in . .	·03″–·05″	+ 200
,, ,, ,, (+) ,, . .	·05″–·07″	+ 65
,, ,, ,, (+) ,, . .	·03″–·05″	+ 190
,, ,, ,, (+) ,, . .	·05″–·07″	+ 105

If we attempt to get still nearer in time to the excitation, and ascertain if a response is perceptible during such intervals as 0″ to ·01″ and ·01″ to ·02″ after the passage of the induction shock, this induction shock itself becomes perceptible. The only method of getting such information is thus to compare the relative deflections produced by passage through the experimental circuit of (+) and (—) induction shocks res[...] If the [...] hundredths of a se[...]

[...]-Rey[...]

present, and his results are thus explained (41). If he did so, then his results were not observed at Arcachon with winter Torpedoes, but perhaps a future investigation will disclose them

Thus a strip of Torpedo, 60 mm. long, gave the following results, when the rheotome was arranged so as to break the galvanometer circuit for good at ·05″ after the passage of the induction shock, the after-effect being thus excluded from observation:—

The two readings were always taken in the same direction, the galvanometer wires being reversed so as to admit this in the case of both (+) and (−) shocks.

FIVE GROVES in Primary Coil.

		Deflections
Secondary coil, 10 centims.	(+) G $\frac{1}{100}$	440
„ „	(−) „	435
„ 0 centim.	(+) G. $\frac{1}{1000}$	185
„ „	(−) „	185

In another preparation experiments were carried out upon the whole organ and a cut strip.

Two Groves in Primary Circuit.

		Deflections.
Secondary coil, 5 centims., G. $\frac{1}{100}$	(−)	420
„ „ „	(+)	410
„ „ „	(−)	408
Whole organ	(+)	415
„	(−)	412
„ 0 centim., G. $\frac{1}{100}$	(+)	622
„ „ „	(−)	626
Cut strip, 0 centim., 0055 R.	(+)	492
„ „ „	(−)	483

An important fact must be taken into consideration in this connection, since ...ot inconsistent with so-called "irreciprocal" conduction, namely that the ...ion current is more efficacious than the (−) one in producing the ... How far this...

into a state of prolonged excitation. Now it is only in cases of very vigorous response to nerve excitation that the peculiar prolongation of the response referred to in Part II. is seen. There is thus every reason to suppose that this prolonged change is the same, whether its precursor be an induction shock or the shock of the nerve-organ response itself. The fact that in the one case the tissue is quiescent, whilst in the other it is in a state of functional activity, does not really affect the matter. The experiments on summation to be now referred to show that the tissue is capable of responding to two stimuli occurring within ·01″ of each other, and that an after-effect may be procured even when the tissue is in a state of activity by leading an induction shock through the organ ·02″ after a response has been evoked by excitation of its nerve. To these experiments we now turn.

Summation of effects.—Before proceeding to experiments designed to show the summation of after-effects, it was first necessary to obtain satisfactory proof of the summation of two successive nerve-organ responses produced by two successive excitations of its nerve. It was found that the results were not satisfactory if the second stimulus followed the first at a very short interval, and was applied to the same portion of the nerve. It appeared that for ·03″ to ·04″ after the passage of the exciting induction current the excitability of the nerve at the exciting electrodes was lowered, and the second stimulus, if occurring in this interval only, produced a small inadequate response of the organ. This difficulty was got over by using two pairs of platinum exciting electrodes, and exciting different parts of the same nerve. The exciting electrodes may be indicated as X_1 and X_2. Of these, X_1 was 35 mms. from the organ, X_2 45 mms., the length of the whole nerve being 55 mms. The transmission time along the intervening 1 centim. of nerve is probably not more than ·002″. The spring myograph was used for the production of the induction shocks, K_1 being placed in the primary circuit of the induction apparatus connected with X_1, and K_2 in that connected with X_2. The traveller could be made to break K_1 or K_2 only, or K_1 and K_2 successively. Although the induction shocks were made equal as far as th galvanometric reading of the deflection caused by their passage through it al it will be noticed that excitation at the more centrally situated electrode pr

d nearer the middle of th

This shows that two nerve-organ responses are summed even when produced ·005″ after one another. It is, therefore, to be expected that the after-effects should also be summed, and experiments were now made as to summation of the after-effect of a nerve-organ response and that produced by the passage of an induction current. In order to obtain this result, it was necessary to lead an induction shock through the organ at a short interval of time after the organ response had been evoked by excitation of its nerve. For this purpose the rheotome was so arranged that K_1 should break the primary circuit of one induction apparatus and thus excite the nerve, whilst after an interval of ·02″ K_2 broke the primary circuit of an induction apparatus arranged so as to allow the break-shock to traverse the organ. Finally, K_3 was arranged so as to short-circuit the galvanometer up to ·05″. The deflections obtained thus showed either the after-effect of the nerve-organ response only, or that following the passage of the induction shock only, or the summation of the two after-effects.

A nerve-organ preparation was made by carefully dissecting out a long piece of the second nerve, and then cutting out a thin strip which should contain columns supplied by it. It was then necessary to ascertain that the preparation gave a large nerve-organ response followed by a prolonged after-effect. With K_1 at 0″, K_2 at ·02″, and K_3 at ·05″, the results were as follows :—

		After-effect.
Break of K_1 and K_3— After-effect following nerve-organ response	15″	+ 220
	30″	+ 100
	45″	+ 50
	60″	+ 35
	75″	+ 18
Break of K_2 and K_3— After-effect following passage of induction current only.	15″	+ 240
	30″	+ 110
	45″	+ 40
	60″	+ 25
	75″	+ 5
Break of K_1, K_2, and K_3— Summation of two effects	15″	+ 500
	30″	+ 250
	45″	+ 100
	60″	+ 50

show

Conclusion.

The work done at Arcachon has thus brought out the different excitatory changes which are produced by different methods of excitation in the organ.

The electrical organ of the Torpedo responds to a stimulus by a change in the electromotive character of the elements which make up its hexagonal columns. These elements are plates composed of nucleated protoplasmic masses and nerve-fibres. An extraordinary number of these are present in, and as it were bound together by, the former. In the active state of the organ the ventral surface of each plate with its contained nerves becomes negative to the dorsal surface; the effect in all the plates of a column when summed up is, therefore, such that the dorsal end of the column becomes positive to the ventral end. This effect may be produced in at least three different ways, and in each case it presents features which are those of the excitatory process.

i The obvious method of production is that of excitation of the trunk of the electrical nerve. The nerve-organ response is characterised by a short period of delay, an extremely rapid development, occupying less than $\frac{1}{100}''$, and a less rapid decline. If the response is very pronounced, the main effect lasts $\frac{6}{100}''$, but is followed by a prolonged after-effect in the same direction.

ii. This after-effect is itself an excitatory change, and may be produced by the passage of a sufficiently intense current of short duration through the organ. This excitatory change is probably developed more or less rapidly, but is especially characterised by its slow subsidence, as it does not entirely disappear until several minutes after its production.

iii. A more prolonged electromotive change of the same character is produced when by mechanical or thermal means a large number of electromotive elements are injured. The adjacent parts are then thrown into a state of prolonged excitation, the effect taking some hours to subside.

Whilst then response i. is an affair of fractions of seconds, ii. is an affair of fractions of minutes, and iii. of fractions of hours. All three effects are characterised [illegible] the column, and thus of the plates; and in all [illegible] slowly.

[illegible] the Torpedo org[illegible] [illegible] throw[illegible]

owing to the direction of the excitatory change. The demarcation-current in the Torpedo is a strong support for the view that in nerve and muscle the negativity of an injured as compared with an uninjured part is the electromotive expression of a prolonged local excitatory process occurring in the neighbourhood of the injury.

In future experiments I hope to follow up the lines of inquiry which have been indicated at various points in this research.

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XVIII On the Tubercular Swellings on the Roots of Vicia Faba.

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[PLATES 32, 33.]

IT has long been known that the roots of the Leguminosæ are commonly provided with peculiar tubercle-like swellings of various sizes, from that of a mustard-seed to that of a hazel-nut, and attention has been repeatedly directed to them of late years. They were observed by MALPIGHI, who seems to have looked on them as of the nature of Galls,* and TREVIRANUS regarded them as undeveloped buds, while A. P. DECANDOLLE considered them as diseased structures Since that time very various ideas have been published with respect to them, and as to their origin and relation to the roots which bear them. One of the most curious facts about them is that, although it is very difficult to find a specimen of our ordinary Leguminosæ (Clover, Lucerne, Beans, Peas, Vetches, &c) the roots of which are free from the swellings, no one has succeeded in showing that they do any injury to the plant: this has been repeatedly employed as an argument against their being due to the influence of any parasite The contrary opinion has gradually gained ground, however, and I am now in a position to prove conclusively that it is the correct one

The first close investigation of these root-tubercles (as they may be shortly termed) is due to WORONIN, who, in 1866,† examined in detail the structure and contents of the similar swellings which are to be found on nearly every Alder, as well as those on the roots of the Lupin. In the cells of the Alder tubercles WORONIN found a curious little Fungus, which was referred to the genus *Schinzia*, founded by NAEGELI on a form which he himself discovered in the roots of *Iris*‡ in 1842. In the cells of the tubercles of the Lupin, WORONIN found multitudes of minute corpuscles, which he took to be Bacteria, or Vibrios, or organisms of that kind.

Similar swellings have from time to time been observed on the roots of various

* 'Botan Zeitung,' 1874, p. 382, and SORAUER, 'Pflanzenkrankheiten,' vol. 1, p 744

† 'Mémoires de l'Académie des Sciences de St Pétersbourg,' vol 10 (May, 1866)

‡ 'Linnæa,' vol. 16, 1842 (Plate xi, figs 1–10).

other plants, those on *Cyperus flavescens* and *Juncus bufonius** being the best-known. Those of *Juncus* are especially interesting, because they have lately been shown to contain a parasitic Fungus belonging to the group Ustilagineæ † The swellings on the roots of *Orobanche* also contain a Fungus, referred to by SCHACHT‡: those of *Sonneratia* and *Taxodium* are not (so far as I can discover) so well known; although there is still so much that is doubtful about them, they probably differ in character from the others.

Mention may be made of the fungi long known in the roots of Orchids, but not causing tubercles of the kind referred to . some of them at least are now known to belong to the genus *Nectria* § These, and other root-fungi, need be no further regarded here, however , nor is it necessary to speak of other root-tubercles.‖

In 1878 WORONIN published his celebrated memoir on *Plasmodiophora Brassicæ*,¶ a Myxomycete which causes the long-known hypertrophies (Club-foot, Hanbury, Fingers and Toes) on the roots of various species of Cruciferæ. It is important to notice this paper, because it seemed to throw an entirely new light on the general question as to the causes of swellings on roots, and at any rate it was the forerunner of several suggestions.

Confining our attention to the swellings or tubercles on the roots of the Leguminosæ (and particularly the Papilionaceæ), as already stated, their more exact study was initiated by WORONIN, who examined those of the Lupin, and found the cells in the interior of the swellings filled with a slimy colourless matrix in which Bacteria-like granules were embedded · the "Bacteria" were found to increase, in proportion to the rest of the matrix, as the section receded from the growing apex of the swelling. In a paragraph at the end of his paper on *Plasmodiophora*,** in 1878, WORONIN practically retracted his previous opinion that the above granular bodies are really Bacteria or Vibrios, and regarded it as highly probable that a *Plasmodiophora*, or similar parasite, would be found to be the exciting agent.

Meanwhile, in 1874, ERIKSSON had published a masterly investigation of the occurrence, form, structure, and growth of the swellings on the roots of Leguminosæ.†† Their forms differ on the roots of different species, but are remarkably constant in the case of any one species. Perhaps the most striking fact about them is their ubiquity, confirmed subsequently by FRANK.‡‡ They are found on the roots of species of all the

* MAGNUS in 'Hedwigia,' 1878.

† C. WEBER, 'Botan Zeitung,' 1884 (No. 24).

‡ Reference in FRANK, 'Krankheiten der Pflanzen,' p 653.

§ WAHRLICH in 'Botan Zeitung,' 1886 (Nos. 28, 29).

‖ Such as those in which nematoid worms, &c., are found, for instance, and the gall-structures caused by various insects.

¶ 'PRINGSHEIM, Jahrb. f. Botan.,' vol. 11, 1878, p. 548.

** *Loc. cit.*, p. 571.

†† "Studier öfver Leguminosernas rotknölar," Lund, 1874. See 'Botan Zeitung,' 1874, p. 381.

‡‡ 'Botan. Zeitung,' 1879 (Nos. 24, 25)

genera, and in all parts of the world, as the examination of herbaria shows · on heights and in valleys, in poor soil and in good land, wet or dry, they may almost invariably be met with.

ERIKSSON found that in *Faba vulgaris* the young tubercle-like swellings arise in the cortical layer of the main root, or of a lateral rootlet, and without reference to the position or age of other parts. Although the development of the tubercle resembles that of a rootlet in many respects, its origin has no constant relation to the xylem and phloem strands—sometimes it begins to develop between the strands, or opposite one or other of them. Although the pericambium joins in the development subsequently, the origin of the swelling is outside the plerome cylinder, and is first evinced by active irregular divisions of the innermost cells of the periblem.

ERIKSSON observed that the inner cells of the swelling contained the vibrio-like bodies described by WORONIN. in the outer part of the swelling he occasionally noticed Fungus hyphæ proceeding radially inwards from cell to cell, and becoming branched and finer as they go inwards. ERIKSSON saw no hyphæ in the cells containing the vibrio-like bodies. he noticed also that the latter are not always simple rod-like bodies, but oftener forked, &c He left it open whether any connection exists between the hyphæ and the vibrio-like bodies.

In 1878 KNY asserted* that the tubercular swellings on the roots of Leguminosæ do not appear on plants cultivated in nutritive fluids, and expressed the opinion that this was because some parasite which causes the tubercles is absent from the fluids—in fact, is evidence in favour of the parasitic nature of the swellings. KNY also stated that he found a *plasmodium* in the dividing cells of the tubercles, especially in *Cicer*. he states that the parasitic plasmodium can be followed from cell to cell in the form of delicate, sparsely branched strands. Where they traverse the dividing cellulose-walls the plasmodial strands are slightly thickened (*loc. cit.*, p 55).

Then followed the Paper by FRANK,† in which he confirmed and extended ERIKSSON's statements as to the ubiquity of the swellings on the roots. FRANK states that the greatest care is necessary to prevent the development of the tubercles, that they occasionally occur on water-cultures as well as on roots in any soil. They did not appear on roots in soil which had been heated, though every plant in soil not so treated had them. FRANK concluded that the roots are infected from outside by a "germ" which must be as ubiquitous as the germs of putrefactive and fermentative organisms.

In the cells of the swellings there are always‡ two elements to be found, (1) fine hyphæ running across the lumina of the cells and through their walls, (2) minute cell-like corpuscles suspended in the protoplasm of the cell as in an emulsion.

That the hyphæ are the same as KNY's plasmodial strands is almost certain, and

* 'Sitzungsber. d Botan. Vereins d. Prov Brandenburg,' 1878 (April 26), p 55

† 'Botan. Zeitung,' 1879 (Nos 24 and 25)

‡ Except in those of the Lupin, p 396, *loc cit*

they are undoubtedly the same as those found by ERIKSSON. FRANK also found the hyphæ here and there in cells containing the minute corpuscles (these corpuscles are the "vibrios" or "bacteria" of WORONIN and ERIKSSON): these corpuscles are cells, and not mere granules; their form varies; some are rod-like or curved, or shaped like a finger-biscuit, others Y-shaped, and so on. They seem to increase by budding, but this is assumed only. They have no proper movement, as WORONIN supposed.

The hyphæ are thicker than the above corpuscles, and grow through from wall to wall across the lumina of the cells. They may end in the cavity, and in any case are rarely traced through more than two cells or so. FRANK and SCHENK both regarded certain short branches in the cells as haustoria.

FRANK then goes into the question as to the continuity of the hyphæ and the bacteria-like corpuscles. Are there two parasites present in the swellings? Without actually deciding the point, FRANK has little doubt that the corpuscles are budded off from branches of the hyphæ.

No results were obtained on trying to cultivate the corpuscles beyond the putting out of a fine hypha from either end in one or two unsatisfactory cases.

FRANK regarded the parasite as allied to that described by WORONIN in the Alder, and named it *Schinzia leguminosarum*.

KNY* replied to FRANK's Paper that he regarded the corpuscles as the spores of a plasmodium, the strand-like extensions of the latter show no membrane, and abut on the wall with curious funnel-like widenings—this is clearly the case in FRANK's figures. SCHWENDENER is also quoted as viewing the strands as membraneless plasmodial strands. KNY also states that DE VRIES and others have seen the tubercles on the roots of water-cultures

This appears to be a satisfactory account of the position of our knowledge of these curious structures up to date: it resolves itself into the following. While it is certain that some organism or other exists in the tubercles, no observer has cleared up the question as to whether there is one constant Fungus present, or whether it is a plasmodium which causes the hypertrophy, or whether two forms coexist in the cells: it is also uncertain how the parasite (if there is only one) enters the tissues—whether from the soil into the root, as seems so probable from ERIKSSON's and FRANK's researches, or whether it is already present in the seed. The latter is a more improbable suggestion, and we may dismiss the idea that no parasite exists at all, in spite of the doubts implied in SORAUER's statement on p. 744 of his book, and of the very general belief to the contrary which exists among agriculturists especially. In any case, however, no one had as yet succeeded in infecting the roots, and producing the tubercles artificially. I have done this, and may now proceed to describe my own observations and experiments.

I first commenced the microscopic examination of the tubercles in 1883, but, although the investigation was renewed each year since, the really important additional

* 'Sitzungsber. d. Botan. Vereins d. Prov Brandenburg,' 1879 (June 27), p. 115

results were only obtained recently. The chief investigation has been almost entirely confined to the tubercles on *Vicia Faba* (L.), the common Broad Bean, no particular regard being paid to the garden varieties employed

At the same time it should be stated that I have seen and to a certain extent examined them on the roots of several Clovers,* Peas, Vetches, and the Scarlet Runner in this country, on Broad Beans in Germany, and on the roots of several Papilionaceæ in Ceylon, and can to a certain extent confirm the much more extensive observations of FRANK as to their ubiquity and general resemblance. In Manchester it was by no means rare to find the tubercles on the roots of *Vicia Faba* as large as a small Hazel-nut, and I have had specimens even larger from the damp heavy soil of some districts. In the light sandy soil of this part of Surrey I have not seen them much larger than a fair-sized pea, though they are often very numerous and crowded on the roots.

I have examined many hundreds of roots of *Vicia Faba* during the last four years, and have only once or twice failed to find the tubercles on plants in fruit, and even in the case of younger plants the percentage of failures has been small

In 1884, at the Owens College, Manchester, several beans grown in pots of burnt soil, and watered with solutions of nutritive salts, developed no tubercles on the roots; and in the majority of cases, then and since, beans grown for laboratory purposes in carefully-prepared nutritive solutions have been devoid of the tubercles. Nevertheless, this has not always happened, and in some instances the water-cultures have developed excellent specimens of the tubercles Considering that these were growing in solutions of chemically pure salts in distilled water,† it is hardly to be wondered at if one sometimes doubted the existence of an external cause for the swellings, and felt tempted to believe that either the tubercles were really due to the roots themselves, or that if a parasite exists it is present in the seed from the first. The possible sources of infection in these experiments were (1) germs in the air, (2) germs attached to the testa of the Bean, (3) the medium (damp sand or sawdust) in which germination was commenced Although it was not demonstrated, I thought the second a more probable source of error than the others. I have this year found the tubercles on a bean which was germinated in "clean river-sand" not heated, and then grown as a water-culture—everything hereafter chemically pure, and in a new building. Of course it is not impossible that a germ might fall into the culture; but it seems much more likely that the infecting agent was attached to the seedling before its roots were placed in the solution of nutritive salts, and possibly came from the sand. In any case, whether the cause of infection is in the medium or on the Bean, it must be very minute and ubiquitous, and the results quite bear out FRANK's comparison with the minuteness and ubiquity of putrefactive and yeast germs. In

* Those of the Lupin may be distinct

† It should be stated that they were exposed to the air, however, and that no regard was then paid to the medium in which they were allowed to commence to germinate.

any case, again, it seems clear from further research that the infecting agent is present in the soil practically everywhere, and it is by no means difficult to suppose that it might attach itself to a seed harvested in the ordinary manner if the roots and soil are disturbed so that germs could be scattered.

The tubercles are formed without any order on the tap-root and lateral roots, the only apparent rule being that they are not developed until the young root-system is fairly advanced,* and they do not arise close to the apex of the root: they seem to especially affect the region where the root-hairs are in full vigour. There may be only a few, or very many, crowded in groups or scattered (fig. 1).

They are at first very small, and much like young rootlets in appearance; but they soon swell, often very irregularly, and may become lobed in various ways. Their colour and texture are quite like those of the rest of the root.

A longitudinal section (figs. 2, 3, 7) through the tubercle shows the structures described by ERIKSSON. The axial chief mass of the tubercle consists of rather large polyhedral parenchymatous cells, passing at the apex into smaller, closely packed, thin-walled cells which constitute a meristem, which would be homologous with the growing-point of a rootlet: several layers of more compressed parenchymatous cells envelope the above tissues, and may be looked upon as a periblem (fig. 7). There is no root-cap.

A short distance from the meristem, rows of cells at the boundary between this periblem and the axial (plerome) cylinder gradually pass over into vascular strands. Perhaps the outer layer of the periblem mantle may be regarded as the homologue of an epidermis and root-cap, but the resemblance is not very obvious. In the lobed or convoluted older tubercles (figs. 1, 2) all these tissues take part in forming the lobes.

It is in the large-celled axial tissue that the parasite is rampant (fig. 3), and the cells which contain the densely crowded corpuscles (figs. 4, 5, 12) are seen on the section as slightly pinkish or buff-coloured masses sending ramifications into the various lobes (fig. 2).

Any cell at the base of this mass may be seen to contain a densely granulated mass of substance (figs. 4, 5), which swells in water, and allows the corpuscles to escape passively, but with the well-known dancing Brownian movement. The separated corpuscles are very brilliant, and vary in size and shape; some are rod-shaped, others have the form of a Y or V, and others are still more branched, as described by FRANK. (Fig. 6.)

These corpuscles are deeply stained by hæmatoxylin, and become yellow-brown in iodine; in chlor-zinc iodine the mass of corpuscles turns bright golden-yellow, the cell-walls of the parenchyma containing them turn blue. All their reactions point to the accuracy of the previous ideas as to their nature; they are unquestionably organised bodies.

I have repeatedly examined them in sections of old dried tubercles gathered the year before. If kept dry during the winter, the tubercle shrinks considerably, and

* Usually when the seedling is about three to four weeks old, if growing luxuriantly.

lowly becomes very hard and wrinkled, turning dark-brown or almost black in the process, it is not easy to cut such tubercles Their consistency is that of stiff horn, nd the razor "drags" unpleasantly in the section. Thin sections placed in water well for several days, and the contents of the cells are, as before, densely crowded brilliant corpuscles in a matrix, which is bright yellow in chlor-zinc iodine. These corpuscles are particularly minute, and like mere points in the section.

In the cells containing these corpuscles the presence of hyphæ is to be observed figs. 12 and 15–18), even in the cells of tubercles which have been dried for a year, and are hard as horn, a few minutes' maceration in very dilute ammonia enables one to detect these hyphæ (fig. 18), which are obviously those described by FRANK. These hyphæ are very curious In the cells filled with corpuscles they are short, often much branched, extremely delicate, and apparently springing from the cell-walls, though really coming from hyphæ running in and through the substance of the cell-walls figs 12, 17).

Sections through young and actively growing tubercles show that the hyphæ branch and pass from cell to cell throughout the meristem of the interior (figs. 7–18). In very young tubercles the cells contain only these hyphæ, subsequently, when the tubercle reaches the dimensions of a mustard-seed, the tiny bacterium-like bodies begin to accumulate.

In sections through very young tubercles, made transversely to the long axis of the root, and passing axially through the tubercle, I have observed the following facts. A fairly strong hypha, several times thicker than the cell-walls in many cases, can be traced through from the epidermis to the origin of the young tubercle (figs 7, 8, 9), *the tip of the tubercle is always directed so as to meet this hypha.*

I had frequently satisfied myself of this fact, before more fortunate preparations showed the facts explained by figs. 9 and 13. Here, as is seen, the hypha referred to passes down the cavity of a root-hair, and thence traverses the cortex of the root, cell by cell, beginning to branch when it enters the mass of tissue which constitutes the young tubercle. It is more difficult to see the branches in the meristem of the tubercle, for two chief reasons. the cells are smaller and more numerous, and their walls are very thin. Moreover, their protoplasm and nuclei are very bright. Nevertheless the difficulty is only relative, and, as already stated, FRANK had already seen the hyphæ passing from cell to cell inside the tubercle, though he did not trace them far.

The isolated thicker hypha in the epidermis and cortex (figs. 8, 9) offers more distinct characters than the finer ramifications which it makes further inwards. The hypha is without septa, so far as can be made out by reagents of all kinds. It has a very delicate membrane, which is quite distinct in specimens treated with osmic acid or with chloral hydrate, or hardened in alcohol, &c. It passes straight across the lumina of the cells, through wall after wall, on its way towards the centre of the root; but a curious and very characteristic feature is the trumpet-like enlargement of the hyphæ at the spots where they pierce the walls (figs. 9–14). This has been

noticed before and is figured by FRANK, and is also a character of the mycelium found in *Juncus bufonius*, and named by WEBER* *Entorrhiza*. In suitable preparations the hypha may be seen to swell up inside the substance of the cell-wall, and it looks as if the widening was due to the cellulose wall itself (fig 14). Two possibilities suggest themselves · the swelling might be due to increased nutrition—a less probable view, or, as I think more probably, the cellulose wall extends by the growth of the cell after the hypha has pierced it, carrying the insertion of the hypha with it as its area increases. As the tubercle becomes older the hyphæ in the cortex of the root turn yellowish and gradually decompose, so that no trace of them can be detected in the larger tubercles which have broken through the cortex of the root.

In the cells of the very young tubercle the finer branches of the above hypha can be seen behaving similarly as regards their passage through the walls and across the lumina; and, although they become too fine to enable the observer to decide as to the presence or absence of a cell-wall and septa, it may no doubt be assumed that the characters are essentially the same. It is often possible to see the hyphæ running in the substance of the cell-wall (figs. 9 and 9A). In one respect these more ultimate hyphæ differ, however; they send out branches which end blindly in the cavities of the cells (figs. 15–18). These branches may be simple, or they may have several rounded or tufted bodies projecting from them, and looking like haustoria. The surface of these haustorium-like projections is often found presenting the appearance shown in figs. 16 and 17. Numerous very minute protuberances stand off from the rest of the mass

In very thin fresh sections of tubercles, about the size of a mustard-seed or smaller, and which are only just beginning to project markedly from the root, the projections just referred to are very numerous, and every cell of the inner meristematic mass of the tubercle seems to be provided with the branches bearing them. A change is also noted in the cell-contents in these cases. In place of the normal-looking protoplasm of the cells in the earlier stages, the protoplasm now becomes extremely vacuolated and frothy, and the tiny bacterium-like corpuscles referred to are found to be gradually increasing in number in the cells (fig. 12). At a slightly later stage these bacterium-like corpuscles have become distributed in dense crowds throughout the frothing mass of protoplasm, and they become so numerous that they obscure the hyphæ, and the appearance is that of a plasmodium, gradually becoming more and more densely filled with granules. At length, the enlarged cell is seen to contain a dense mass of the granules arranged around a large central vacuole (figs. 12 and 4). *The nucleus remains in the protoplasm.* So dense is the mass of granules in the protoplasm at last, and so sharply defined the vacuole, that extremely thin preparations hardened in osmic acid or picric acid, and then in absolute alcohol, can be washed and stained with hæmatoxylin and mounted in Canada balsam in the usual way, without destroying this arrangement (fig. 4).

* 'Botan. Zeitung,' 1884 (No. 24).

The next question is, what are these bacterium-like corpuscles, or granules, and how do they arise and thus crowd the protoplasm? They seem to be budded off from the projections of the hypha in the cells although I have failed to see an undoubted case of the actual formation and separation of one of these minute corpuscles, still there is not much room for any other view of their origin in the face of preparations such as yielded figs. 12, 15, 16, and 17 These extremely minute bodies are, according to my view, budded off in large quantities from the hypha.* They then multiply further *by budding in the cells.* This explains their shapes, which are Y and V-shaped, or branched still more, in the younger states, but more uniformly ellipsoidal when older, and also accounts for their rapid increase in numbers.

The bacterium-like corpuscles are thus to be looked upon as extremely minute, elongated, yeast-like cells or gemmules, and it is not difficult to suppose that their sudden and rapid increase in the juices of the cell would give rise to the vacuolation of the protoplasm. It seems impossible to doubt that the so-called plasmodium seen by various observers in these tubercles is really the protoplasm of the cells themselves, *stimulated into increased activity by these parasitic gemmules.*† In any case this seems a fair explanation of the fact that the protoplasm of the cell becomes exceptionally frothy and vacuolated, and the cell itself enlarges quickly, coincidently with the appearance and rapid increase of the tiny gemmules At this time, also, vigorous specimens have these cells abundantly supplied with starch at the period when the above increase in size and coincident multiplication of gemmules occur (fig. 12) The only explanation seems to be that the parasitic gemmules and hyphæ are stimulating the protoplasm to greater activity, and thus making the cell act as a centre of attraction for the plentiful supplies of carbo-hydrates sent down from the leaves, and other substances coming up from the root-hairs.

Tracing the hyphæ from the cortex into the young tubercle, then, and the changes which go on in the cells of the latter, the following facts are to be noted The single stout hypha (fig. 9A) first breaks up into numerous branches (figs. 8–12) which distribute themselves in all the cells of the dividing mass of tissue—the incipient tubercle. While the younger branches continue this process the older ones form the short branches, and begin to bud off numerous tiny gemmules into the cell-contents (figs. 12–17): the struggle between the protoplasm and these gemmules—which themselves go on budding—is evinced by the frothing and activity of the protoplasm, which thus comes to resemble a plasmodium, and by the accumulation of starch, and perhaps other substances also. This causes the cells to grow (hypertrophy), and by the time the hypertrophy has ceased the gemmules in that particular cell have ceased

* FRANK was also inclined to the same view ('Botan. Zeitung,' 1879, p. 385), but he has since retracted his opinions on the whole subject in a most unaccountable manner See 'Berichte d Deutschen Botan. Gesellschaft,' 1887, pp. 56 and 57.

† In order to avoid the inference that I suppose these cells to be true Saccharomycetes, they may be conveniently termed *gemmules.*

to multiply. The ferment is now over. The gemmules come to rest in the matrix, which they have first stimulated and then exhausted, and a large vacuole may be formed (figs. 4 and 12), in which lie the remains of the nucleus. It should be noted that at this stage the nucleus is often very bright and fatty-looking, and stains black in osmic acid. By this time the filaments which budded off the tiny yeast-like gemmules are so obscured by their progeny that they are almost certain to be overlooked, and special methods are necessary to detect them. They are there, however—at least, recognisable remains of them are (fig. 18).

The tubercle, when all its cells have undergone the above changes, now passes into a state of rest it is a mass of cells full of yeast-cells—gemmules, germs—so tiny* that they might well be, as they were, mistaken for Bacteria The rotting of the root and tubercle liberates these into the soil, and an extended acquaintance with these "germs" and their numbers leads one to feel no surprise if they turn out to be the ubiquitous germs which it has been suggested must exist to account for the universality of the root tubercles.

Speculation apart, however, I have the following facts to offer. I made several attempts last year to infect the roots of the Bean by laying pieces of the tubers on the young rootlets of water-cultures. The success was only partial and doubtful, and the results did not seem sufficiently satisfactory to be worth recording. At the end of the summer I collected a number of the Bean-roots which had tubercles on them, and dried them; these I kept through the autumn and winter, and made further experiments in January, February, and March of this spring (1887). Several of the infected plants had already developed several typical tubercles in March and April, and indeed the preparations from which figs. 7–9 and 11–13 were made have been obtained from these artificially infected tubercles. I have since repeated these experiments with marked success.

The most curious feature about the matter is, perhaps, the long "period of incubation" (as the doctors would term it). I infected the roots by placing very thin sections of the dried tubercles on the young tap-root; but no signs of the tubercles were noticed before the root was five weeks or so old, and had developed an abundant outgrowth of lateral rootlets.† It was from a section of one of the younger of these tubercles that the preparation of the root-hair in fig. 13 came. It will be noticed that the hypha running down the root-hair starts from a minute bright dot; unless this dot is one of the above-named "germs," I do not know what it can be. I have now seen *the root-hair thus containing a hypha starting from a mere bright point many times*, and have several permanent preparations of such infected hairs with the hypha passing down the cavity and across into the root. It very commonly happens that the

* FRANK gives their measurement as about 0·001 mm. thick. WORONIN gives length ·0016 to ·0028 mm. Many Bacteria are larger.

† They were, probably, at least a week or ten days old, however, and I now know that this specimen (growing slowly in January and February) furnished late examples.

root-hair thus infected is curiously coiled and twisted at the spot whence the hypha takes its origin—*i e*, at the point of infection—looking as if the growth of the cell-wall had continued after the attack, but was profoundly affected at the point of entry (fig 13A). It is sufficient to germinate a Bean in ordinary garden soil for a week or ten days, and then place the seedling as a water-culture, to obtain such preparations in nearly every case, provided the *very young* tubercles are examined.

It is now necessary to say a few words as to the systematic position of the above Fungus. The behaviour of the mycelium is so like that of some Ustilagineæ (especially the *Entorrhiza* found in *Juncus bufonius*) that an alliance to that group might readily be claimed, the difficulty of determining the membrane of the hypha, the glistening character of the protoplasm, the course straight through the cell-walls and across the lumina of the cells, are all points which remind one of the Ustilagineæ. The dilated abutments on the cell-walls I have explained as due to the extension of the cell-walls by growth after the perforation.

On the other hand, there are no septa discoverable in the hyphæ; nor can I detect anything of the nature of a true spore like the resting-spores typical for the ordinary Ustilagineæ, unless the haustorium-like branches are to be regarded as aborted spores. The "yeast-like" minute corpuscles (gemmules) which are sprouted off from the short branches in the cells, and which then go on budding until they fill up the space, appear to be the only reproductive organs which exist

But, in view of BREFELD's recent discoveries as to the extensive "yeast" formation which goes on in the Ustilagineæ,* it seems not at all improbable that this character may be in favour of, rather than against, the alliance of this Fungus to the Ustilagineæ

In fact, I regard the Fungus as one of the Ustilagineæ, which has become so closely adapted to its life as a parasite in the roots of the Leguminosæ that it has come to stimulate and tax its host in an exquisitely well-balanced manner, and has lost its needless true resting-spores because the more numerous and tiny sprouting yeast-cells (gemmules) are kept in the cells of the tubercle through the dry summer and autumn, and freed during the rotting in the soil in the winter and spring. Their very minuteness and numbers enable these "germs" to become as ubiquitous as "Bacteria" or ordinary "Yeast" forms, thus explaining the ubiquity of the tubercles.

BREFELD has shown that the ordinary resting-spores of the Ustilagineæ, which usually take a long time to germinate in pure water, will germinate much more rapidly in nutritive solutions, and that several species can be made to develope their promycelia in nutritive solutions which had hitherto been cultivated in vain. The curious fact comes out, however, that when thus germinating in nutritive solutions the promycelia go on budding off yeast-like cells, which multiply still further by budding, and thus extend the parasite outside the plant enormously. According to BREFELD's discoveries,

* BREFELD, 'Botanische Untersuchungen,' Heft 5, 1883.

whereas some forms, such as *Ustilago destruens*,* first develope a definite mycelium from the resting-spore, and the yeast-like cells are budded off from this into the air, in other cases the budding commences at once on the germination of the spore. Thus, in *Ustilago antherarum*,† the budding begins direct from the spore, or immediately the promycelium is protruded. Similarly with *Ustilago intermedia*,‡ *U. Maydis*, and others. Now, if we supposed the spore to germinate in this manner while still attached to the mycelium inside the plant, we should have something very like what happens in the Fungus I have described. In fact, I regard the swollen parts of the mycelium which give rise to the gemmules as the homologues of the spores proper (*i.e.*, resting-spores) in the ordinary Ustilagineæ. Anyone who compares the development of the spores in those Ustilagineæ which have been sufficiently studied§ cannot fail to notice the remarkable similarity to the development of the branches which bud off the gemmules in this Fungus. If, for any reason, the future resting-spore of an *Ustilago* began to put forth buds (gemmules) before it developed its thick coats, we should have the very process I have been describing. In the case of the Fungus causing the tubercles on the roots of Leguminosæ it is not difficult to suggest that it might obviously be a disadvantage to the parasite to develope resting-spores, of (comparatively) large size, which should be set free on the rotting of the root, since they would not easily be disseminated in the soil; on the other hand, it would seem to be a great step gained if the habit of forming yeast-like gemmules no larger than Bacteria was thrown back, as it were, earlier into the life-history. There is no lack of equally curious adaptations in the life-history of Fungi. In fact, steps towards such retrogression are already found in the asexual development of the resting-spores of the Ustilagineæ if we accept DE BARY's views as to their homologies; and it does not seem possible to refute them.

These extremely minute gemmules will be disseminated between the interstices of the soil as easily as the well-known Schizomycetes found everywhere; every stream, every puff of wind, and many slighter movements will carry them from place to place, and hence their ubiquity.

I have been much exercised with the question as to whether these gemmules bud and multiply in the soil, *i.e.*, in artificial nutritive solutions, and have repeatedly made efforts to cultivate them in hanging drops of pure water, PASTEUR's solution, and other nutritive media. The general result has been disappointment. In some cases I have thought that certain dense flocculent groups which develope in PASTEUR's solution were clouds of the buds, but they never form rapidly, and in the interval (four to eight days, or longer) Bacteria have always made their appearance. The suspicion that these flocculent clouds might be colonies of budding gemmules is not an absurd one: very similar, I believe identical, clouds form on the root-hairs of my water-

* *Loc. cit.*, Plate VII., fig. 24.

† *Loc. cit.*, Plate I., especially figs. 14–17.

‡ *Loc. cit.*, Plate VI. and Plate IV.

§ *E.g.*, DE BARY, 'Morphol. und Biol.,' fig. 82, p 189 (Engl. Edition, p. 175).

cultures, and even close to where a hypha enters the root-hair. However, the whole of this matter is as yet too uncertain for me to say more than that it seems a point well worth investigation to determine whether the gemmules will go on budding in the soil or on root-hairs. It may, perhaps, be permitted to make another suggestion here. Even if, as is usually accepted, the protoplasm of the root-hairs of plants cultivated in nutritive solutions—and, if so, presumably of plants in soil—is confined to the interior of the cell-walls, these cell-walls are very thin and delicate. There are facts which seem to support the view that the germinal hypha starts its course at the root-hairs for this reason, and it is not impossible that the gemmules will only germinate on the root-hairs of the host-plants. At any rate, I never find them elsewhere.

It is a well-known and very popular view that the Leguminosæ enrich the soil in nitrogenous substances. There is no doubt that the plants of this natural order contain abundance of nitrogenous substances, and it is a fact that the roots of our ordinary field and garden Papilionaceæ go deep, and thus bring up nitrogenous compounds from below to enrich the surface soil when the plants are turned into it by the spade or plough. It is also unquestionable that the root-tubercles which have been described above are extraordinarily rich in nitrogenous substances. Apart from the evidence already given, analyses of the tubercles of the Lupin show an enormous excess of proteids as compared with the other parts of the roots,* and, since these tubercles are apt† to be on the upper portions of the root, it is not improbable that these circumstances have contributed to the views so favourable among agriculturists. Of course there may be more in this matter than meets the eye, if future investigations demonstrate that the gemmules can live as saprophytes or ferment-organisms in the soil outside the plant.

The above results were already on paper, and had been shown to friends, when the 'Berichte der Deutschen Botanischen Gesellschaft,' Heft 2, 1887, came to hand, containing a paper by TSCHIRCH on the root-tubercles of the Leguminosæ ‡ The Paper is somewhat long, and not always clear, but, since it contains important and sweeping statements as to the biological significance of these root-tubercles, it will be necessary to devote some time to its examination. TSCHIRCH has investigated chiefly the anatomy of the tubercles on the roots of the Lupine and *Robinia*, and expressly states that the research was undertaken at an unfavourable season (October to December), a statement which makes it the more surprising that his generalisations should be so sweeping in character. He does not seem to have examined *Vicia Faba* particularly, and the following criticisms must be regarded as bearing only on the points common to *Vicia Faba* and other Leguminosæ; these are no doubt numerous, but it

* TROSCHKE, cited by SORAUER, 'Pflanzenkrankheiten,' vol 1, p 746

† Not necessarily so, as my cultures prove

‡ 'Beitrage zur Kenntniss der Wurzel-Knöllchen der Leguminosen, (1),' pp 58–98 The same also contains a paper by FRANK, advocating similar views to those of BRUNCHORST and TSCHIRCH.

is as well to bear in mind that until other forms have been investigated on the basis of the knowledge now to hand, there is always a certain amount of assumption in supposing that the tubercles of the roots of Leguminosæ are all due to the Fungus I have described. TSCHIRCH, like others, has by no means hesitated at assuming a common cause for these structures, however, and I am strongly of opinion that they are so far right, but that the causal agent is in all cases a Fungus.

TSCHIRCH regards the tubercles as normal structures, differing in form and anatomy according to two types. In *Robinia* and the majority of the Leguminosæ the anatomy, &c., is so like what occurs in *Vicia Faba* that we need not dwell on the details. He finds the development of the tubercles on the roots begins with that of the first leaves above ground, and traces coincidence of developments of tubercles and leaves onwards. at the close of the period of vegetation the tubercles become emptied, in whole or in part, as the seeds ripen.

This emptying is considered to be due to the dissolution (and absorption into other parts of the plant) of the inner core of tissue containing the gemmules. Here comes in an important difference between our descriptions. TSCHIRCH accepts a view, already propounded by BRUNCHORST,* that the gemmules belong to the root, and are formed by the protoplasm of the cells, as will be shown shortly, the two writers differ somewhat in their interpretations of these bodies, but both employ the same name for them—*Bacteroids.*

In *Robinia* TSCHIRCH finds that the "bacteroid tissue" is partially or nearly wholly emptied in autumn, and states that this is the rule The phenomenon thus assumed to be a resorption of the bacteroids I take to be the escape of the gemmules from the tubercles; they are not absorbed, they are set free.

TSCHIRCH finds much starch in the young bacteroid tissue, but less as the cells become older, this is in accordance with my observations, but is subsequently explained differently.

In annuals the tubers are found to attain a maximum of development, and then become emptied as the seeds ripen.

TSCHIRCH agrees with BRUNCHORST that the gemmules or "bacteroids" are not organisms, but bodies which arise by differentiation from the protoplasm of the cell. This view he bases on their "development" (though no account of their development appears), their fate (the assumed resorption into the plant), and the normal occurrence of the tubers in all genera of the strictly defined group Leguminosæ. TSCHIRCH also adds that every attempt to cultivate the "bacteroids" has failed, and that they could not enter the tubercles, because the latter are covered by cork.

Reference is made in a footnote (p. 67) to the observations of WORONIN and FRANK as to doubtful development of the gemmules (bacteroids) in artificial cultures. Why it should be assumed (p. 67) that "im vorliegenden Falle hat man es, wenn

* 'Berichte der Deutschen Botanischen Gesellschaft, 1885,' p 241; and 'Untersuchungen aus dem Botanischen Institut zu Tübingen,' vol. 2, Heft 1, 1886, p. 151.

uberhaupt mit einem Pilze, mit einem Schizomyceten zu thun" is not clear, but the author admits no other alternative.

BRUNCHORST supposes the "bacteroids" to arise by differentiation from the protoplasm of the cell, and TSCHIRCH accepts this view, the "bacteroids" (*i.e.*, gemmules) are supposed to be remnants of a protoplasmic network (p. 68) Insuperable difficulties against accepting the view that the tubercles arise by infection from without are found in that it involves the assumption that the germs must be in all soils and in water, and TSCHIRCH states (p 69), "Thatsachlich sind derartige geformte Pilze aber gar nicht im Boden vorhanden." The "bacteroids" are thus denied autonomy as parasitic bodies, and are relegated by TSCHIRCH and BRUNCHORST to the protoplasm of the root as "geformte Eiweisskorper."

TSCHIRCH then goes further, and classifies these bodies with the vegetable caseins related to Legumin.

The filaments (hyphæ) observed by ERIKSSON and FRANK, and which I have traced from the exterior, through the root-hairs and into the tubercle in *Vicia Faba*, are said to be absent from the cortex and outer tissues "Ich habe in der ausseren Rinde niemals Faden gefunden" (p. 72) They were observed in abundance, however, in the outer limits of younger "bacteroid tissue."

TSCHIRCH denies their fungoid nature entirely, and refuses to see in them either hyphæ or plasmodial strands His account of their "development" (pp. 73, 74) is quite unintelligible: among other things he regards the cell-walls of the tubercle as cutting these filaments (hyphæ) in two. He thinks that no connection exists between the "bacteroids" and the filaments, but this is admitted to be doubtful. The view accepted—so far as any definite idea about the hyphæ is accepted—is that the filaments dissolve, and the protoplasm of the cell then gives rise to the "bacteroids" by differentiation.

Looking at TSCHIRCH's statements so far, it seems clear that, apart from the small amount of direct observation which has been brought to bear on these bodies, two chief points come in to explain the mistakes. In the first place, he has not examined tubercles sufficiently young to determine the course of the hyphæ from without inwards; and, secondly, he has failed to observe the relations of the hyphæ to the cells and protoplasm of the interior of the tubercle.

Apart from the dying-off of the older hypha which enters the young root-tubercle—often, at a time when root-hairs are still present—it is, of course, obvious that many sections will not take in the point of entrance and the course of the hypha shown in my figures 8 and 9. TSCHIRCH's figures 1, 2, 4, and 44 show that the sections were made in the wrong direction to give the required information, and, although the tubercles examined are described as "young," they are far too old for the purpose referred to. Again, in the absence of some intelligible suggestion as to what the filaments are, if not hyphæ, there is nothing gained by reiterating that "mit einer Pilzhyphe haben sie also nichts gemein." The assumption that they are cut in two by

the cell-walls required some evidence. Anyone who has worked with such mycelia as those of the Ustilagineæ must have seen hyphæ as bright and delicate as these.

The point as to the budding of the gemmules from the hyphæ is a more difficult one, because it seems impossible to witness the process of budding off itself. The view proposed by BRUNCHORST, and accepted by TSCHIRCH, that the gemmules ("bacteroids") are remnants of a broken-up protoplasmic network, is quite untenable in the cases examined, and TSCHIRCH gives no figures illustrating such a process. I have shown that the gemmules arise in connection with the hyphæ in the cells, and not as points scattered in a matrix. BRUNCHORST himself seems to have held at one time the idea that these "bacteroids" escape from the interior of the hyphæ like minute sporules, but the account is by no means clear. It seems impossible to explain the frequent cases shown in my figures 15, 16, and 17 on any other view than that they are the gemmules budding off from the hyphæ, and it ought to be noticed that others have seen similar cases, but without properly explaining them. For instance, TSCHIRCH himself says (p. 76), "In einem Falle nur schien der Kopf eines Fadens mit unzähligen behauften rundlichen Ausstulpungen besetzt," and in a foot-note he remarks, "Diese Bildung ist vielleicht mit der von PRILLIEUX beschriebenen Auflosung der Faden der warzigen Massen identisch" (p. 76).* Again, it is difficult to attach any other meaning to FRANK's figures ('Botan. Zeitung,' 1879, Plate V., esp. fig. 11) than the one he then ascribed to them, though he has since retracted his explanation in a most unaccountable way—erroneously, I am convinced.

It is still more difficult to accept TSCHIRCH's assumptions as to the "bacteroids" (gemmules) themselves It may be true that their variable shapes militate against their being *bacteria*, but bacteria are not the only alternative, as he assumes, and these very shapes are fully in accordance with their being gemmules—tiny, yeast-like, budding organisms—as a comparison of FRANK's figures and my own with those of TSCHIRCH's amply testify. TSCHIRCH's assumption that the filaments (hyphæ) dissolve, and the protoplasm then breaks up into "bacteroids," is certainly not true for the tubercles of *Vicia Faba*. In fact, the whole of this purely hypothetical explanation of the nature of these bodies is contradicted by the facts observed in *Vicia Faba;* whereas the phenomena are simply and naturally explained when we recognise that the hyphæ bud off the gemmules, which then multiply further by budding like tiny yeast-cells. These then pass into a dormant state in the matrix, and escape into the surrounding soil when the tubercles decompose at the end of the season of growth, and are there distributed in readiness for contact with root-hairs of other seedlings in the following season.

It now remains to examine the rest of TSCHIRCH's paper. He expressly states, and repeats, that water-cultures are useless in the investigation, though no satisfactory reasons whatever are given in support of this view He admits that the

* PRILLIEUX ('Bull. Soc. Botan. de France,' 1879, p. 98) regards the cause of the tubercles as a plasmodium.

tubercles occur on water-cultures, and I have now shown that most satisfactory results can be obtained by means of water-cultures. TSCHIRCH also states that they are commoner on the roots of plants growing in poor soil (as regards nitrogen) than on those in rich humus-soils, a statement I fail to confirm Darkness, drought, and other disturbing influences of the kind hinder the proper development of the tubercles ; and the rule is that the tubercles flourish in proportion to the thriving of the plant

It is obvious, on a little reflection, that there would be nothing in these facts which contradicts the explanation I have given. Since the gemmules stimulate the protoplasm of the cells to greater activity—as indicated by the increased supplies of starch and the hypertrophied growth—it is clear that the needs for the increased activity will be satisfied only in proportion to the ability of the host-plant to furnish supplies darkness would cut off the supplies of carbo-hydrates, and drought those of water and minerals, and thence the parasite and protoplasm in the tubercles suffer. This can be directly shown in water-cultures, as well as in the case of plants growing in the open. As regards the generalisation that the tubercles are commoner in soils which are poor in nitrogenous substances, this might be because there is more oxygen supplied to the very large active root-systems in such soils ; a view, moreover, which is in accordance with the well-known fact that the tubercles are apt to be more abundant and larger on those parts of the roots which are nearer the surface of the soil, though this is not necessarily the case. There is another point to note here. TSCHIRCH himself observes (p. 90) that the Leguminosæ are deep-rooted plants which do not thrive well in rich humus-soils, abounding in nitrogenous materials. Now, without agreeing with him to the extent of his generalisations (for it is by no means established that the Leguminosæ as a class are dependent on one kind of soil), it is obvious that nothing in this contradicts the explanation of the tubercles as hypertrophies due to the stimulating action of a symbiotic Fungus. TSCHIRCH lays great stress on the facts that the analysis of the tubercles shows them to be relatively very rich in potassium, phosphorus, and nitrogen, and claims that the abundance of nitrogen is due to the "bacteroids". no doubt this is the case, since the "bacteroids" are of the nature of Fungi—yeast-cells—structures known to be particularly rich in potassium and phosphorus as well as nitrogen.* TSCHIRCH employs the evidence of the analysis in support of the hypothesis that the tubercles are normal organs used as storehouses for reserve materials · the evidence, however, distinctly supports the view that the protoplasm becomes surcharged with materials employed in the hypertrophy and for the good of the parasite. Other facts, quoted by TSCHIRCH in a different sense, are equally in favour of this · for instance, the absence of sugar and the presence of fatty substances in the winter.

At the end of his memoir TSCHIRCH gives a summary substantially as follows :—

The Leguminoseæ are plants which require much nitrogen, but only towards the

* *Cf.* HUSEMANN, 'Die Pflanzenstoffe,' 1882, vol. 1, p. 279.

close of their period of vegetation, and especially as the seeds ripen . in accordance with this, their root-system is abundant and far-reaching in the soil poor in nitrogen. The nitrogenous materials not employed at the time are stored up in the root-tubercles, which are emptied as the seeds begin to ripen, the nitrogenous contents being absorbed for the good of the latter

As I have shown, this hypothesis is utterly untenable with regard to the tubercles on the roots of *Vicia Faba*, and everything points to its being equally so for the other Leguminosæ : not only is TSCHIRCH's attempt to explain away the hyphæ and the gemmules a failure, but his drawings indicate that he is not in possession of the histological facts necessary to constitute him an authority on the subject of the development and physiology of the root tubercles of the Leguminosæ, whence his repeated assumptions lose in value.

It now remains to give some account of the experimental cultures which have had so much importance in leading to the discovery of the infecting hypha, and to the general results given above I found some time ago that Beans grown in a soil which had been burnt did not develope the tubercles, and that the same results followed as a rule when the cultivations were made in solutions of nutritive materials. I have since observed that if cultures in burnt soil are watered with washings of common garden soil, or of compost heaps, the tubercles are almost sure to be developed in the course of a few weeks , even river or pond water is dangerous—at least, in one case I traced the infection to the water with which I had watered the plants. Moreover, as I have shown, no experiment can be trusted if the seedling has been in contact with ordinary sand, turf, or leaf-mould

I find that a very convenient process for obtaining the tubercles is to allow the seed to germinate for a week or ten days in common garden soil, and then carefully lift the seedling, wash its root very gently, and place it in a split cork so that its root goes on developing in a nutritive solution. In the majority of cases the young tubercles are quite evident some time during the fourth week following, and a little experience enables one to detect them at least a week sooner than that.

Now, seedlings carefully germinated in the same garden soil, previously sterilised by being burnt, and subsequently treated similarly, do not yield tubercles at all.* This fact is in itself strong evidence of the infection coming from the soil.

It may be well to state the composition of the nutritive solutions I have employed, and the mode of treating the plants grown in them.

In all the cases referred to the solution employed was that given by SACHS in his 'Vorlesungen uber Pflanzenphysiologie,'† care being taken that all vessels and utensils were clean, and the water and reagents pure.

* Or only sporadically—a fact explained by chance infection during the culture.

† P. 342 English edition, p 284.

Water	1000	c. cm.
KNO_3	1	grm
NaCl	0 5	,,
$CaSO_4$	0 5	,,
$MgSO_4$	0·5	,,
$Ca_3(PO_4)_2$	0·5	,,

The seedlings were placed in a slit cork, so arranged that the whole of the root was submerged in the solution, and that the shoot could grow up freely into the light and air, which, together with the temperature and other conditions, were carefully regulated.

I followed SACHS' plan also in placing the cultures in new solutions when necessary, and in some cases of allowing their roots to remain for some hours in a solution of gypsum or salt between the changes However, these precautions are scarcely needed if the roots are kept darkened and the solution changed once a fortnight or so In cases where chlorosis was beginning to set in, traces of a salt of iron were added as necessary.

It may remove all cause for doubt to state that *it is just as easy to obtain the tubercles in the roots of these water-cultures* as it is on the roots of the plants growing in soil, in pots, or in the open; and, conversely, it is in both cases possible to prevent their formation by removing the infecting germs (by heating the medium, &c). There are in my greenhouse and laboratory at the present hour no less than 32 water-cultures and 26 cultures in pots, growing under conditions so well controlled that it is possible to predict with great accuracy when and where tubercles will be developed

I will confine myself to the following inferences from my Table of 81 experiments made this year, merely reserving the opinion that the ambiguous results in six or seven cases were due to faulty sterilisation

The chief points to notice are (1) the all but invariable development of the tubercles within a month, when the Beans were germinated in sand or soil not previously heated; (2) their non-development when the medium was sterilised by being heated; (3) the number of times I succeeded in infecting the roots by means of pieces of old tubercles placed among the root-hairs; and (4) the number of times the infecting hypha was discovered entering the cortex by means of the root-hairs.

It is partly on these grounds that I infer that the tubercles so common on the roots of the Bean are due to the action of the Fungus, the very minute germs of which are all but universally distributed in the soil, and it will be conceded that there is evidence for believing that germ to be the gemmule developed in the cells of the tubercle. The further physiological effects of the symbiosis must be reserved for future discussion.

No. of Bean.	Date of sowing.	Medium in which Bean germinated.	Date when put as water-culture.	Date when infected.	Date when tubercles were seen.	No. of days occupied in infection	Remarks
1	Jan. 26	River sand	Feb.	.	Feb. 28		No record of date when put as a water-culture
2	Feb. 28	Heated sand	.			..	Dug up on March 31, showed no signs of tubercles
3	”	”	.				
4	”	”	..	..			
5	”	”	.		..		
6	”	”				..	
7	”	”		..	..	.	
8	March 7	Heated sand	March 14	March 14	March 31	17	
9	”	”	”	”	”	17	In Nos 9, 10, and 11 I found the hypha passing down the root-hair, the tubercles were already several days old
10	”	”	”	”	”	17	
11	”	”	”	”	”	17	
12	”	”	”	”	April 2	19	
13	”	”	”	”	..		Bean No. 13 rotted away in a few days
14	”	”	”	”	April 2	19	
15	”	”	”	”	..	.	No tubercles developed on No 15, and the Bean rotted off.
16	March 16	Washed sand	March 30	March 30	April 9	10	Traced hypha down root-hair, more than 50 tubercles found
17	March 31	Garden soil	April 11	.	April 25	.	
18	”	”	”	.	April 18		Traced hypha down root-hair of No 18.
19	”	”	”		April 19		
20	”	”	”	..	”	..	Traced hypha down root-hair of No 20.
21	”	”	April 20	..	April 20	.	
22	”	”	”	..	”		The 8 Beans had tubercles when dug up, except No 24, which developed them 4 days later (The tap-root of this specimen was injured, and growth delayed) I traced the hypha down the root-hairs in every case.
23	”	”	”	..	”		
24	”	”	”	.	April 24	.	
25	”	”	”	..	April 20	.	
26	”	”	”	..	”		
27	”	”	”	..	”	.	
28	”	”	”	..	”	.	
29	April 2	Compost soil	..	..	”	.	These 15 plants, on being dug up on April 20, all showed tubercles on the tap-roots, and some on the lateral roots as well
30	”	”	..	..	”		
31	”	”	..	..	”	..	
32	”	”	..	..	”	..	
33	”	”		.	”		
34	”	”	.	..	”		
35	”	”	.	..	”	..	
36	”	”	..	..	”	..	
37	”	”		.	”		
38	”	”	..	..	”	..	
39	”	”	..		”	..	
40	”	”			”		
41	”	”	.		”		
42	”	”	..	..	”	.	
43	”	”	..	..	”	..	

No of Bean.	Date of sowing	Medium in which Bean germinated	Date when put as water-culture	Date when infected.	Date when tubercles were seen.	No of days occupied in infection	Remarks
44	April 2	Old sand	April 9		May 11		
45	,,	,,	,,	April 9	April 22	13	Traced hypha down root-hair in 3 cases.
46	,,	,,	April 10				No traces to May 21.
47	,,	,,	April 20				
48	,,	,,	.		.		No 48 failed to germinate
49	April 2	Heated sand	April 9				No 49 was lost by an accident on April 22
50	,,	,,	,,	April 9	May 21	42	Small tubercles appearing all over the roots.
51	,,	,,	April 10	April 10	April 23	13	
52	,,	,,	,,				
53	,,	,,	,,		.		No traces visible to May 21
54	April 5	Heated soil	April 23		.		
55	,,	,,	,,			..	
56	,,	,,			..	..	No 56, dug up on May 4, showed no traces of tubercles
57	,,	,,			..	.	Nos 57, 58, dug up May 11, showed no signs of tubercles
58	,,	,,	..	..	..	..	
59	,,	,,	April 13	.	May 12	..	
60	,,	,,	,,		,,		
61	,,	,,			May 14		No 61, dug up May 14, had developed tubercles, but few and small
62	,,	,,	..				Nos 62, 63, dug up May 21, showed no traces.
63	,,	,,				..	
64	April 13	Garden soil		..	May 9	.	
65	,,	,,	..		,,		
66	,,	,,		..	,,	.	Nos 64–69 were dug up on May 9, and had many well-developed tubercles
67	,,	,,			,,		
68	,,	,,	.	..	,,		
69	,,	,,			,,		
70	April 14	Heated soil	..	..	.		Nos 70, 71, dug up May 11, showed no traces of tubercles
71	,,	,,	.	..			
72	,,	,,	..	..			No 72, dug up May 17, showed no trace of tubercles
73	,,	,,					
74	,,	,,	..	.	..	..	Dug up May 21, showed no signs of tubercles
75	,,	,,	..			..	
76	April 15	Heated sand	April 22		May 14		Nos. 76–81 stood under the same bell-jar The question is, was sterilisation effected, or did after-infection occur? In all cases the hypha was traced down the root-hair
77	,,	,,	April 23		,,		
78	,,	,,	,,		,,	..	
79	,,	,,	,,	April 23	May 11	18	Tubercles on No 79 very fine and numerous May 11.
80	,,	,,	,,	,,	May 21	28	
81	,,	,,	..	..		..	No 81, dug up May 11, showed no traces of tubercles

Explanation of Plates

PLATE 32

Fig. 1. Portion of Bean-root with the tubercles or root-swellings of various ages, showing the different stages of development; a very young one is seen on the rootlet to the left above. (Natural size)

Fig. 2. One of the larger swellings cut in two. The buff-coloured portion is the part where the parasite is rampant · the paler portions towards the apex contain hyphæ only. The narrow zone *a* at the apices of the lobes consists of meristematic cells scarcely affected as yet by the hyphæ. (Natural size)

Fig. 3. Half-diagrammatic section of a root, with tubercle and normal lateral rootlet. The latter arises opposite a xylem strand. The tubercle originates in the cortex, and also (in this case, but not always) opposite a xylem strand. The various tissues of the tubercle are indicated. The apical meristem contains very few hyphæ numerous branched hyphæ are found in the cells lower down; the dark-shaded cells in the lowest central parts are hypertrophied and filled with the bacterium-like corpuscles—gemmules or yeast-cells.

Fig 4 Three cells of the dark inner portion of the tubercle (fig. 3). The corpuscles (gemmules) have multiplied to an enormous extent, causing the protoplasm to become vacuolated and plasmodium-like, and the cells to enlarge · the dark body embedded in the mass of yeast-cells and protoplasm is the nucleus. E/4.

Fig. 5. Similar cells after the macerated tubercle has rotted . the cells are separating, and appear full of bright (dark) granules in a matrix. The granules are the now dormant yeast-corpuscles (gemmules), the dark body is the nucleus of the cell. E/4.

Fig. 6. The gemmules—granular bodies in figs. 4 and 5—more highly magnified. *a* and *b* are from still active tubercles, and are more highly magnified than *c*, which is from an old tubercle of the preceding year. *a* and *b* = L/4 ; *c* = J/4.

Fig. 7. Transverse section across a rootlet, and tubercle about the size of the smallest one in fig. 1. The tissues of the tubercle are seen to have hyphæ in them; these hyphæ are branches from the thicker hypha which passes down the root-hair *a* (to the left of the figure) and across the cortex of the rootlet. (B/3.)

Fig. 8. Portion of fig. 7 more highly magnified, and showing how the hypha from the root-hair commences to break up into branches as it passes into the young tubercle. (D/4.)

Fig 9. A portion of the last figure still more in detail, and showing the curious trumpet-shaped widenings where the hyphæ pass through the cell-walls. One branch of the main hypha is attached to the nucleus *n* of one of the cells—this is not uncommon. The first branching of the main hypha clearly takes place in the cell-wall. The further branchings are not clear, because the course of the hyphæ varies in different planes J/4.

PLATE 33

Fig 9A. A preparation similar to fig 9, but showing a much younger stage of the development of the tubercle The infecting hypha is crossing the cortex from cell to cell: soon after leaving the root-hair, it branches, the branch soon running in the cell-wall. The cells traversed by the hypha are hypertrophied. The young tubercle is arising by meristematic division of the innermost cortical cells opposite the entering tip of the infecting hypha, which is in contact with a nucleus. (E)

Fig. 10. Part of the central tissue of a young tubercle (near the apex in fig. 3), showing the characters of the hyphæ, and their branching in the cells. The haustorium-like branch in the centre is very characteristic. E/4.

Fig 11. Similar preparation from nearer the apex of the tubercle The hyphæ are thicker, and their course more direct. This and the last figure from preparations treated with very dilute ammonia. E/4.

Fig. 12 Similar preparation (fresh) from the lower part of the central tissue of the tubercle, corresponding to the part shaded dark in fig. 3. The protoplasm of the cells is nearly filled with the budding gemmules, and has become vacuolated and plasmodium-like, the nucleus being driven to one side. Branches of the hyphæ are seen in the cells, and dense tufts of budding gemmules can be seen shining through the matrix (*e.g*, the lower cell) in some cases : in others everything is obscured by the dense matrix. Starch often accumulates in the cells at this stage, and the nucleus in some cells undergoes a sort of fatty degeneration. (J/4.)

Fig. 13. Hypha traced down the root-hair into the cortex. It takes origin from a brilliant granule—presumably one of the gemmules From a water-culture the root of which was infected by a section from a tubercle of the preceding year. The remainder of the course of the hypha is clear from figs. 7, 8, 9, 10, 11, 12. The hypha in the cortex subsequently turns yellow, and shrivels, and only remote traces are to be found in older tubercles. J/4.

Fig. 13A. Further preparations, resembling that from which fig. 13 was drawn. The

infecting hypha clearly arises from a brilliant point, around which the growth of the cell-wall of the root-hair is affected, causing distortions. L/4.

Fig. 14. Hypha passing through the cell-wall, showing the curious trumpet-shaped widening where the passage is effected. L/2.

Figs. 15, 16, and 17. Ultimate branches of the hyphæ in the cells of the tubercle, at the commencement of the budding process. The curious Y and V-shaped gemmules are well seen in fig. 17. The haustorium-like branches in figs. 15 and 16 are very like the incipient spores of an *Ustilago* (*e g*, DE BARY, fig. 82, p. 189), but they do not become spores, and in many cases seem to give off the gemmules. (J/4.)

Fig 18. Part of a section of a last year's tubercle—very thin, and treated with dilute ammonia—showing that the remains of the mycelium are still there, though obscured by the dense aggregation of gemmules. (E/4)

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PHILOSOPHICAL TRANSACTIONS (B)

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HARRISON AND SONS, PRINTERS IN ORDINARY TO HER MAJESTY, ST. MARTIN'S LANE.

G. H. Barrow, del
W. H. Wesley, lith.

West, Newman & Co imp

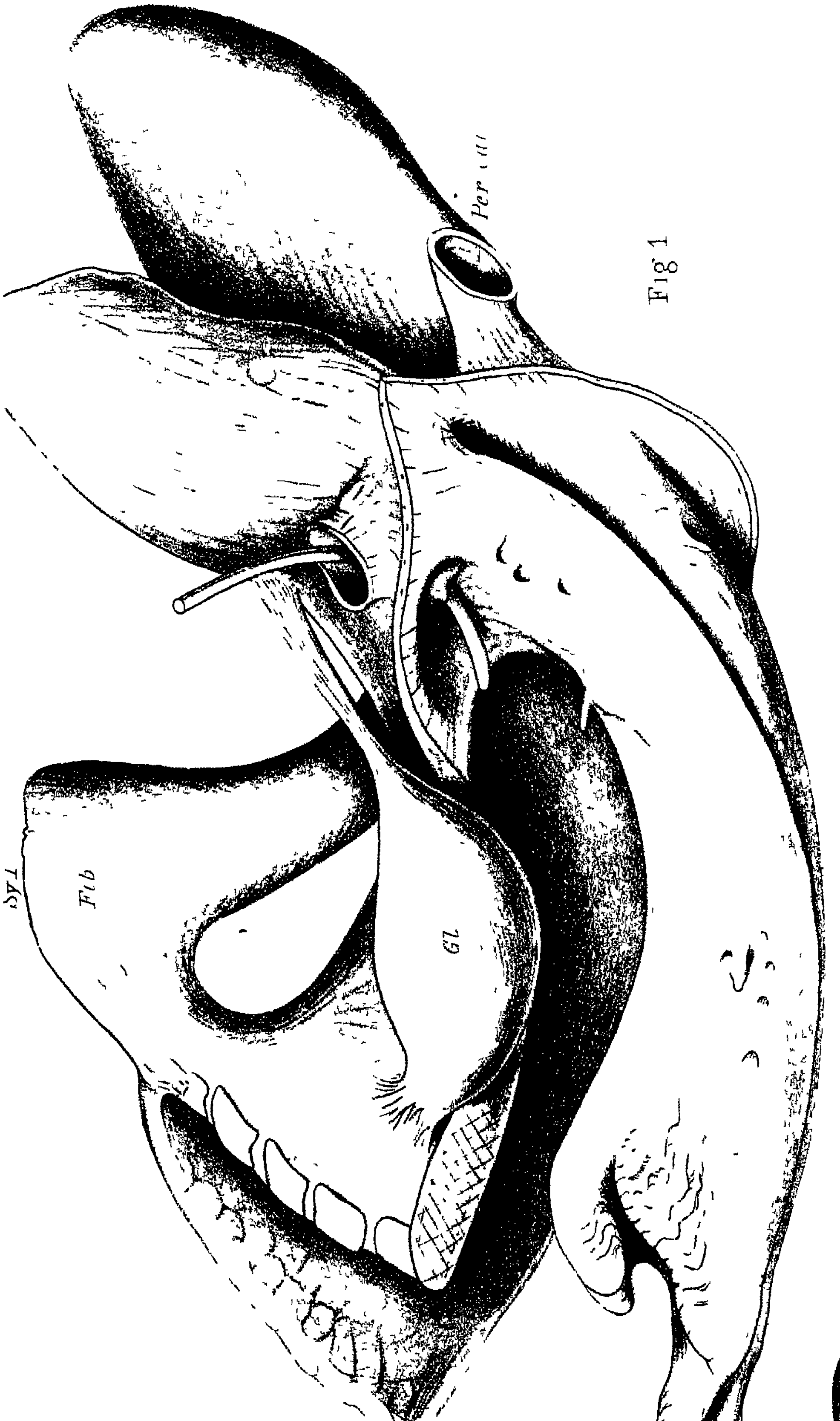
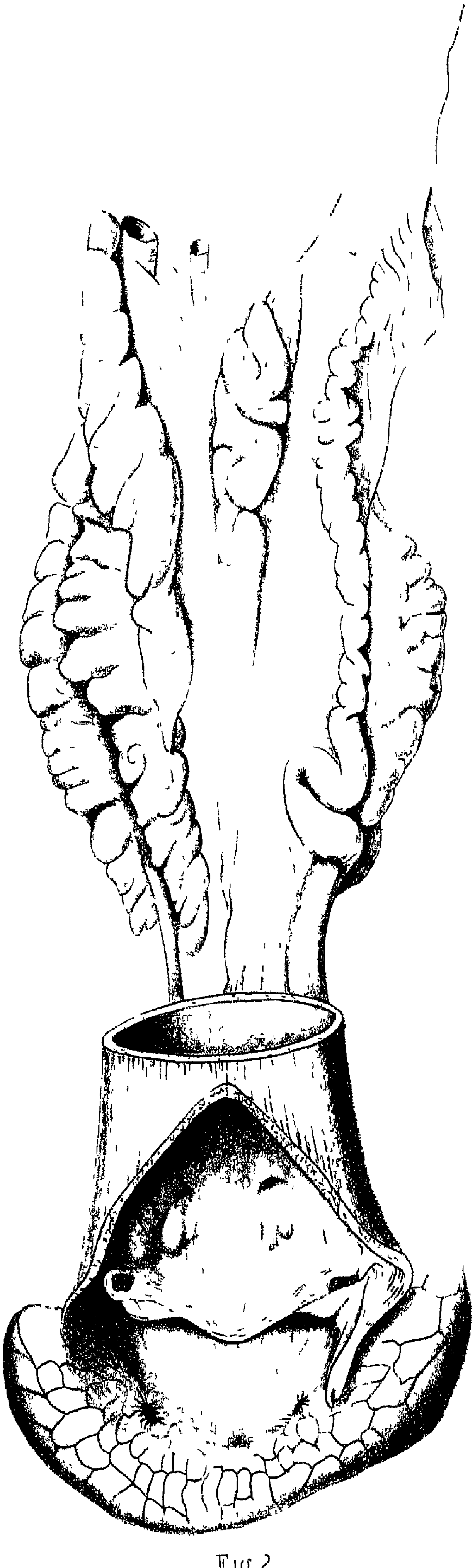


Fig 1



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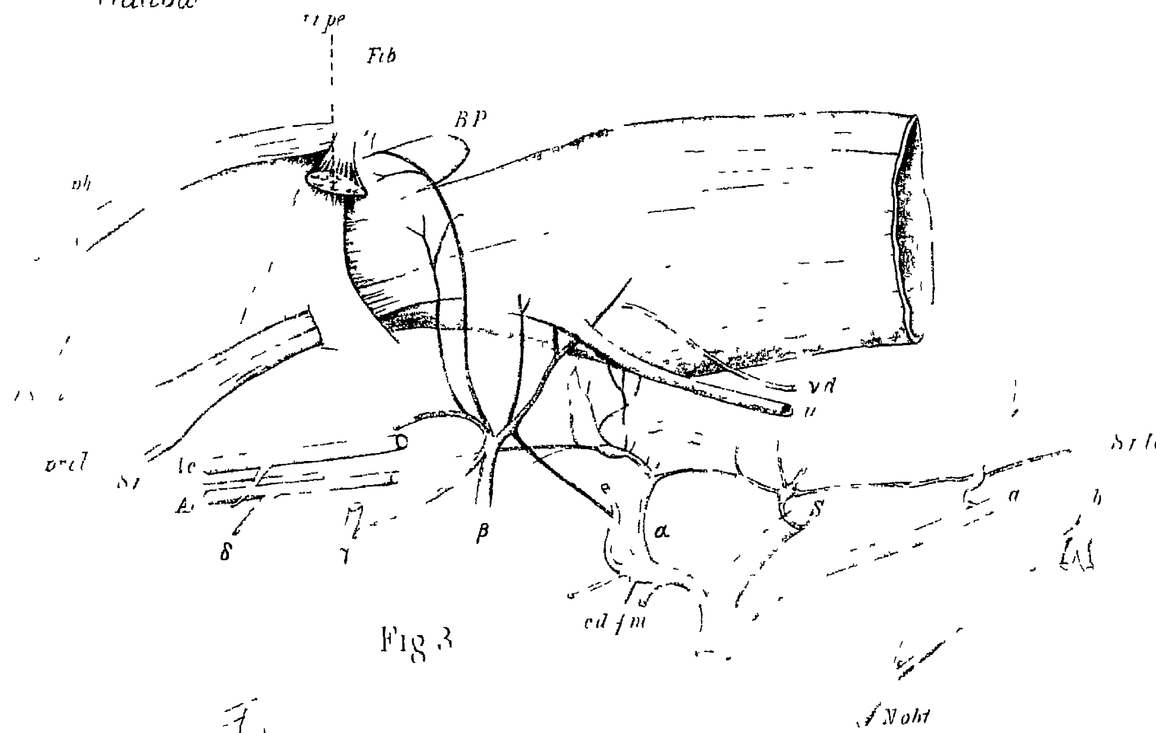

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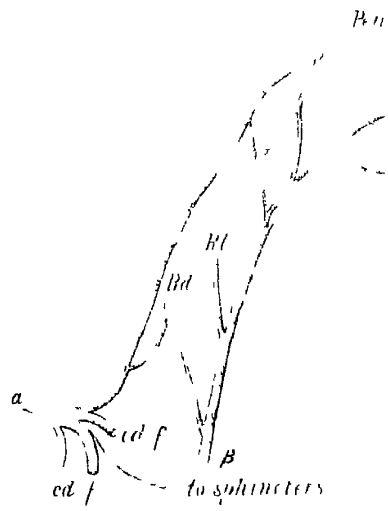


Fig 4

Fig 5

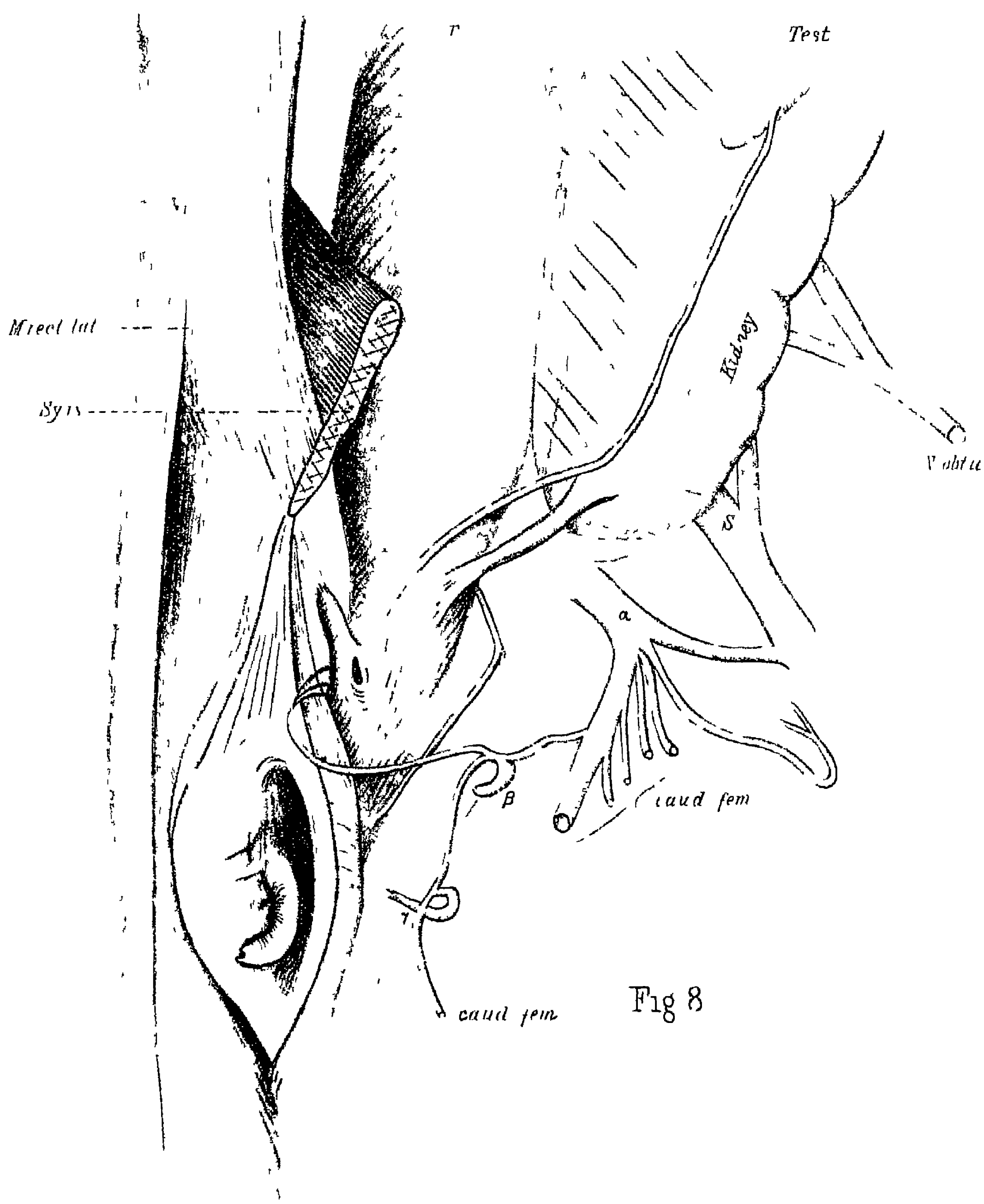
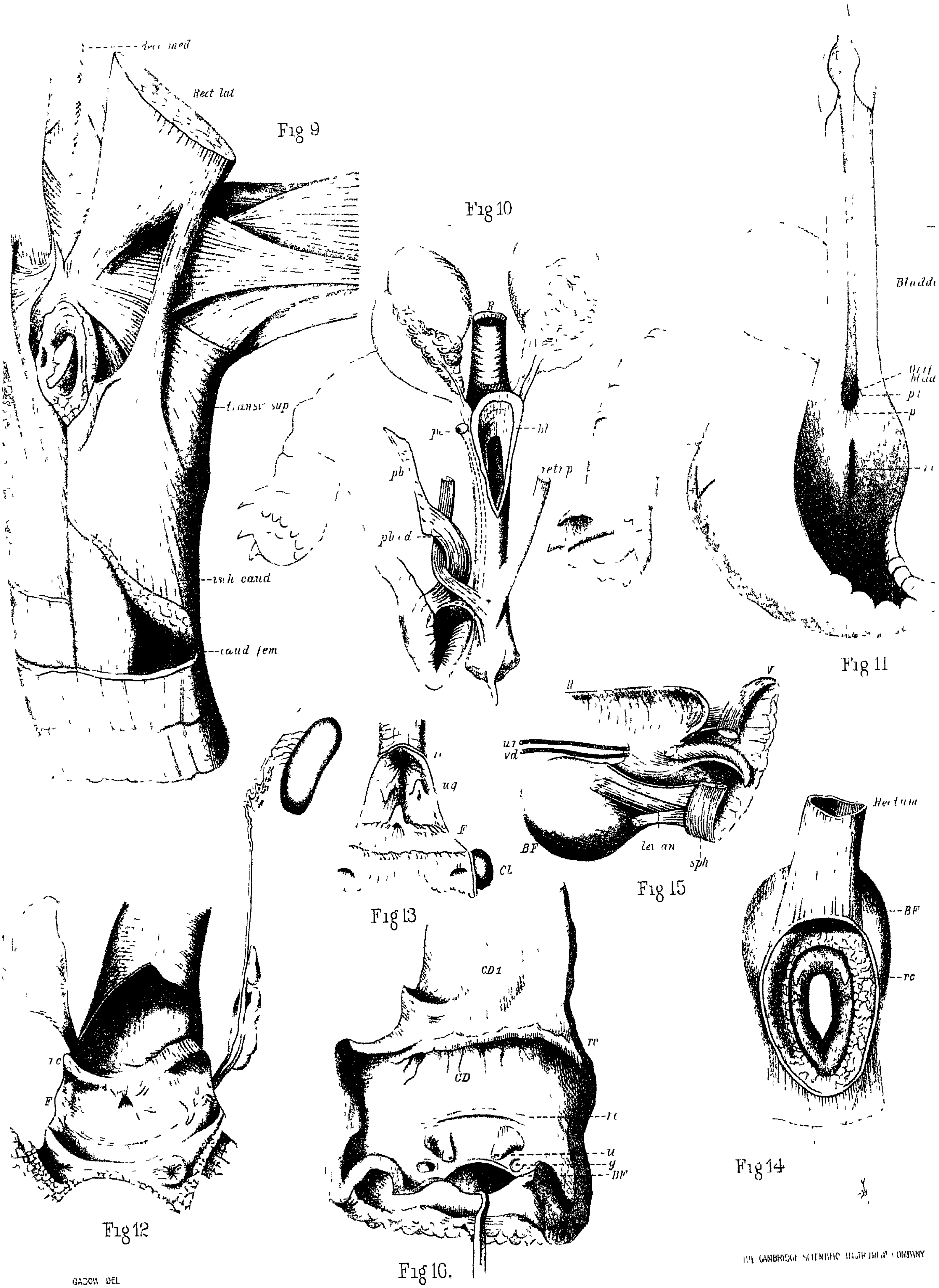


Fig 8

Fig. 6

Fig 7



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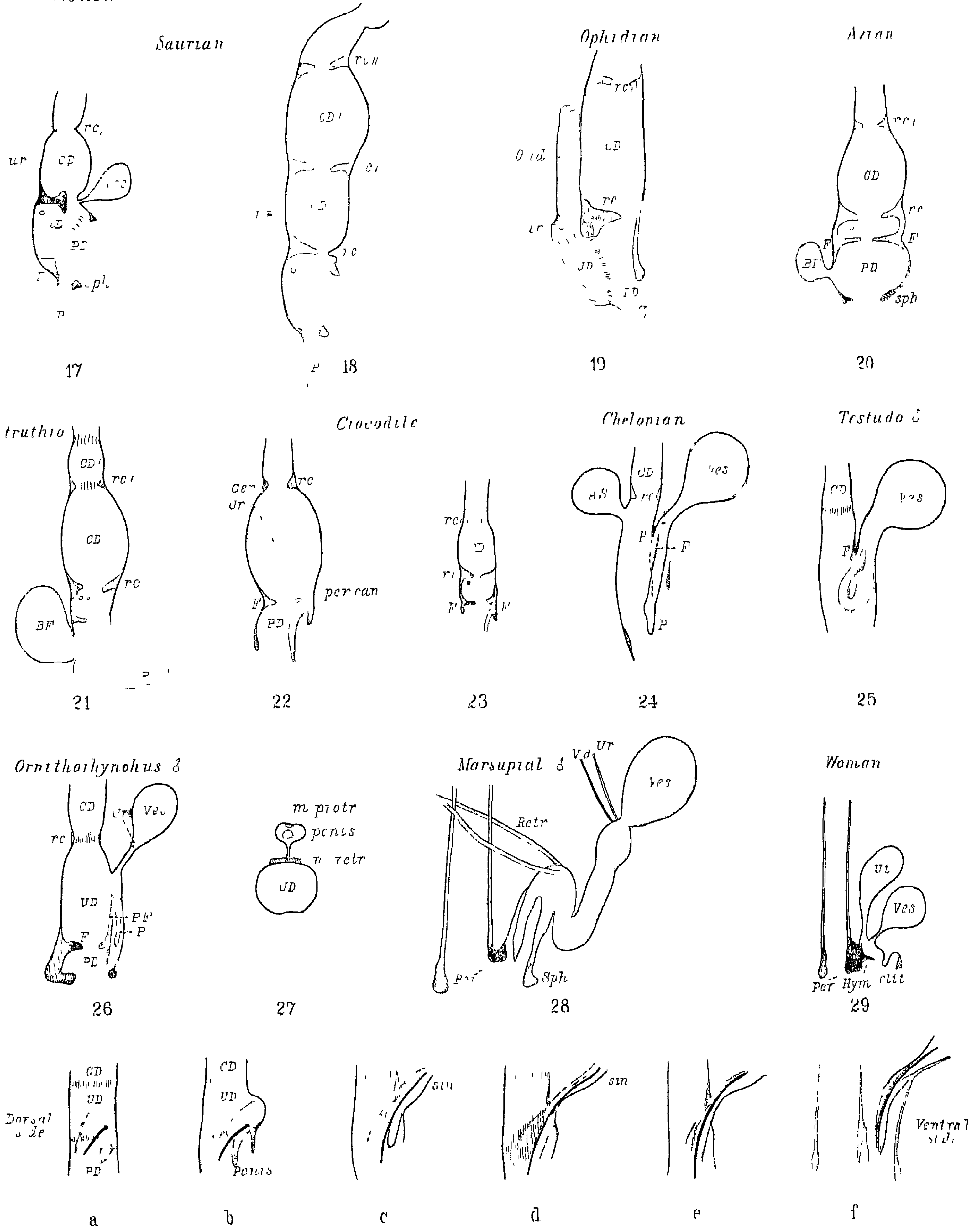
Saurian
Ophidian
Avian
17
18
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20
Crocodile
Chelonian
Testudo ♂
21
22
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Ornithorhynchus ♂
m protr
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Marsupial ♂
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Ves
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Dorsal side
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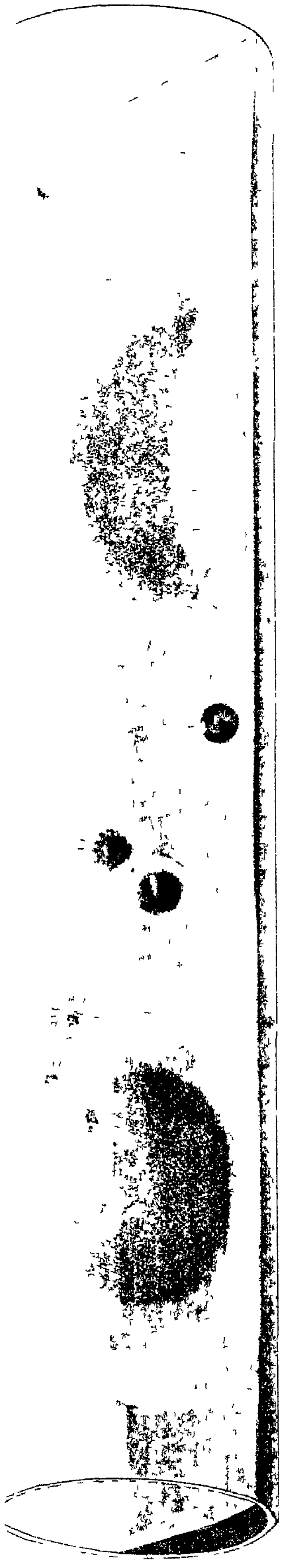
Fig 30



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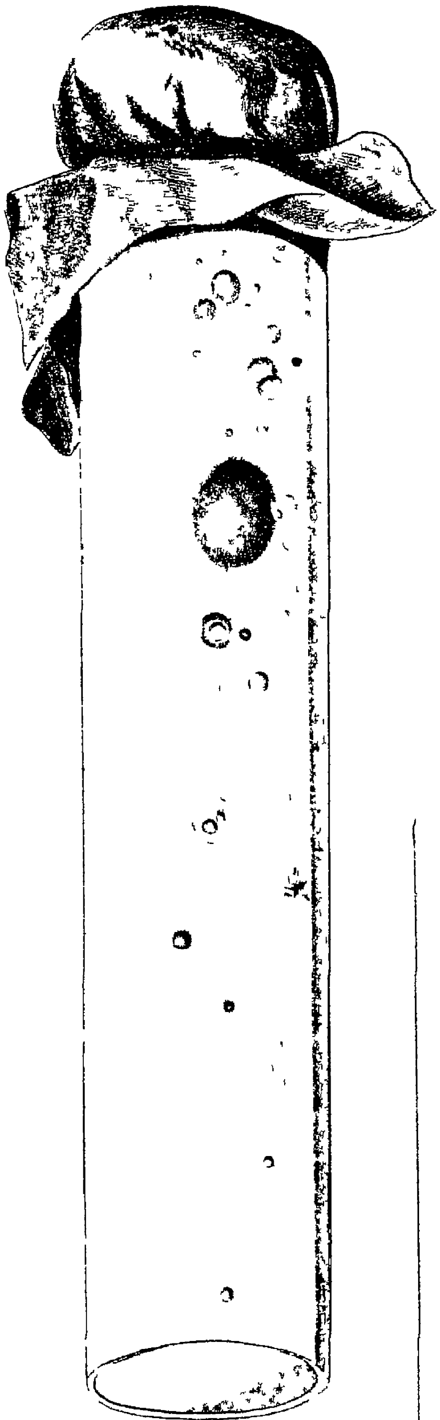

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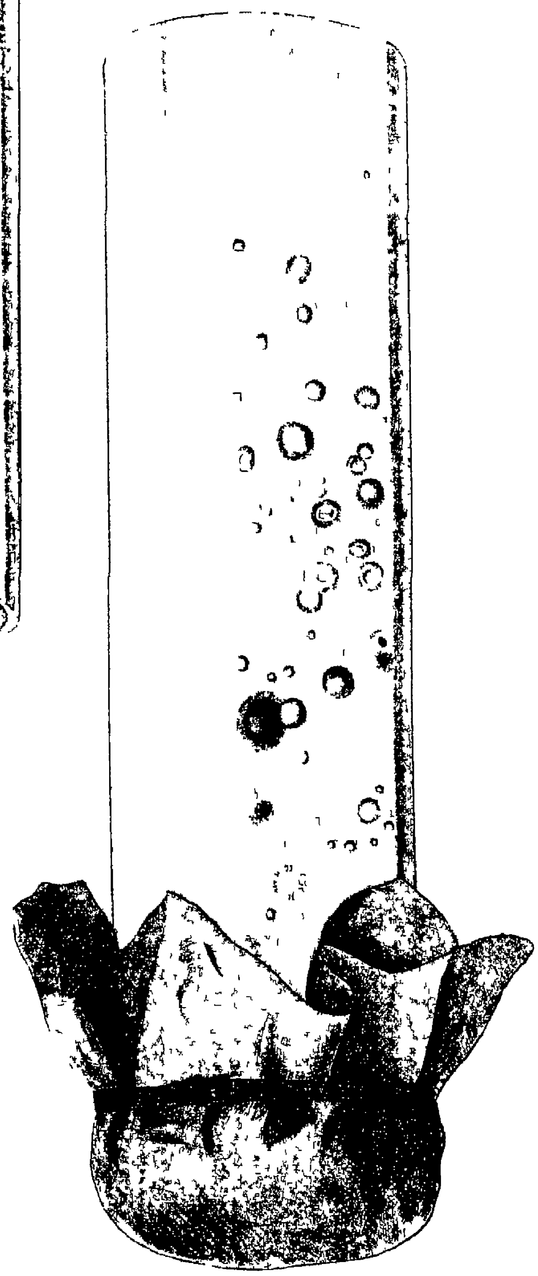

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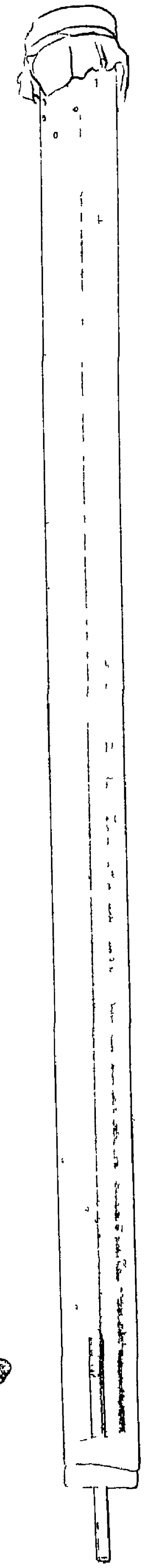

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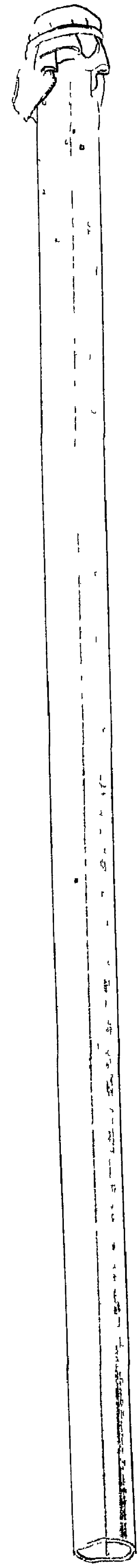

Fig 5

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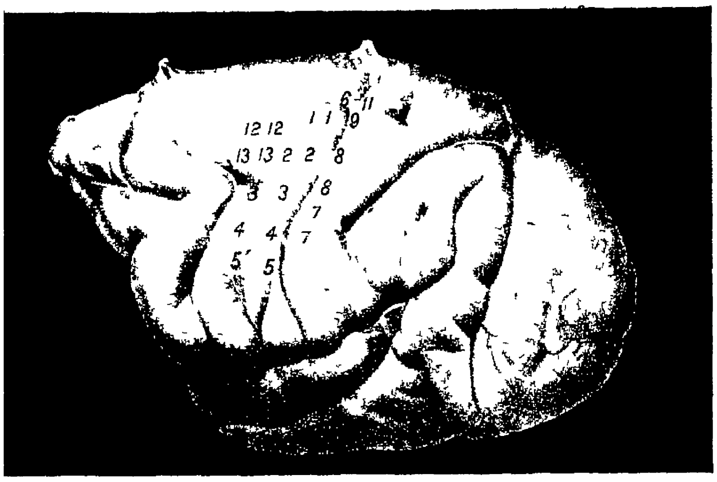
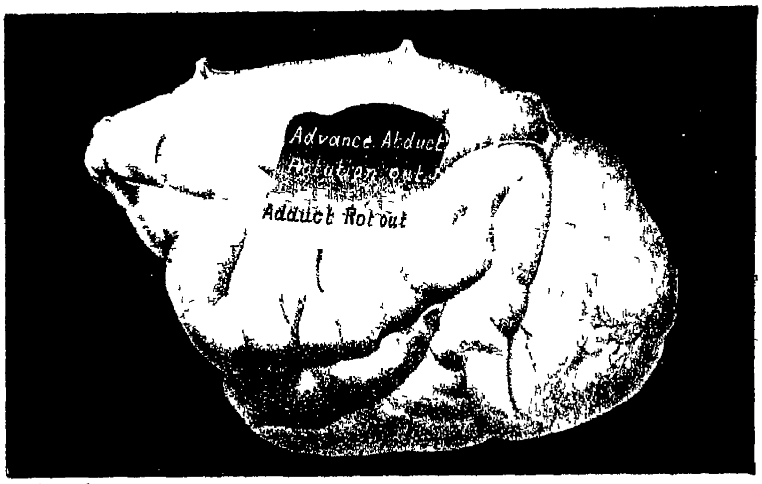


Fig 1



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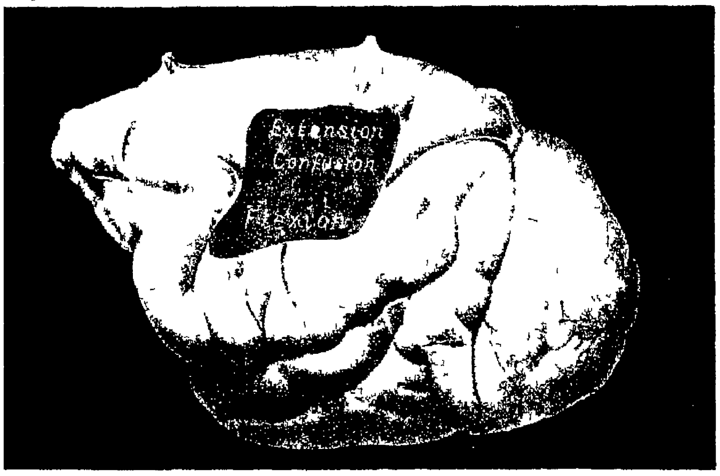

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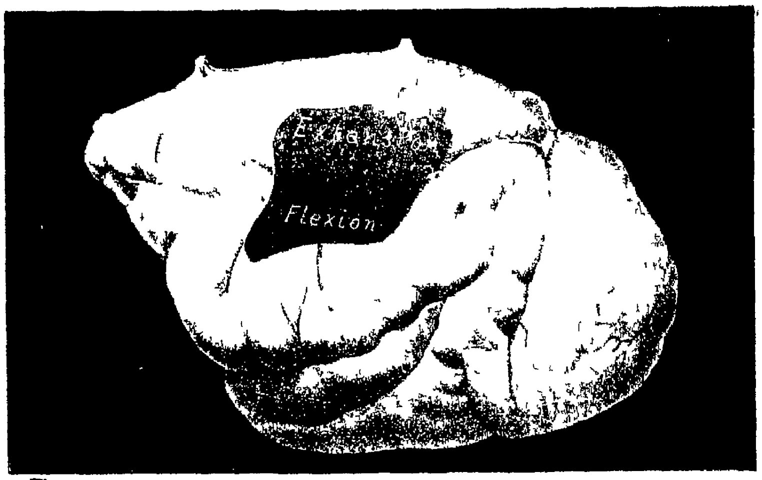

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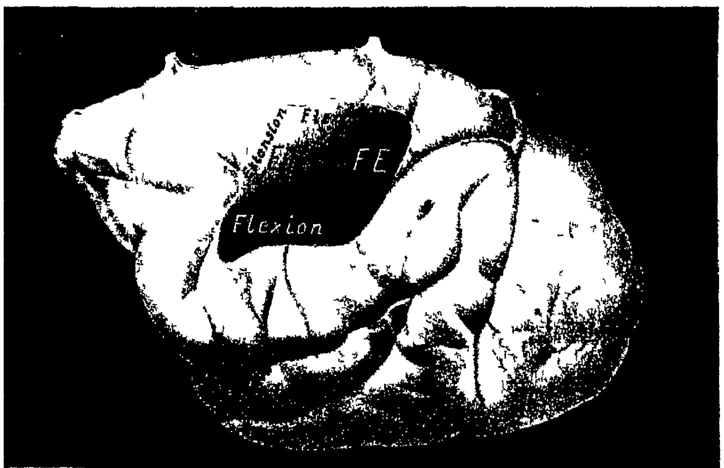

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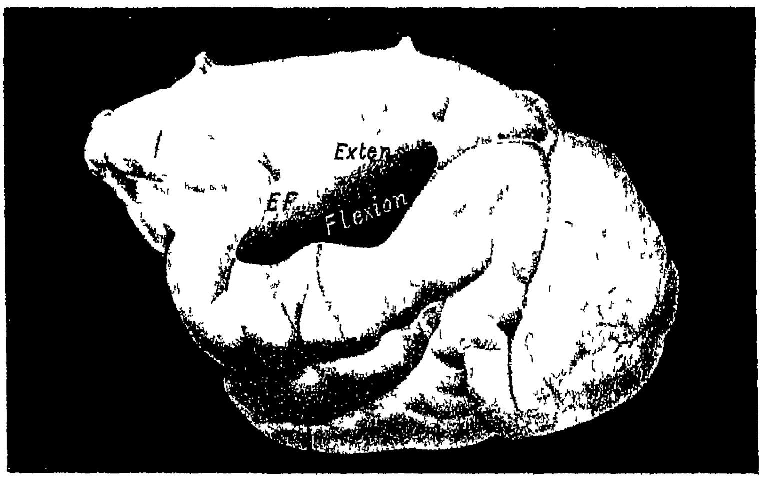

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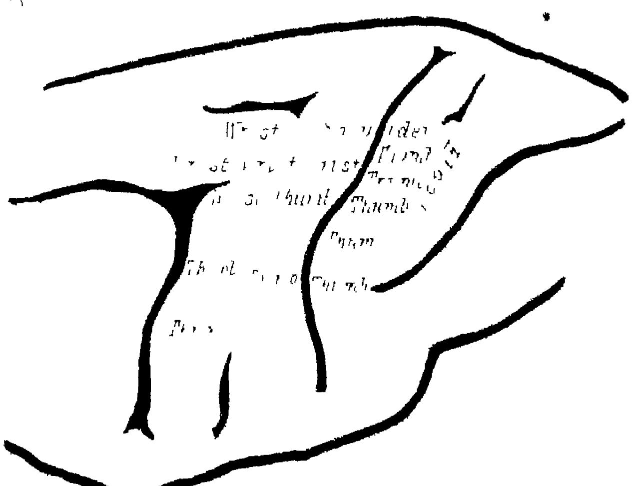

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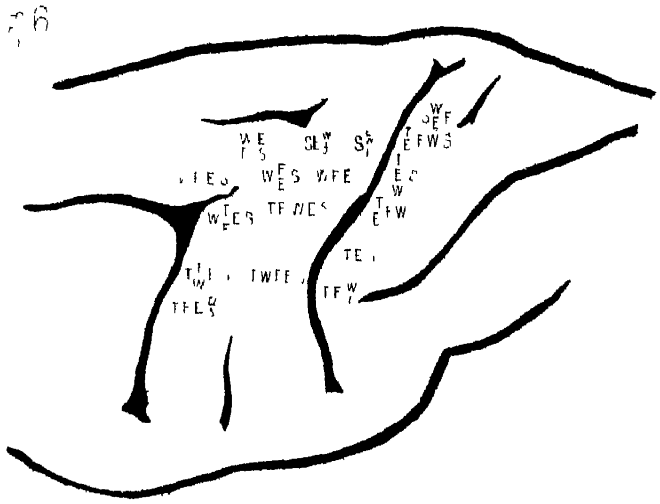

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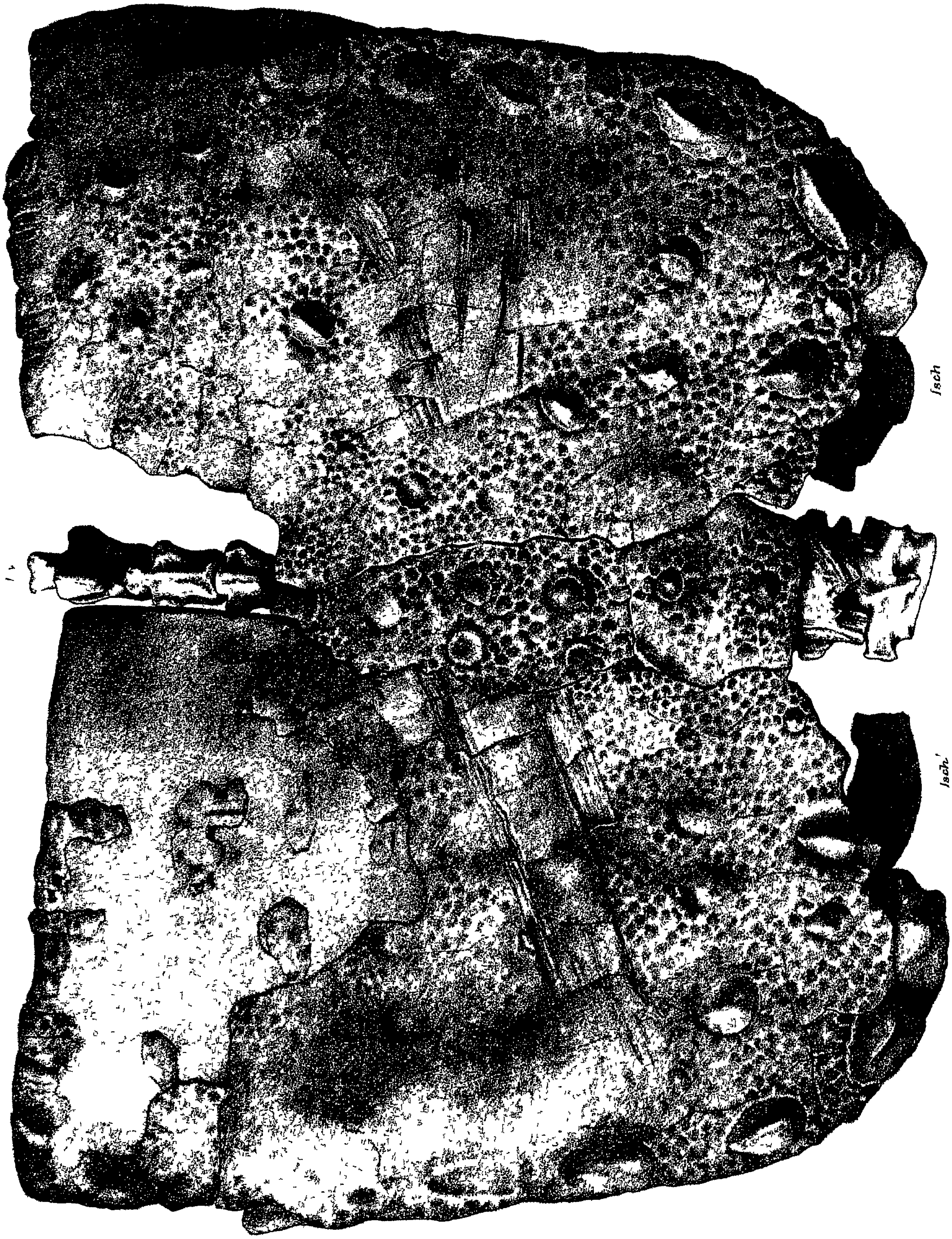

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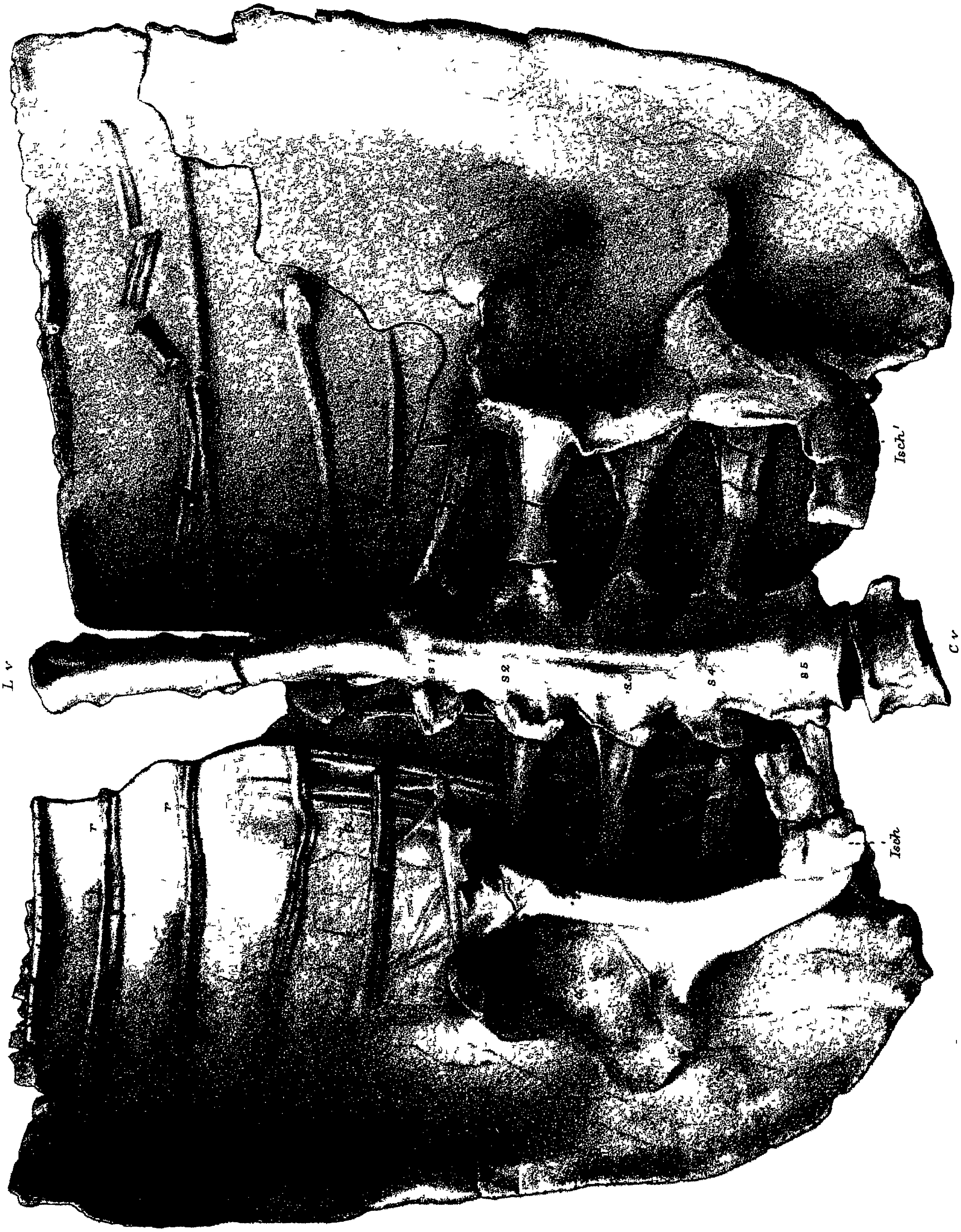
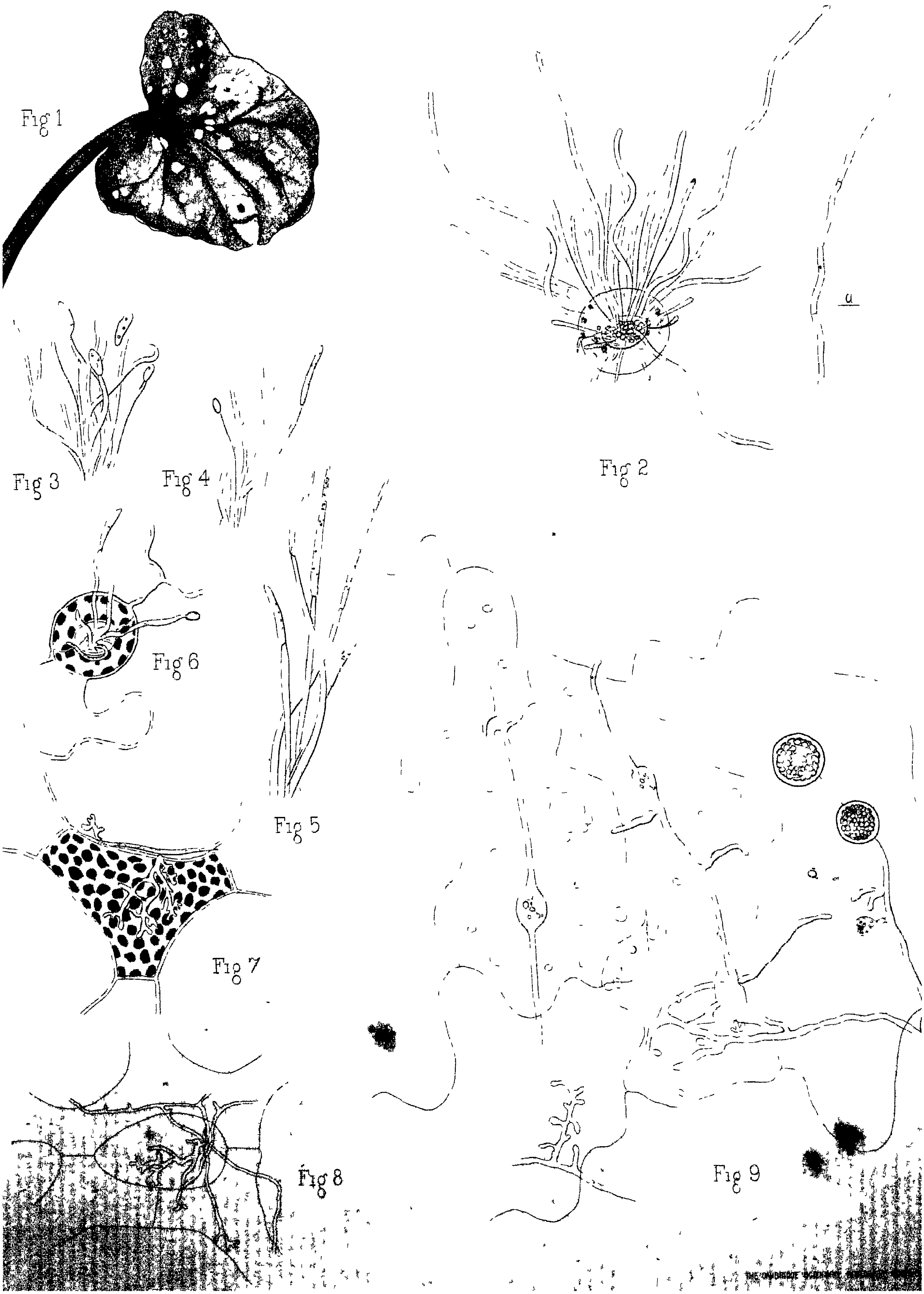


G.M Woodward del et lith. West, Newman & Co. imp.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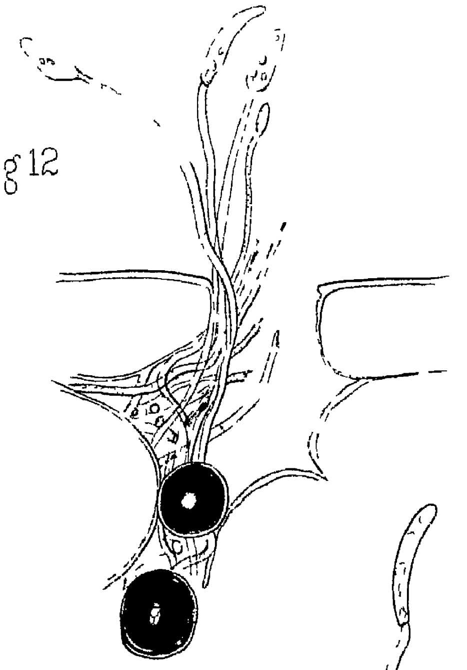

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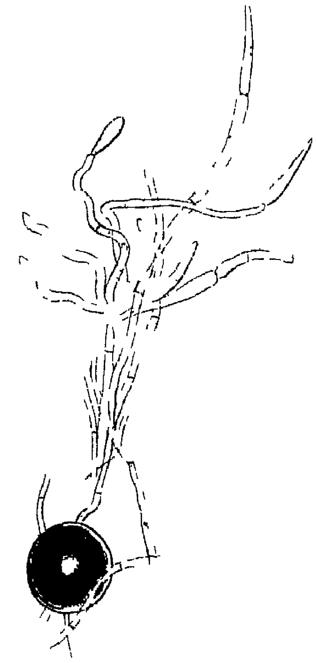

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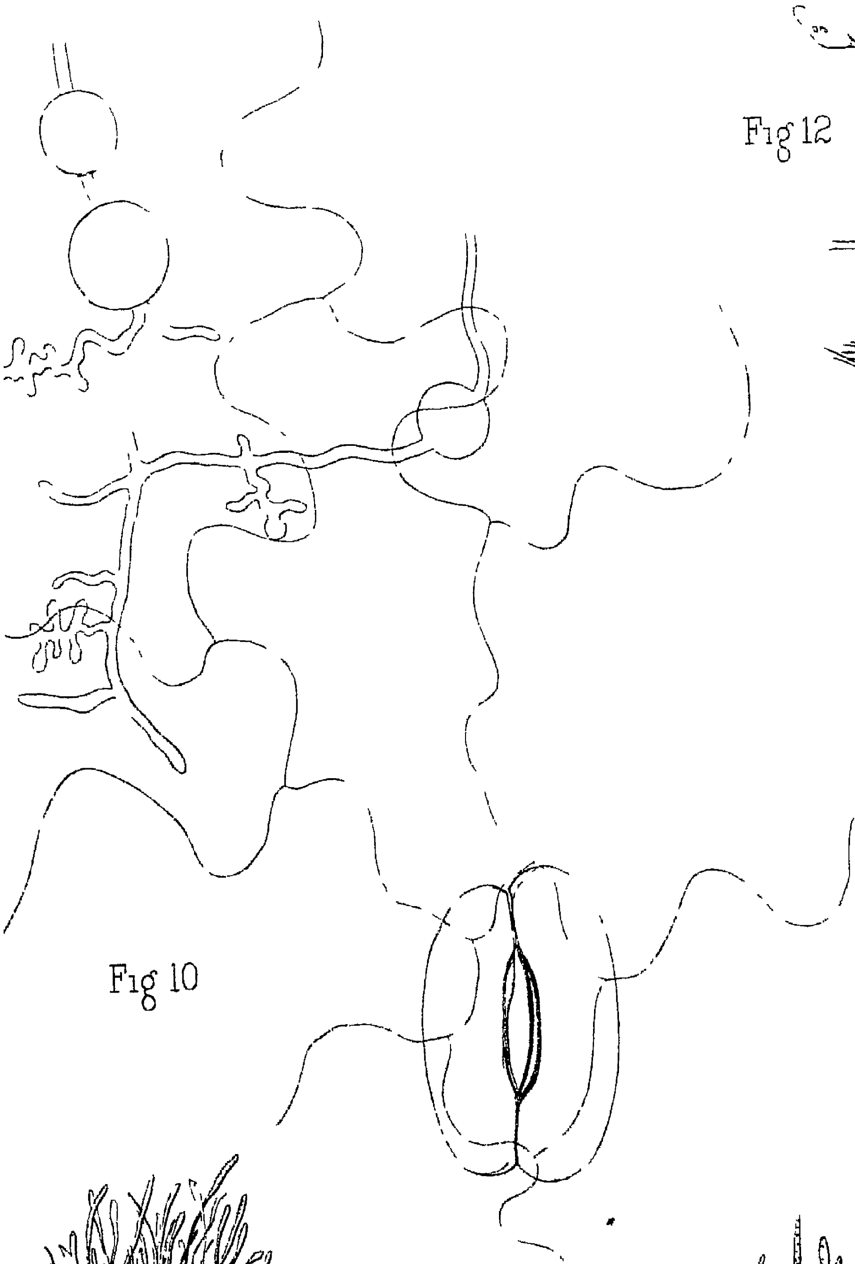

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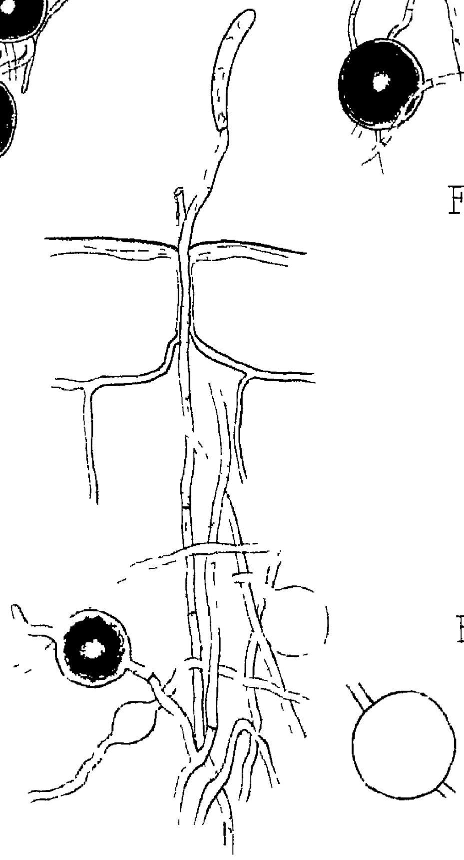

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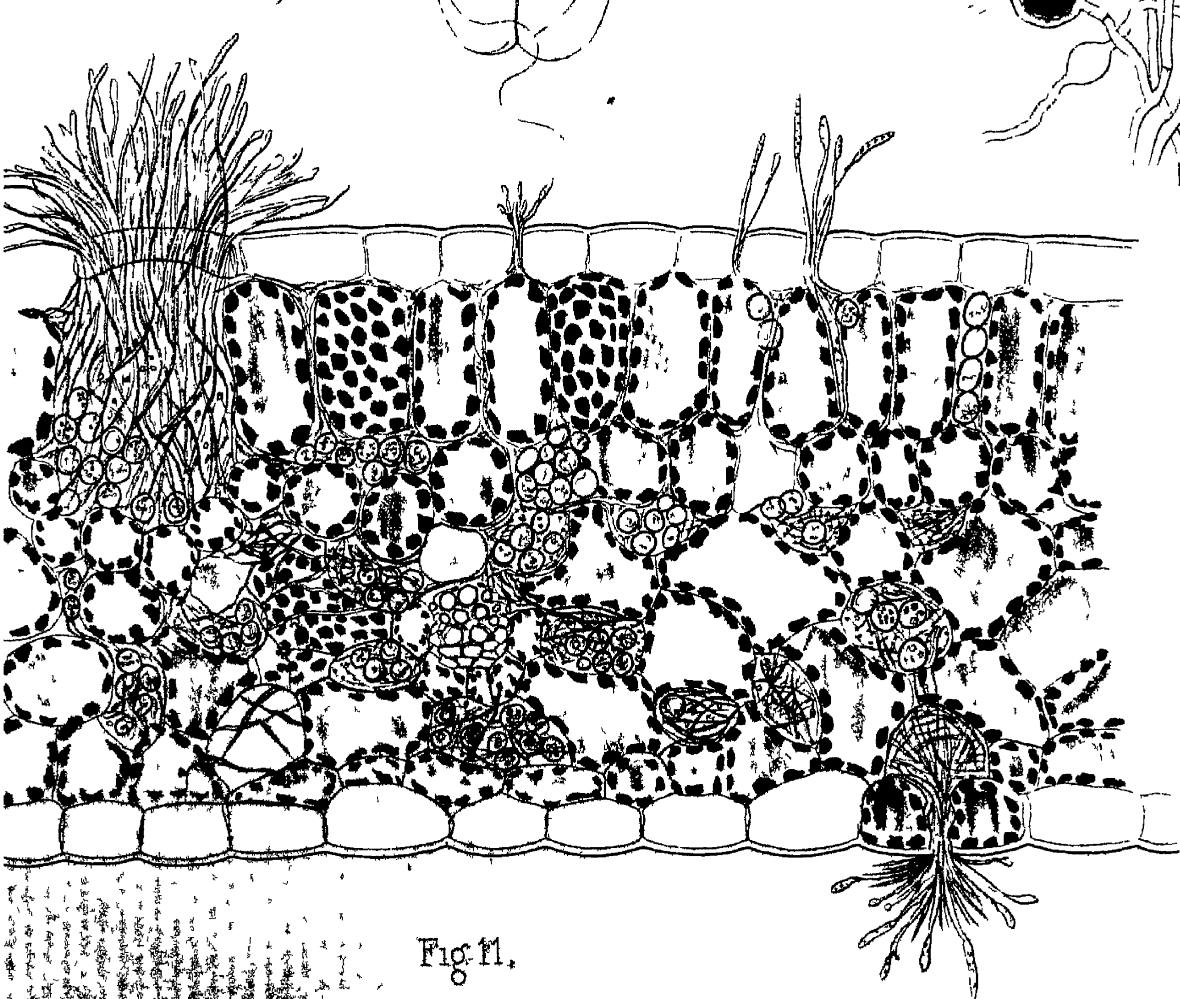

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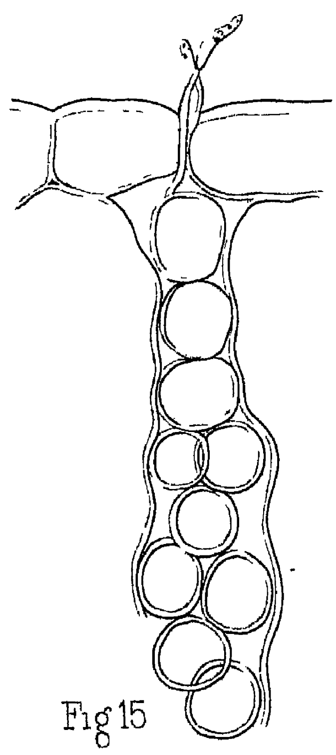

Fig 15

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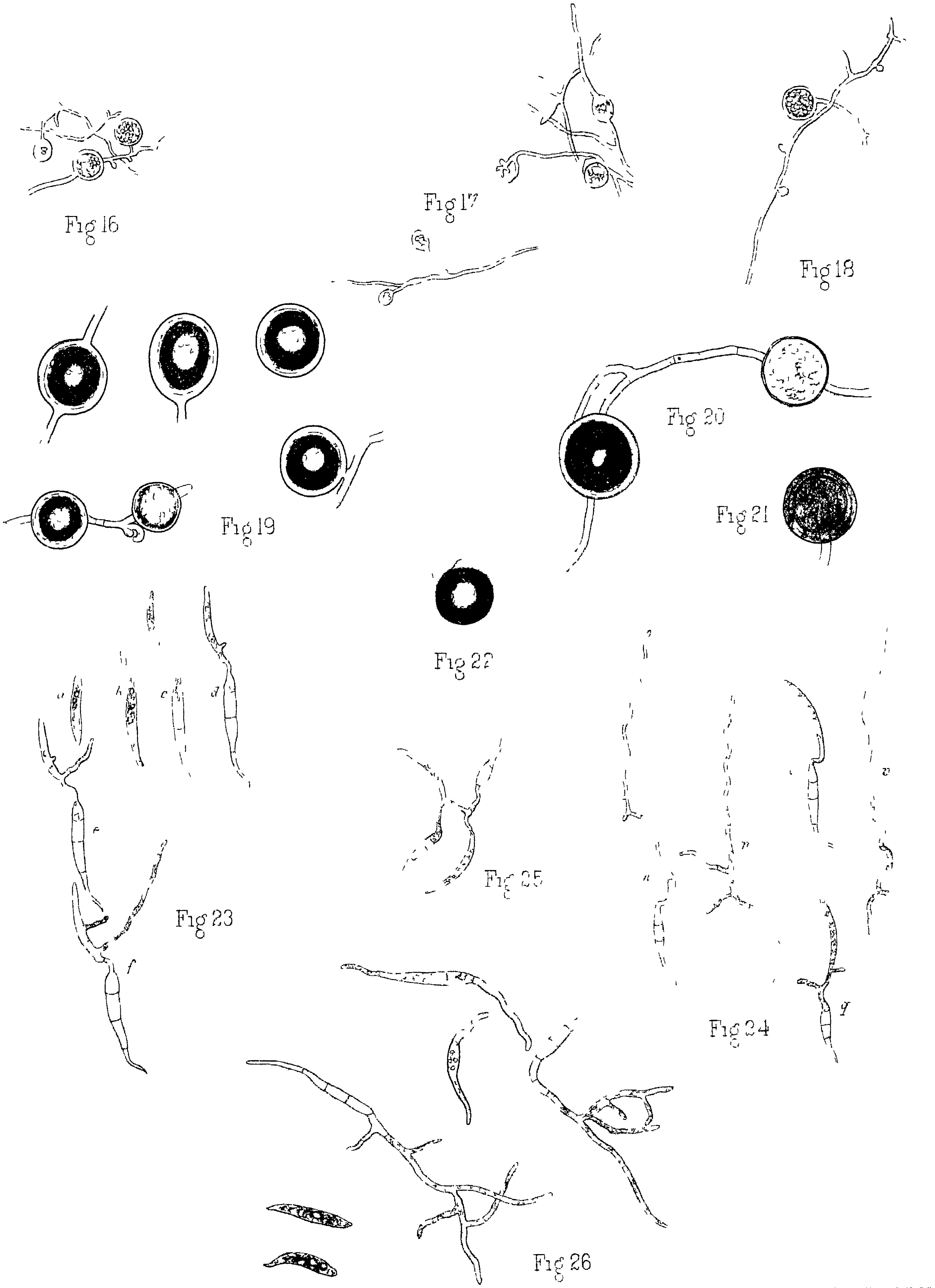

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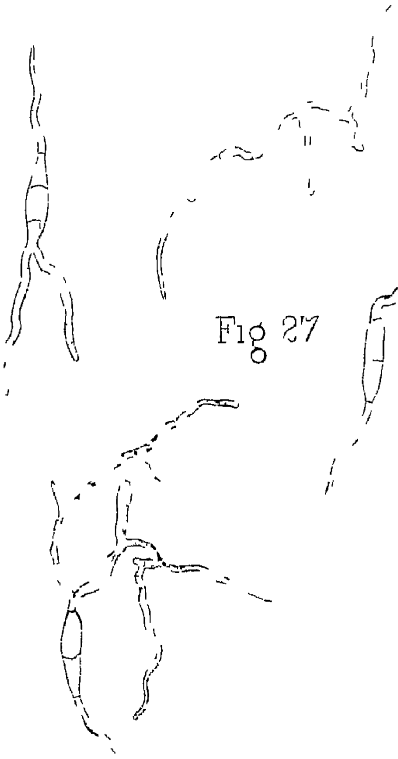

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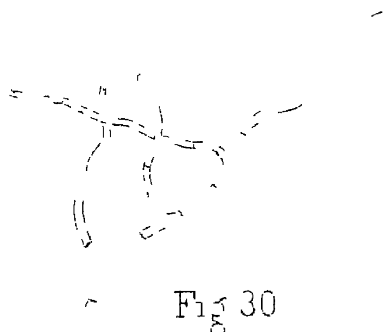

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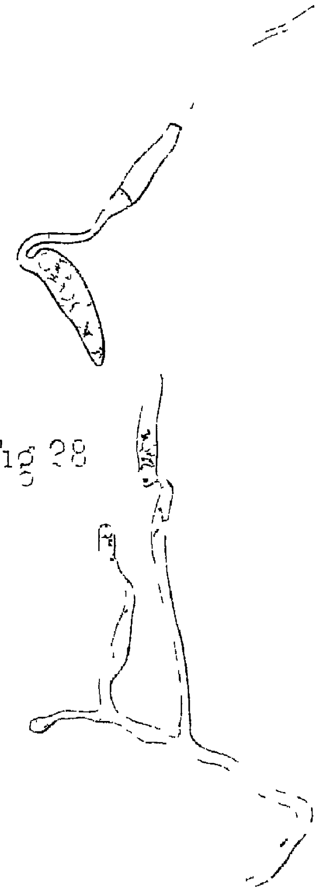

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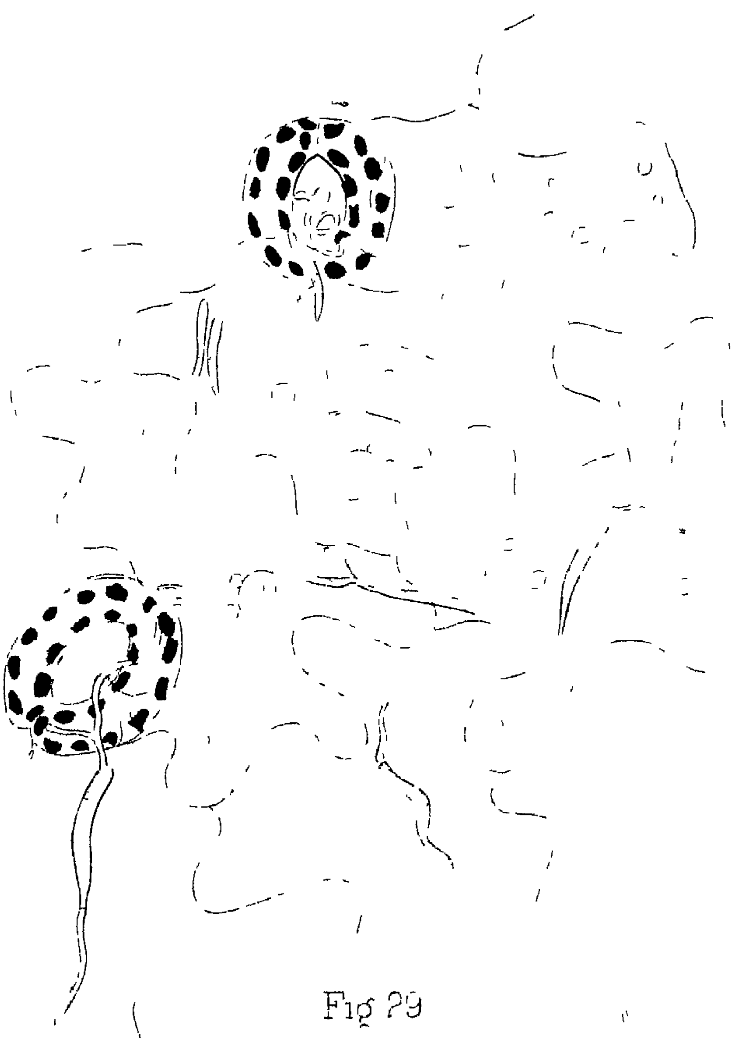

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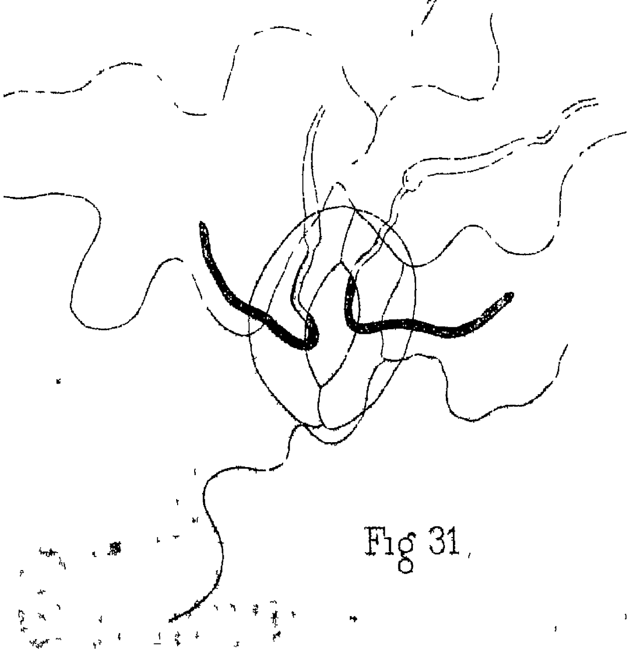

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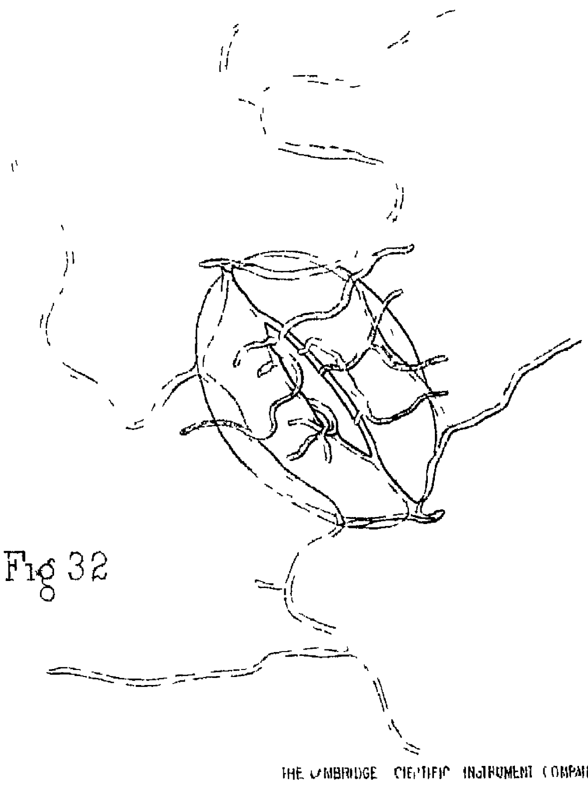

Fig 32

THE CAMBRIDGE SCIENTIFIC INSTRUMENT COMPANY

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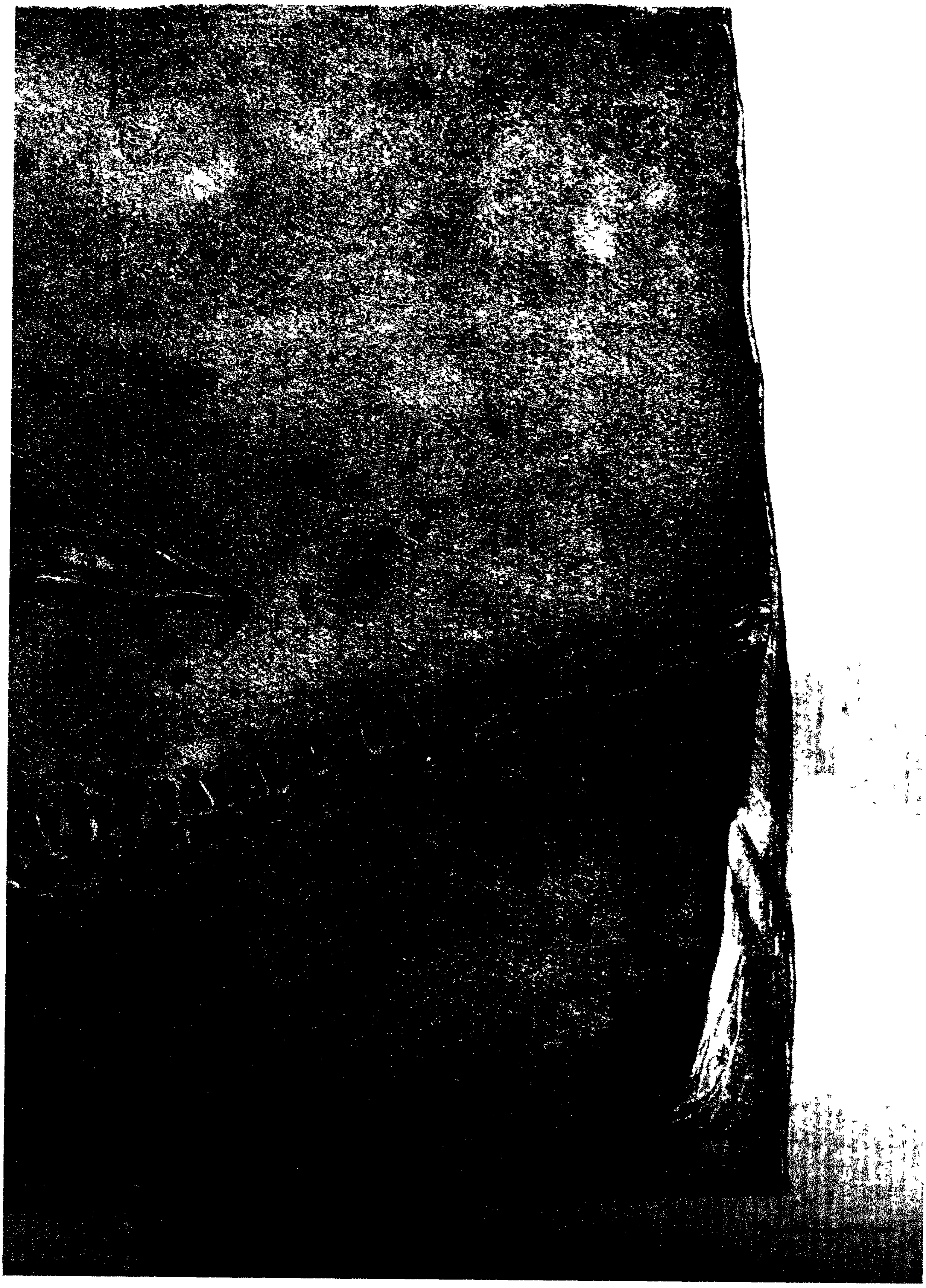
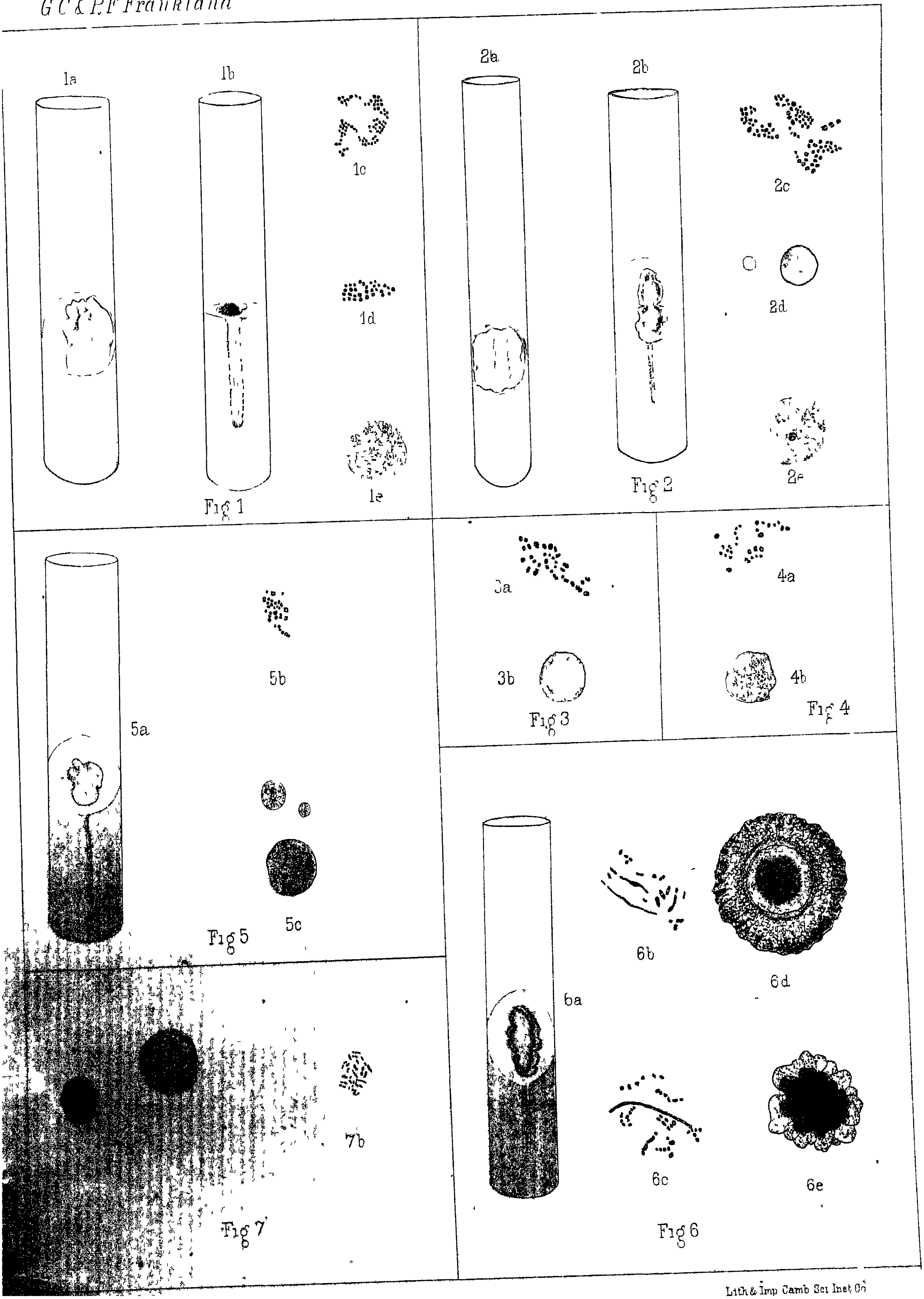


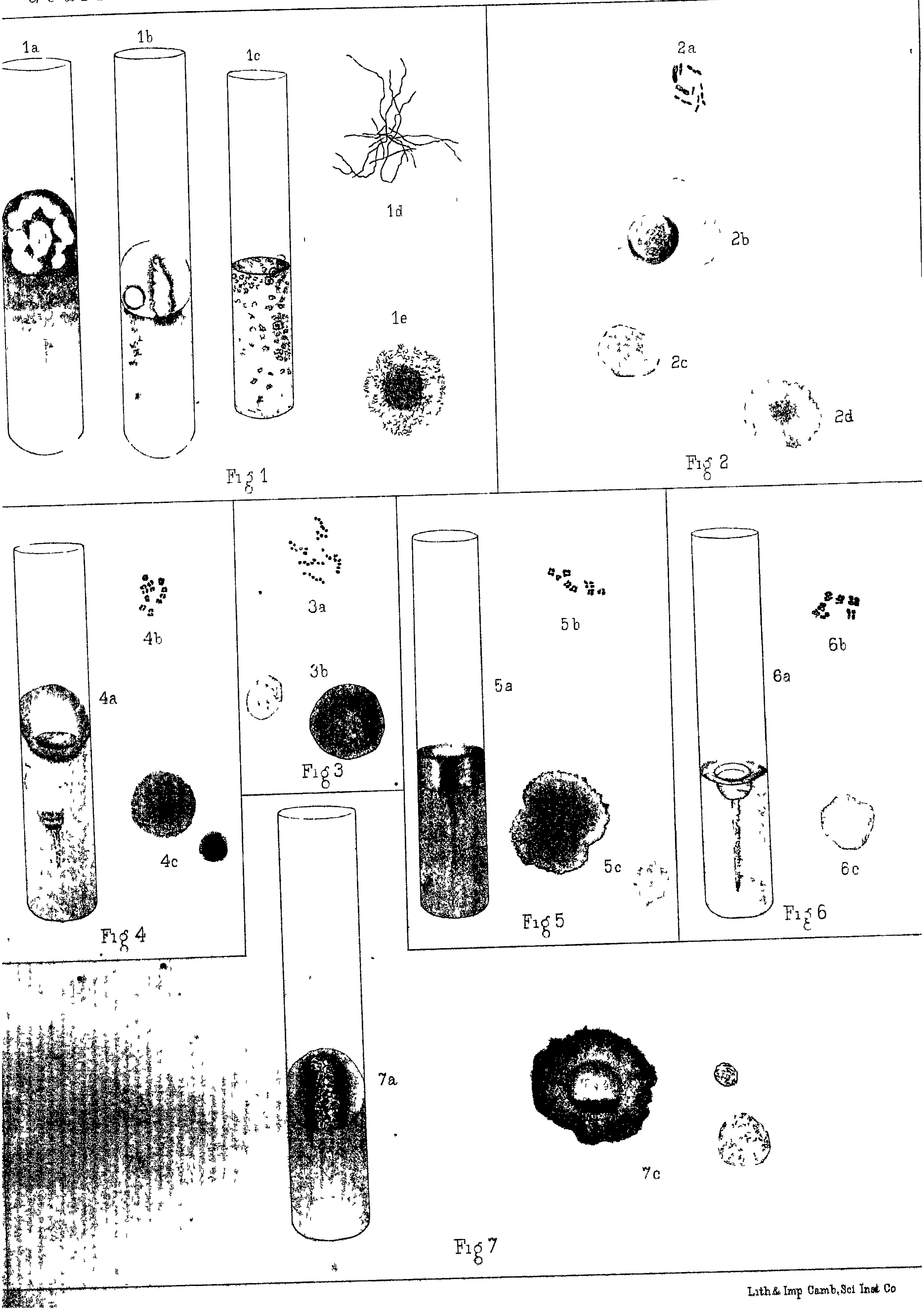
del et lith

Skull of Protorosaurus Speneri

West, Newman & Co. imp.



Lith & Imp Camb Sci Inst Co



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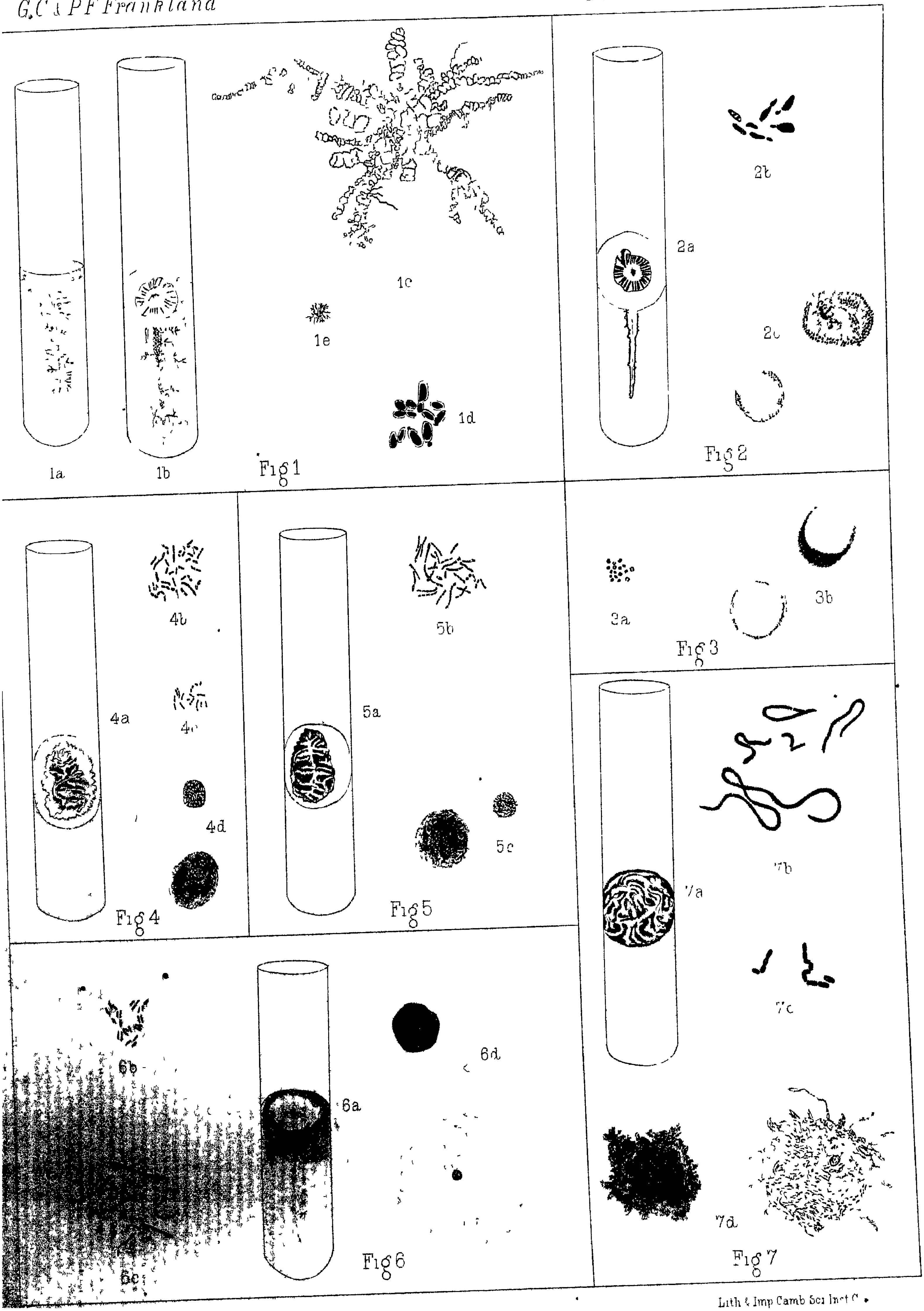
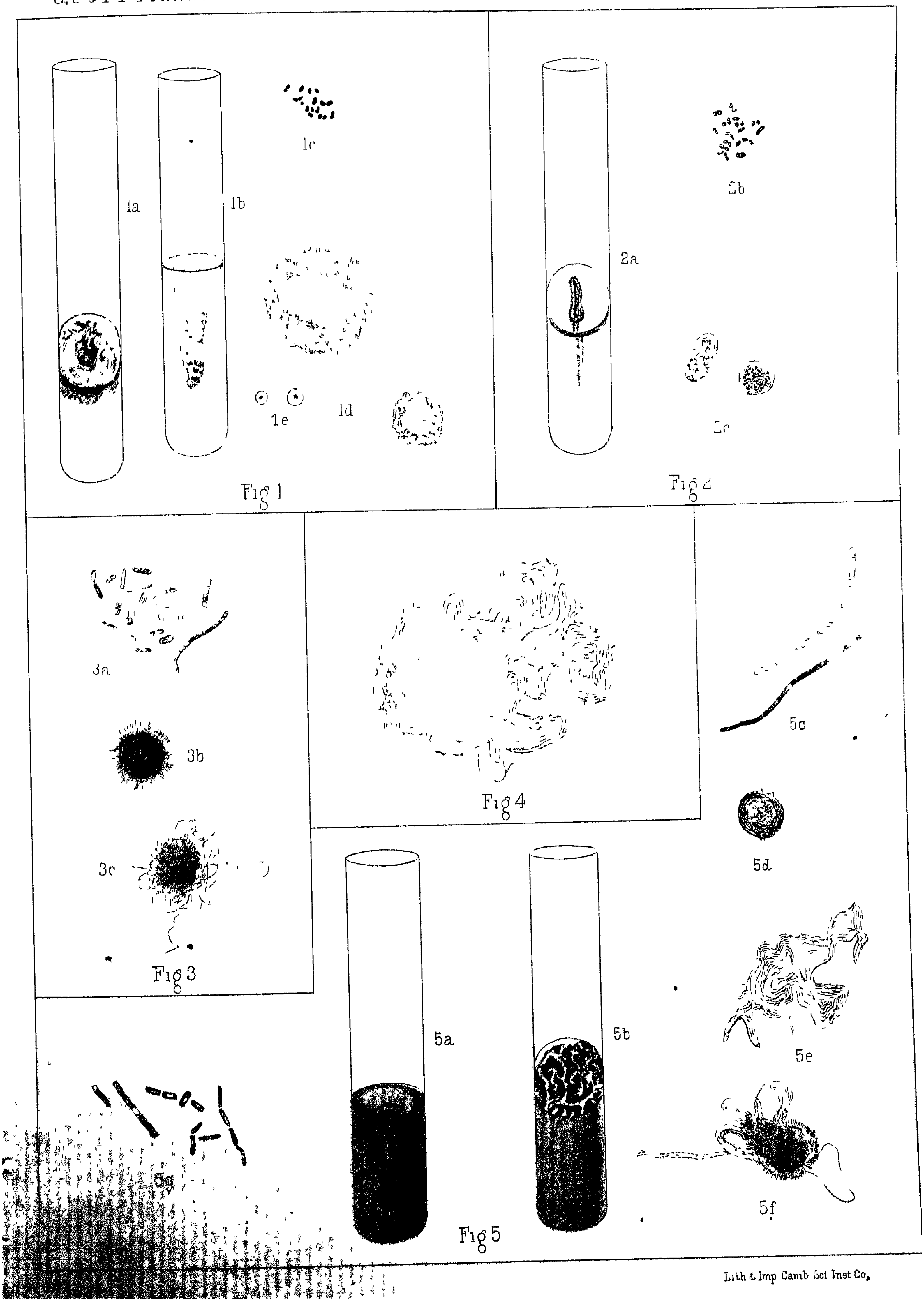


Lith & Imp Camb Sci Inst C°



Lith & Imp Camb Sci Inst Co.

Fig 1

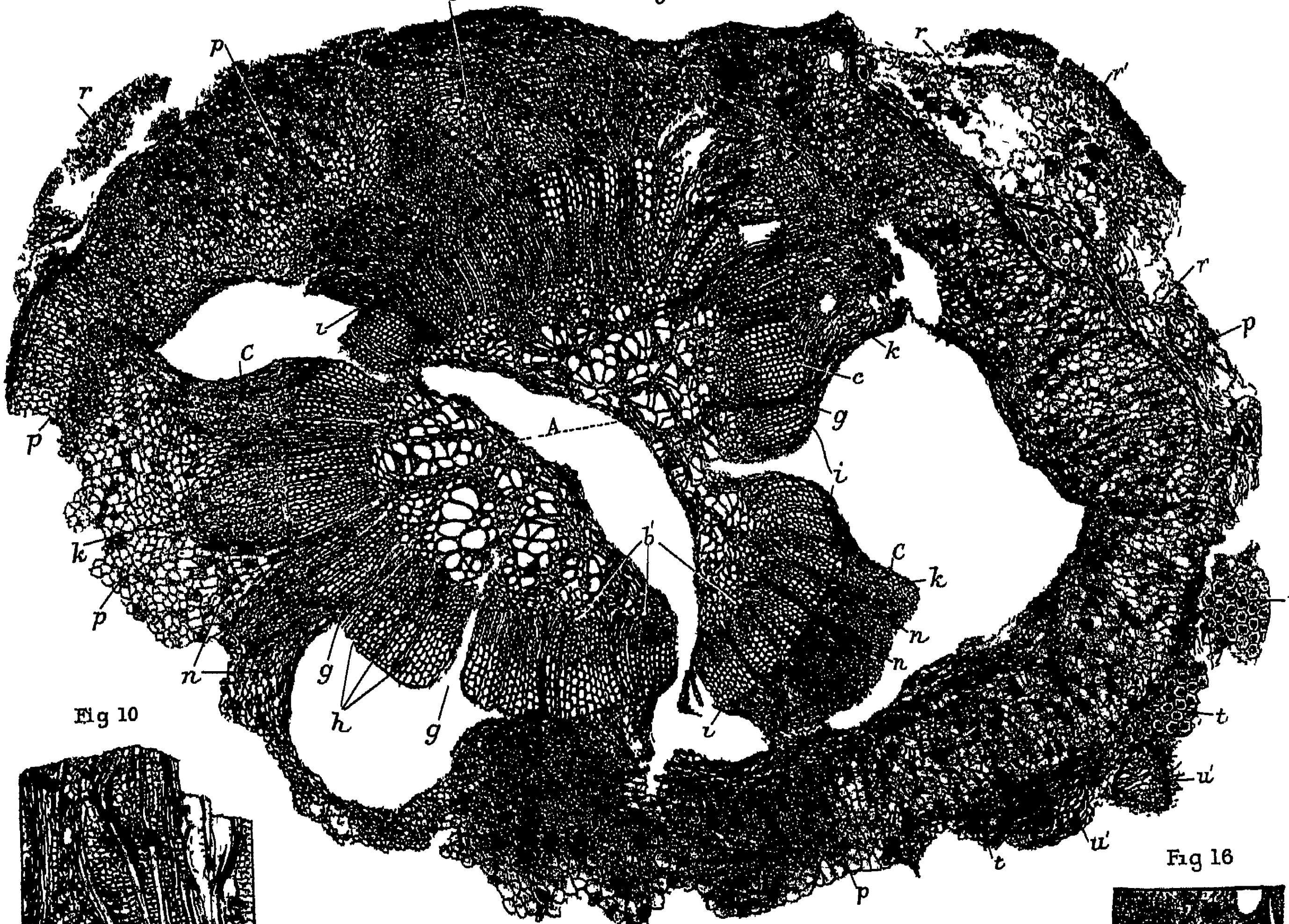
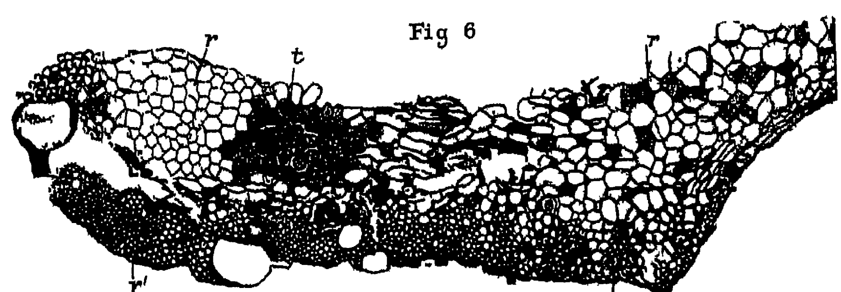


Fig 10

Fig 6



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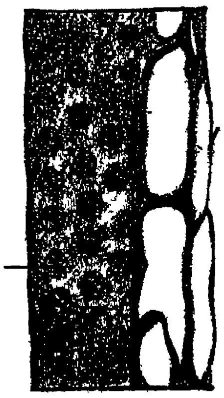

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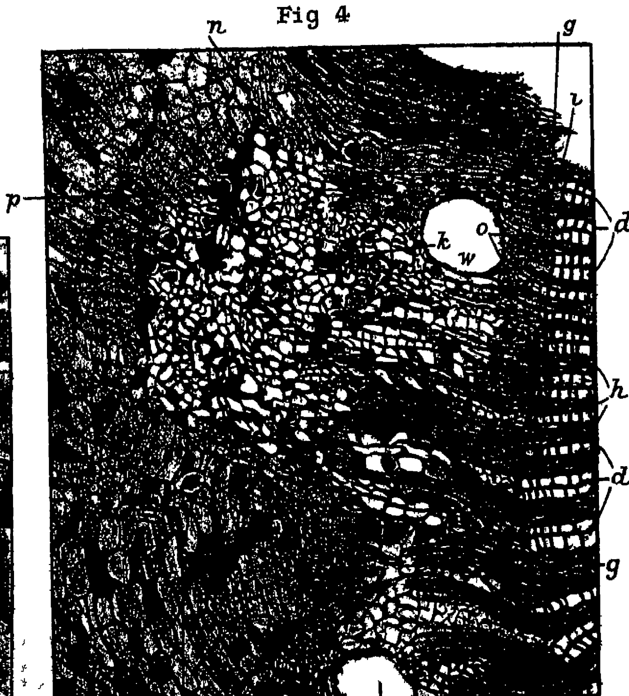

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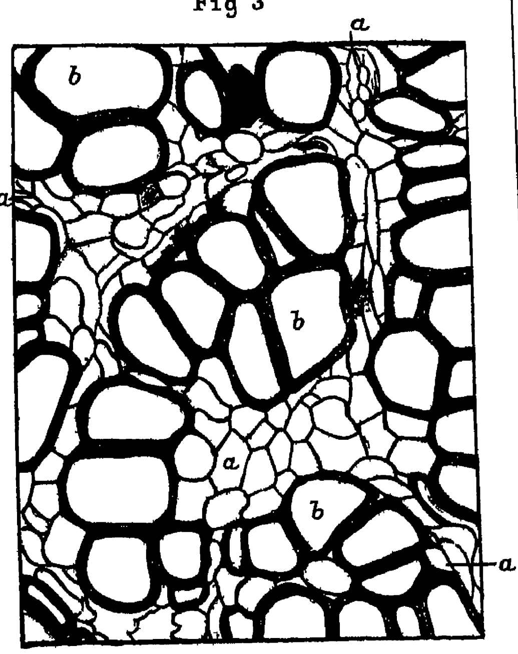

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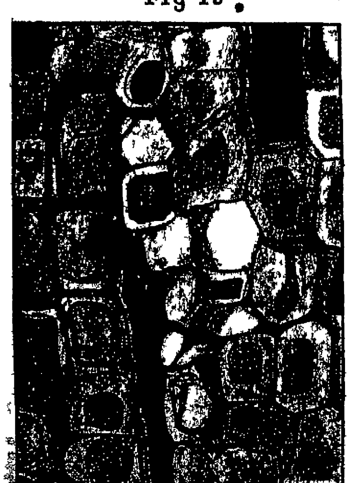

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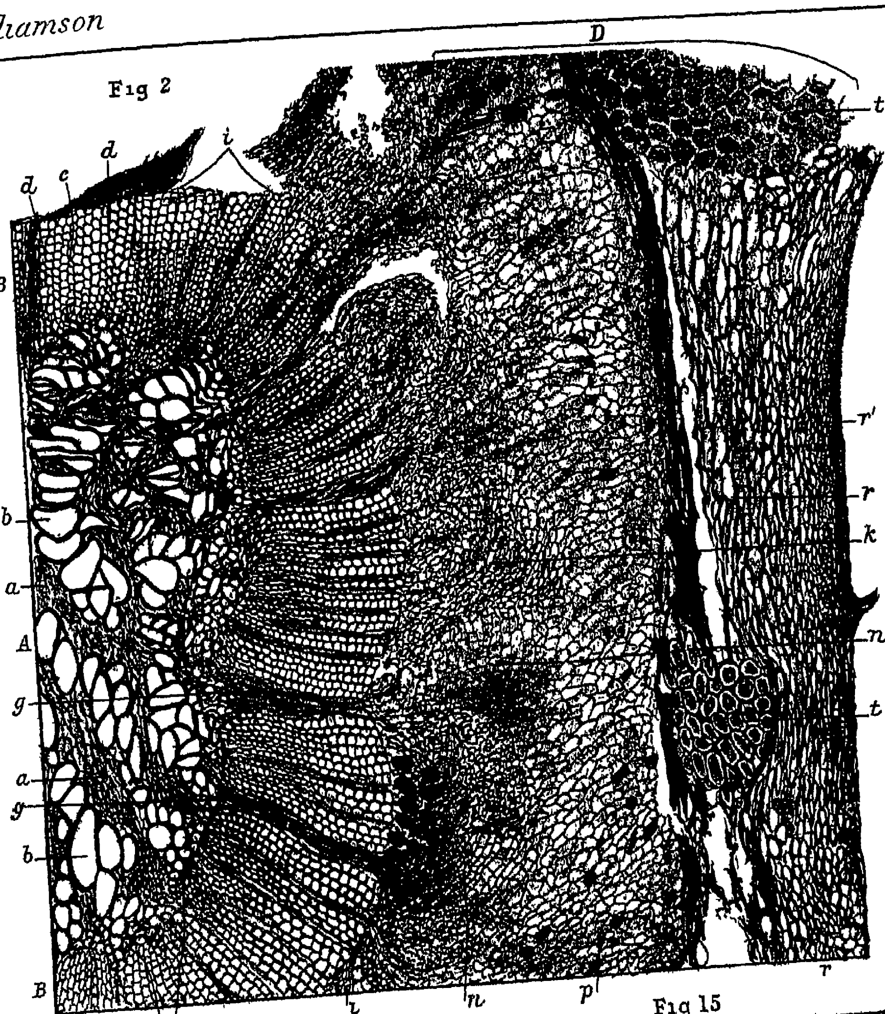
Fig 2



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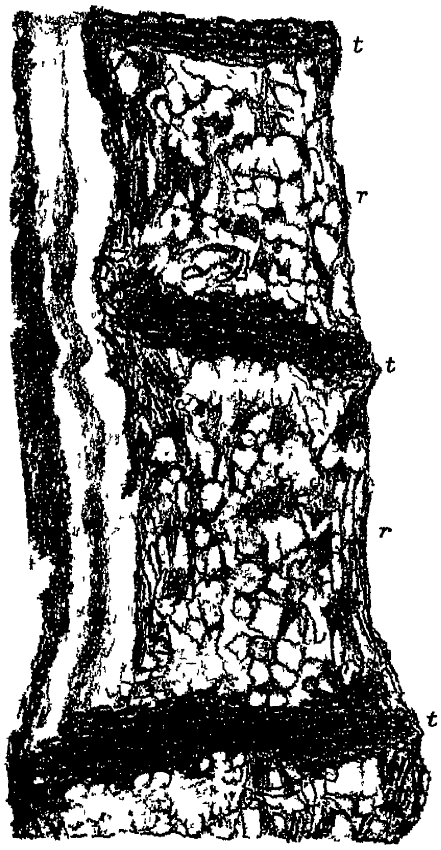

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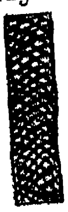

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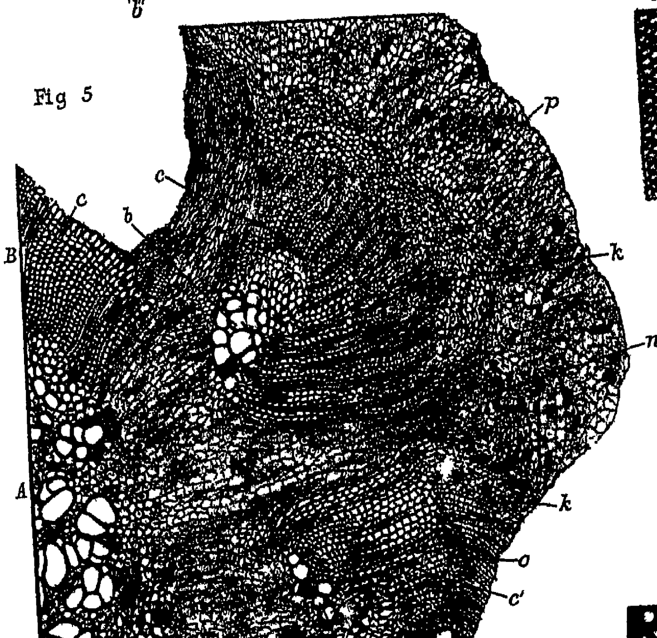

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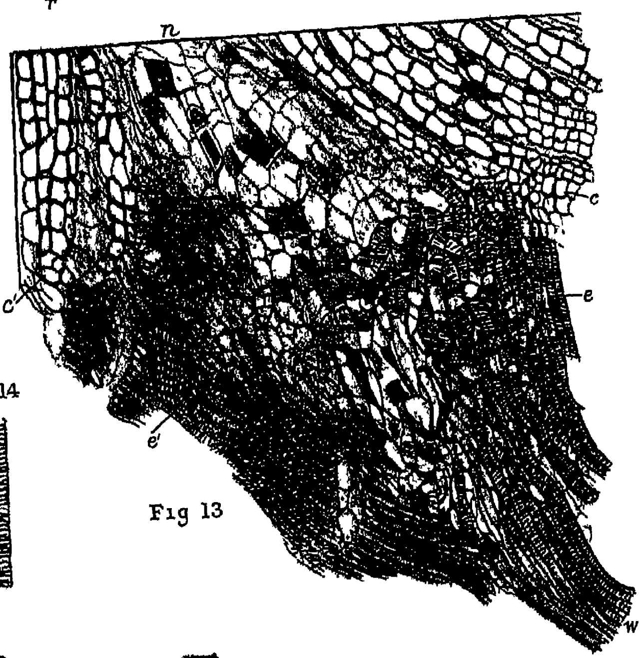
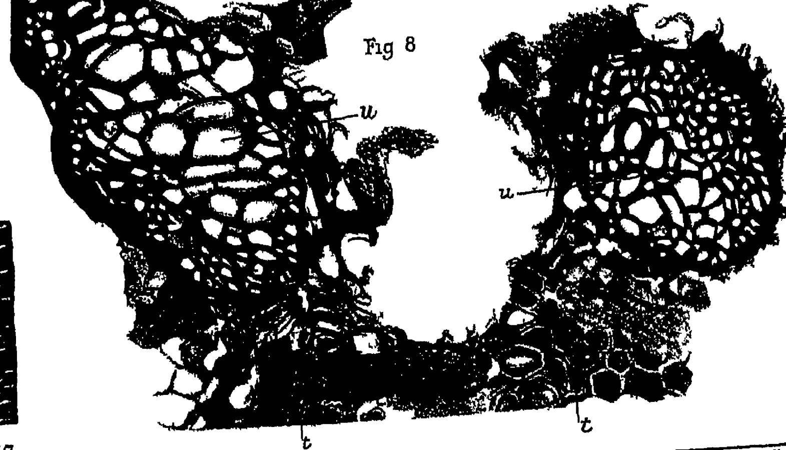
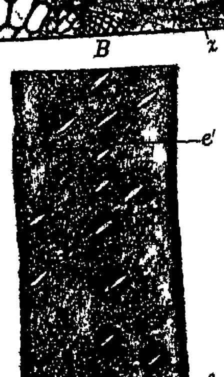


Fig 13

Fig 8





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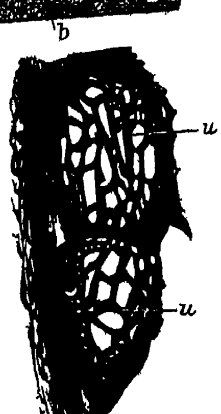

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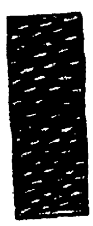


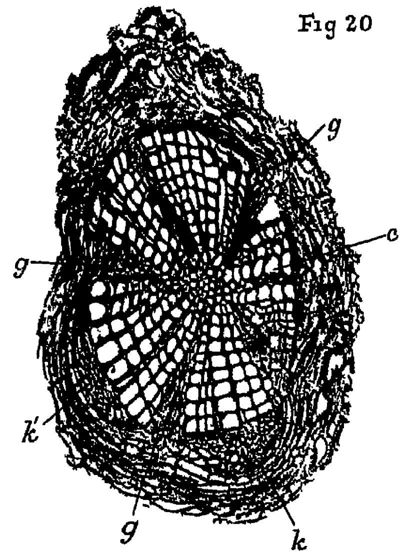
Fig 17

Whiteman & Bass, Photo-Litho London.

W-C Williamson, Del

Fig 9

Fig 20



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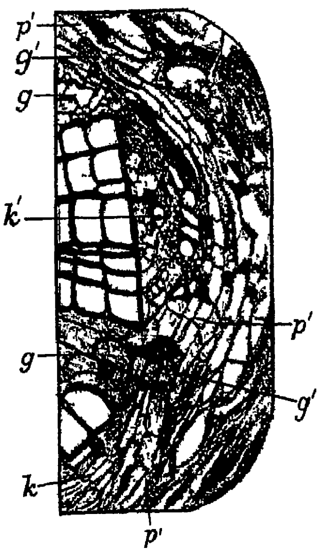

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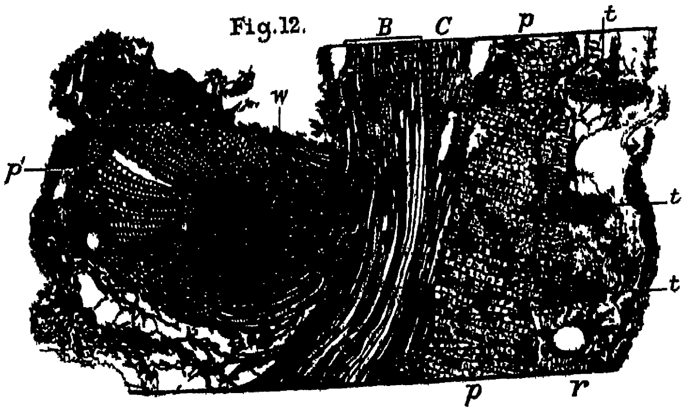
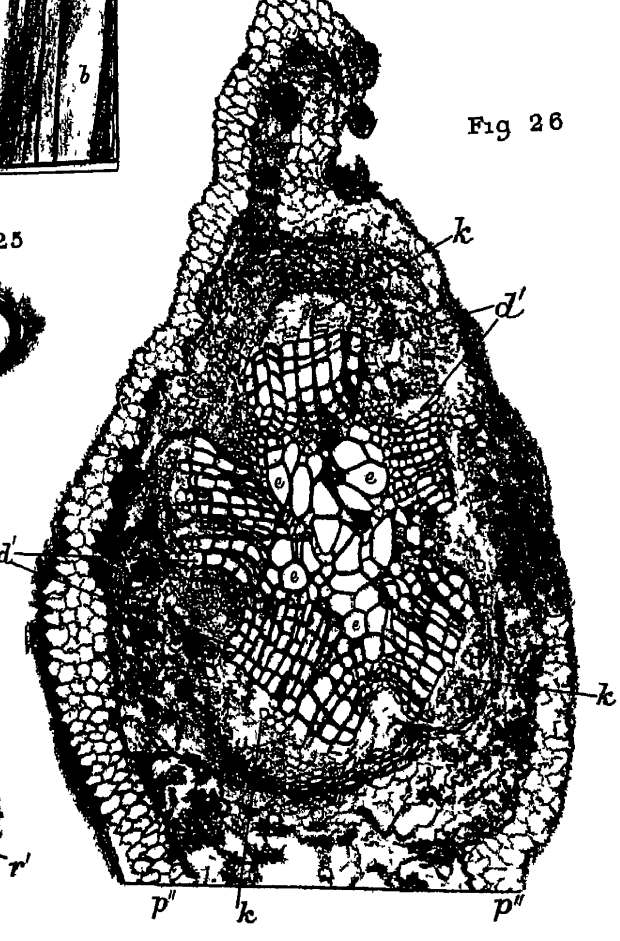


Fig 25

Fig 26



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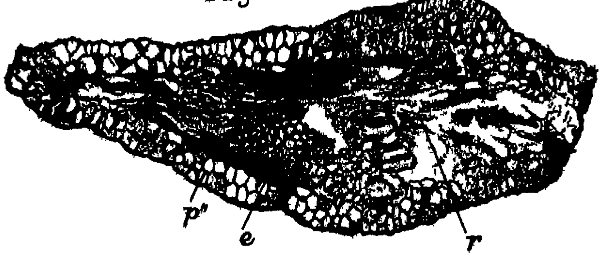

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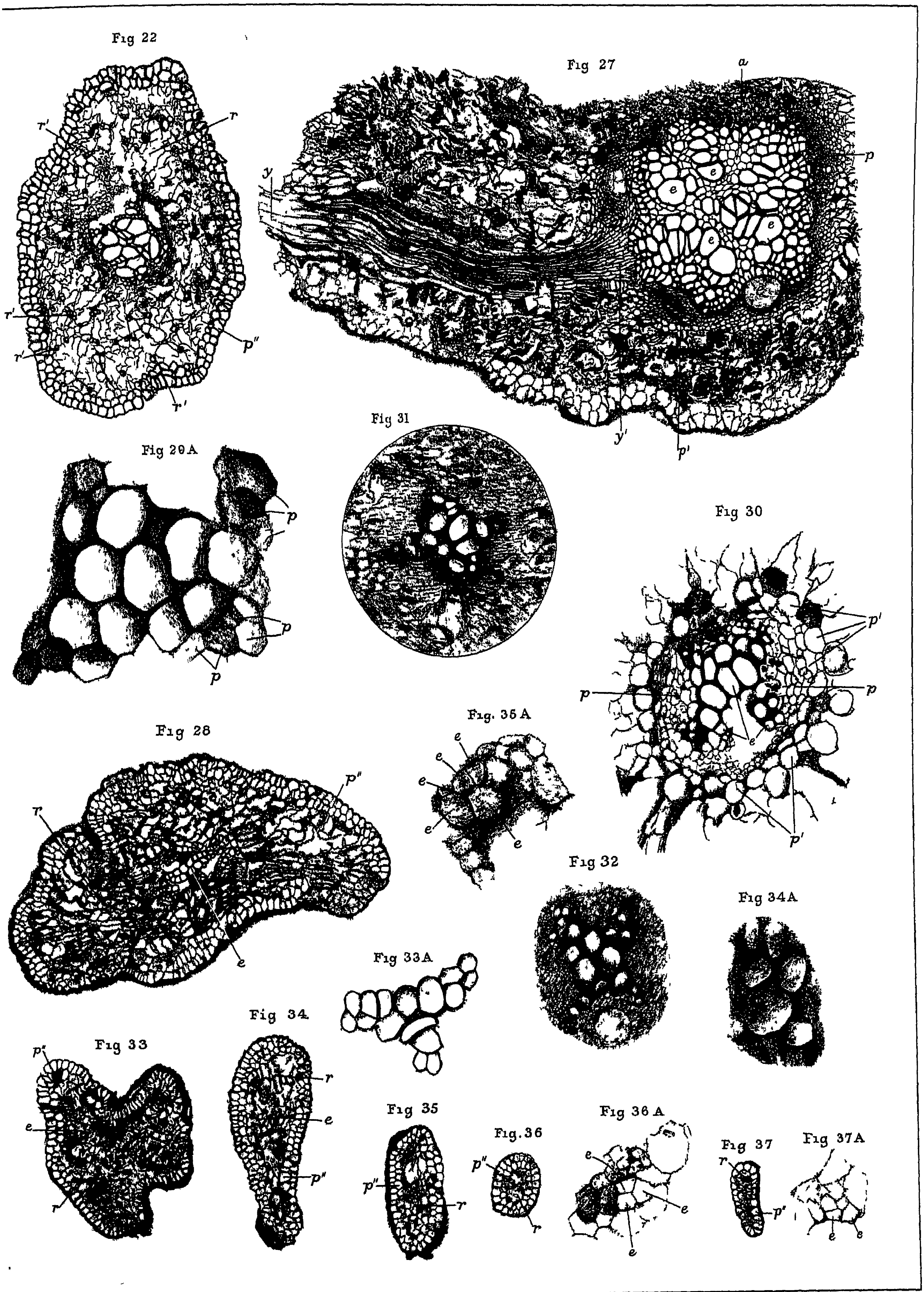
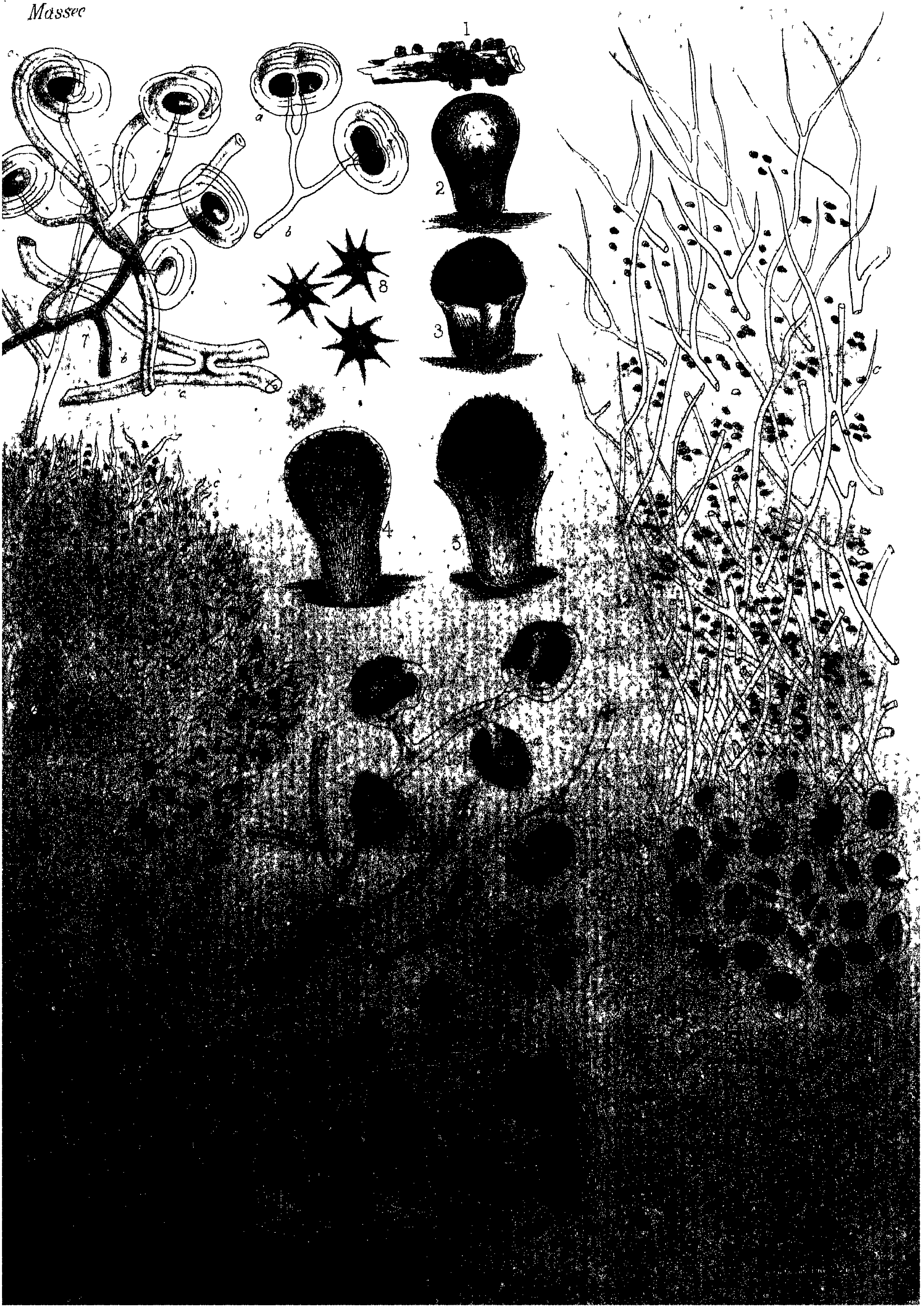


W C Williamson, Del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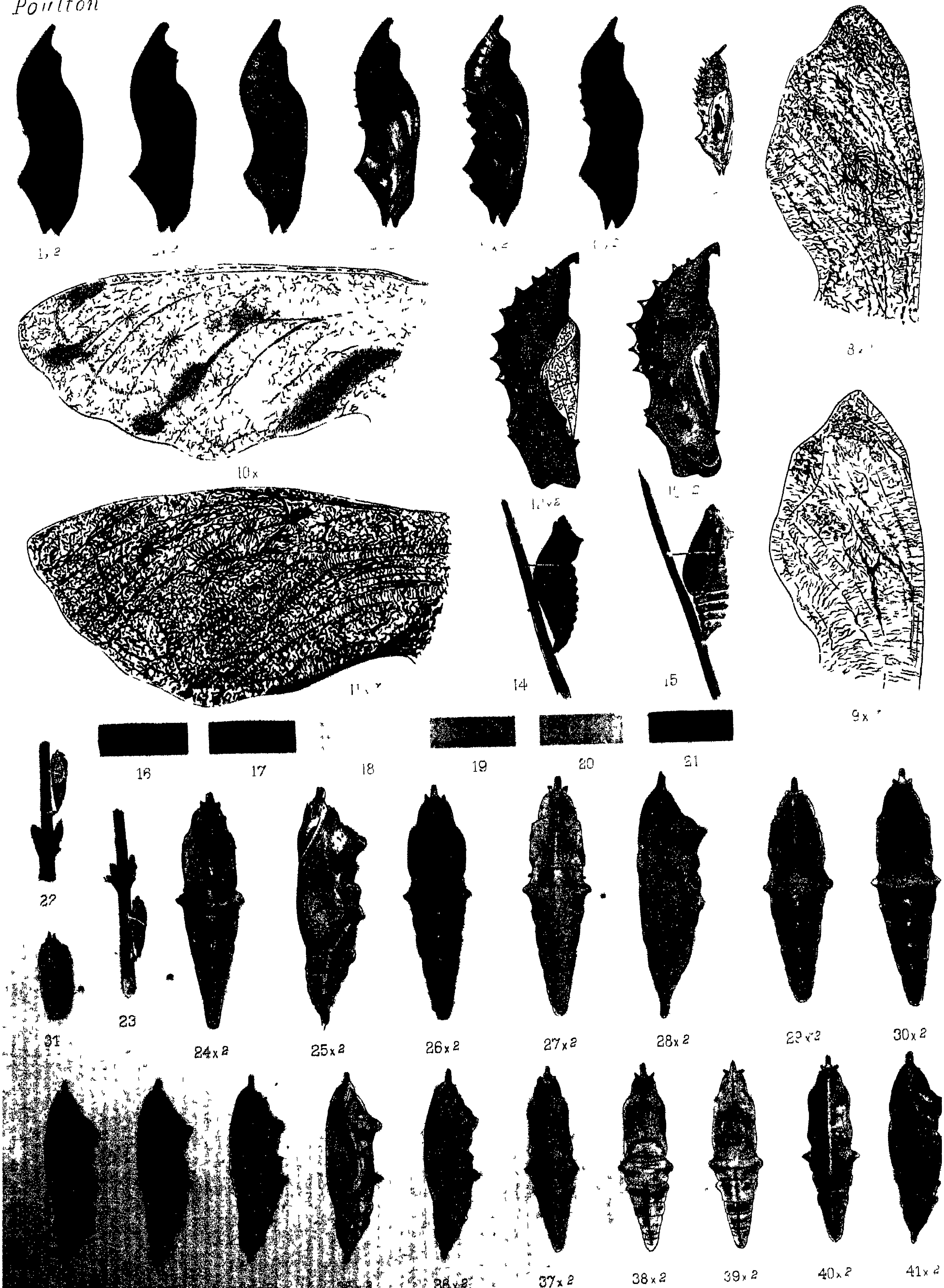
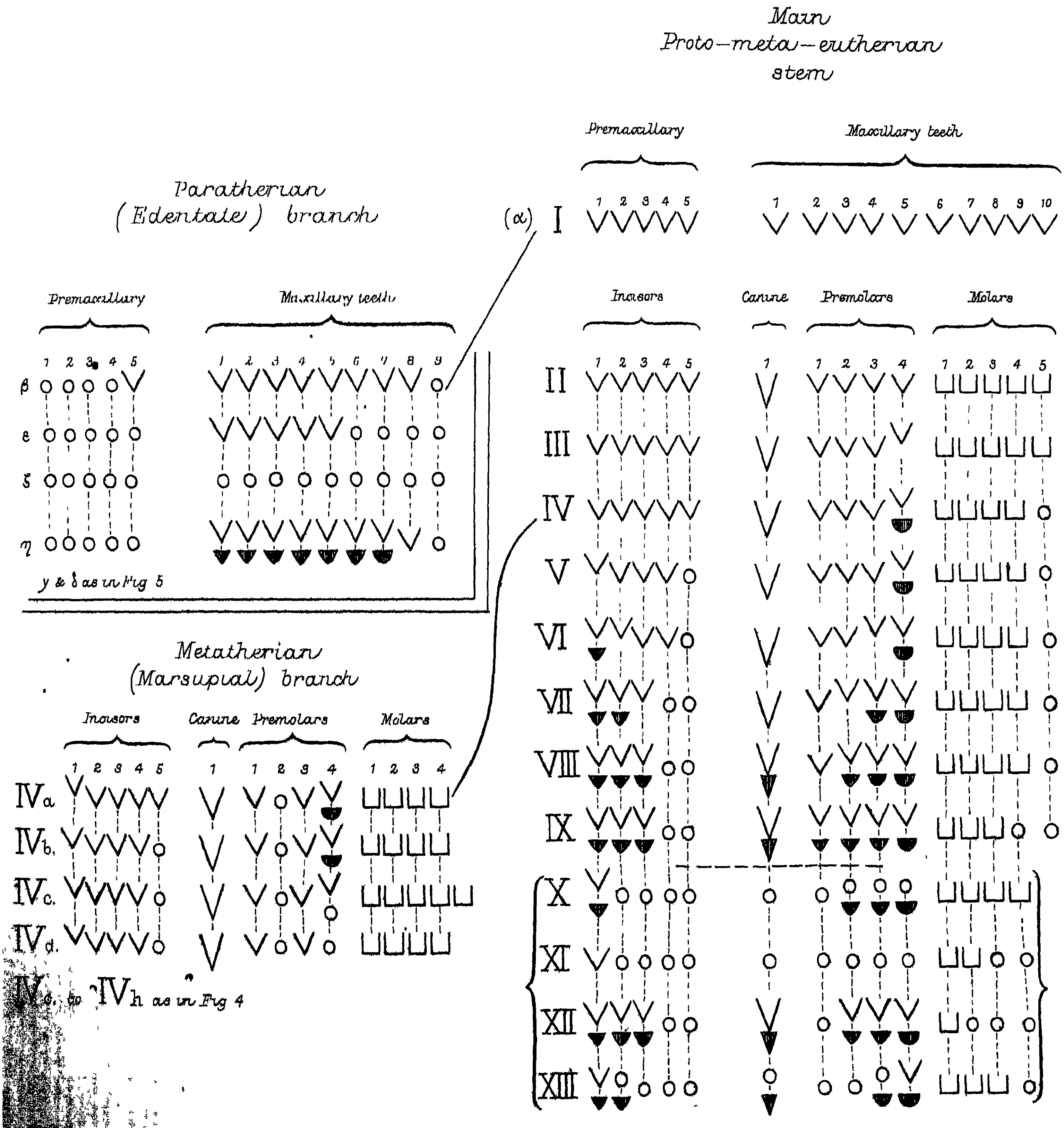


Lith & Imp Camb Sci Inst C



I. Generalized Prototherian dentition. II & III. Intermediate stages IV Generalized Metatherian dentition V–VII Intermediate stages VIII & IX. Generalized Eutherian dentition. X. Elephas XI. Hydromys XII. Felis XIII. Chæromys.

β. Dasypus. ε. Bradypodidæ, &c. ζ. Manidæ, &c. η. Tatusia

IV a. Didelphys, &c. IV. b. Thylacinus, &c. IV c. Myrmecobius IV. d. Dasyurus

West, Newman & Co. imp.

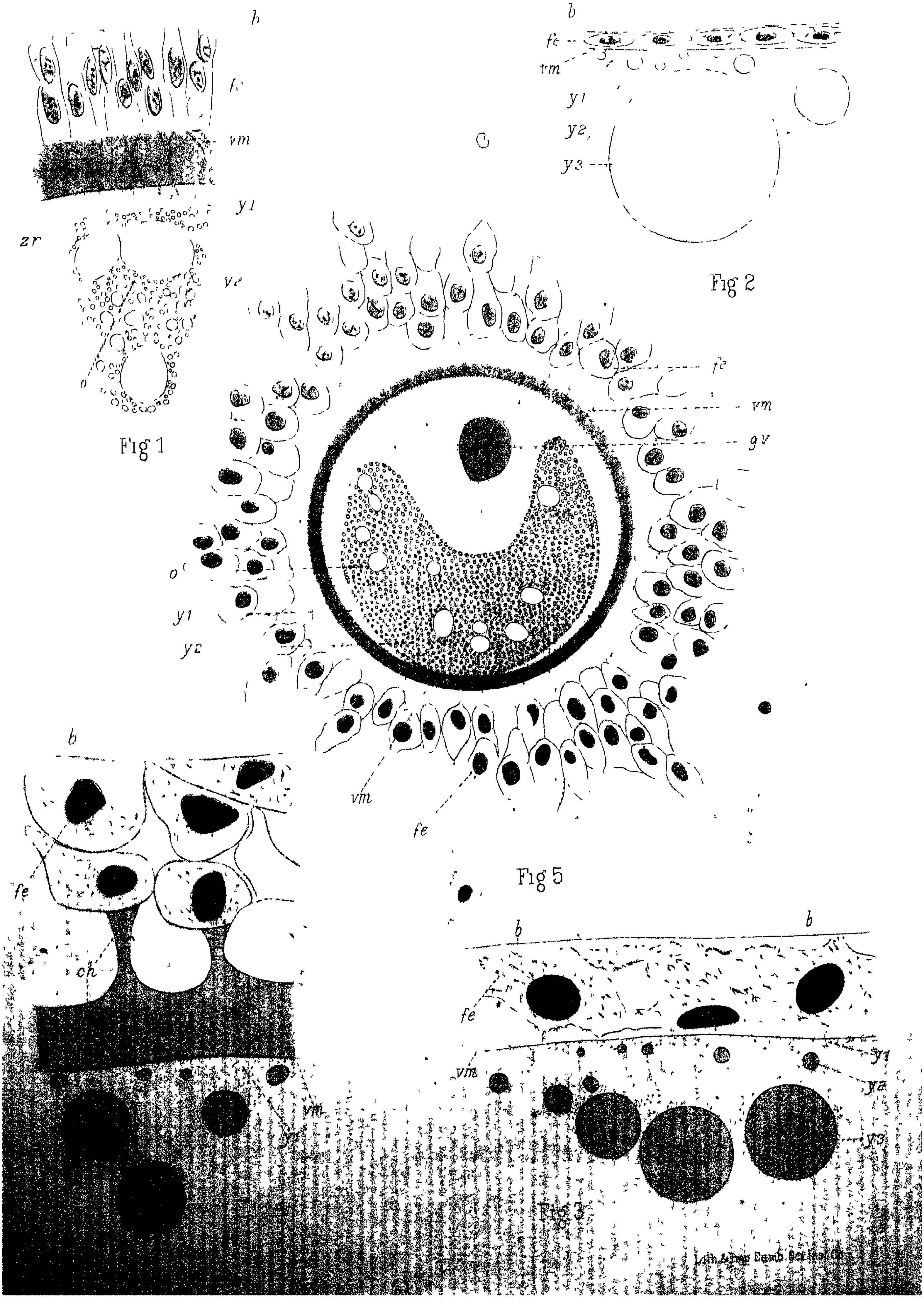


Fig 1

Fig 2

Fig 5

Fig 3

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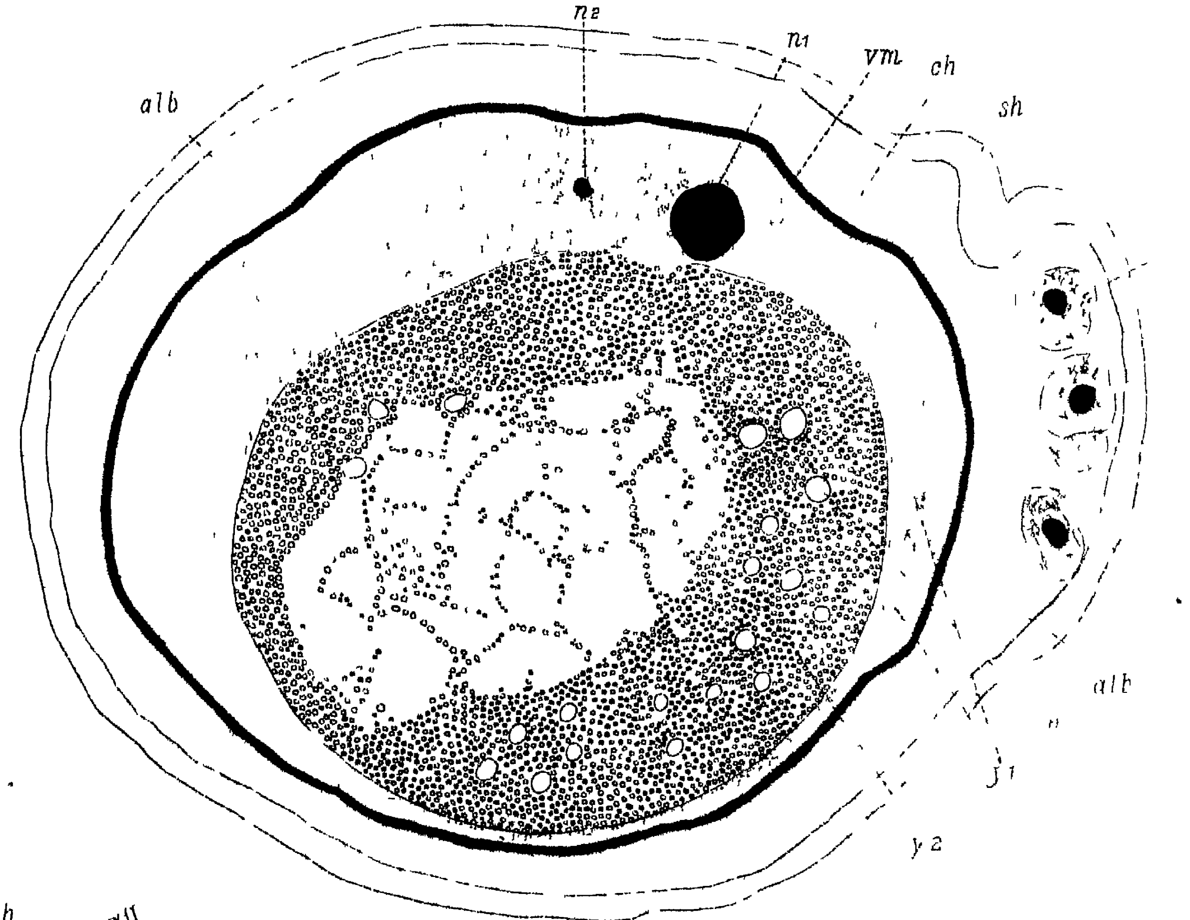


Fig 5

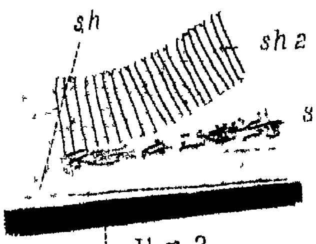


Fig 3

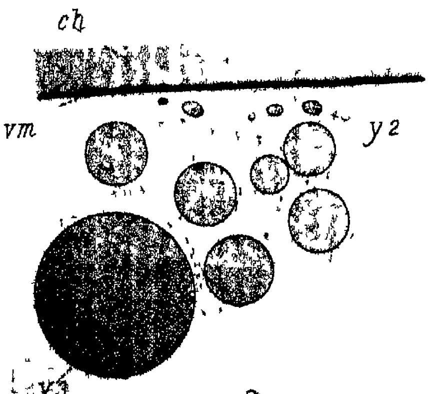


Fig 1

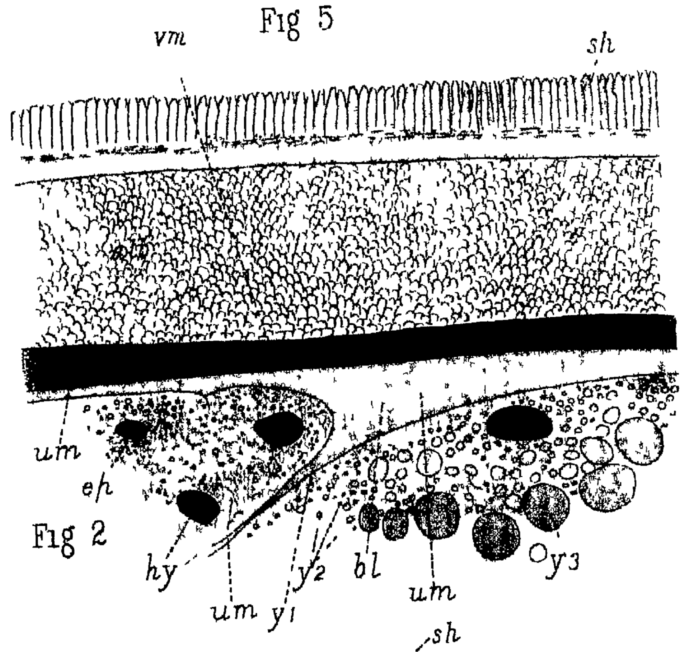


Fig 2

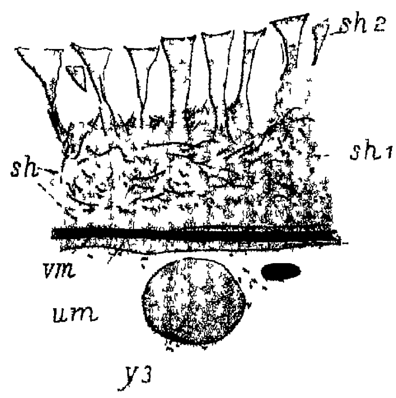
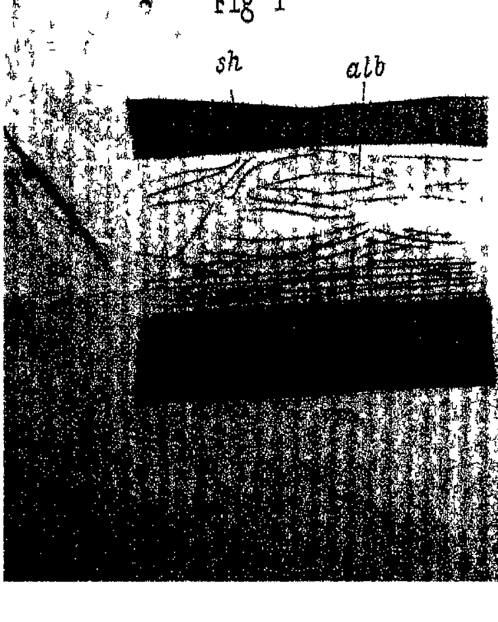
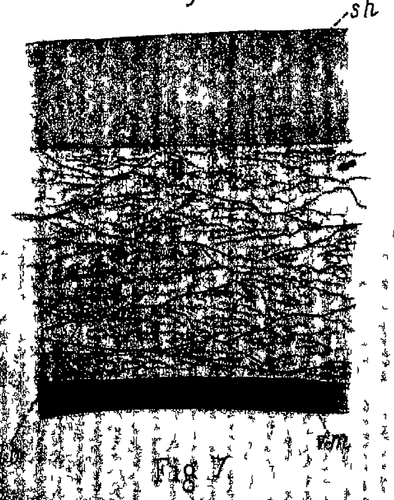


Fig 4





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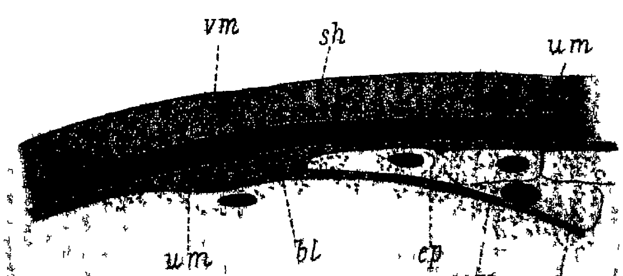

Fig 8.

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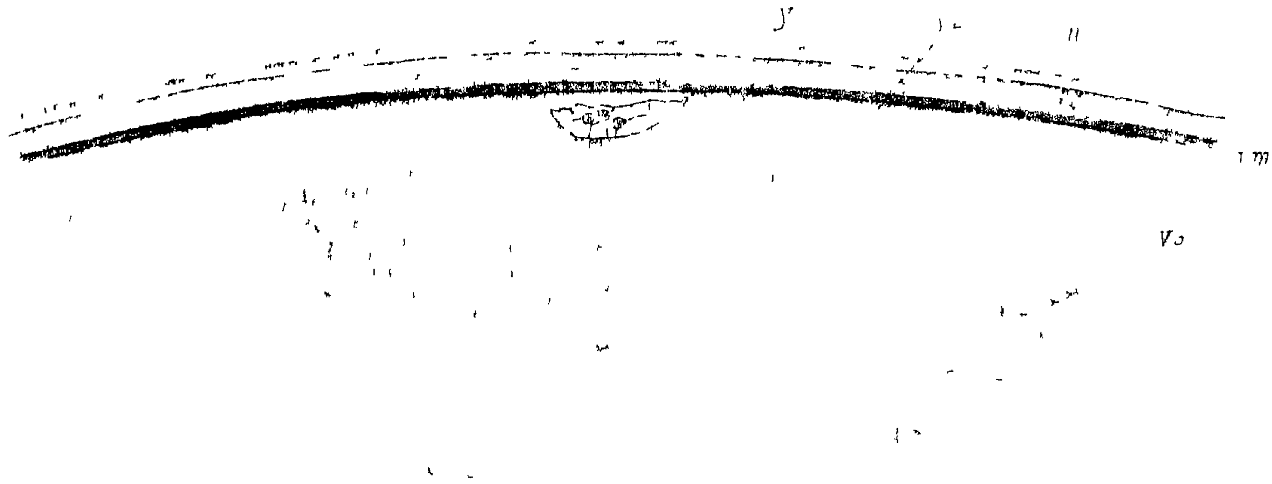
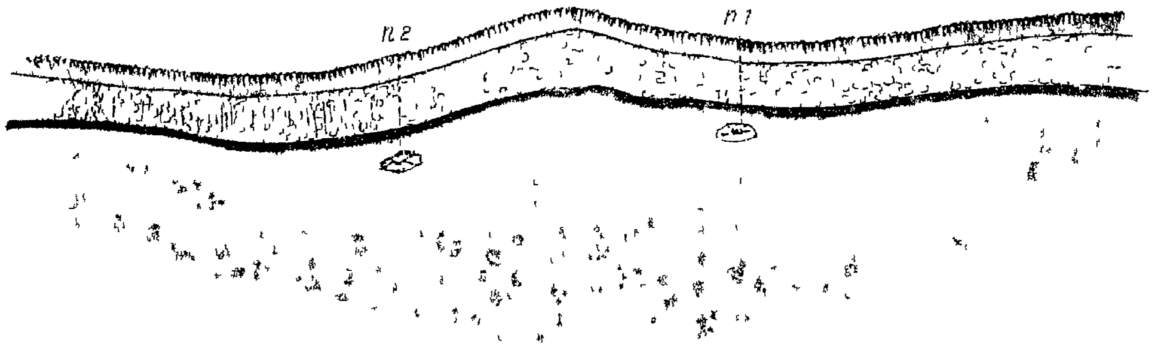


Fig 1



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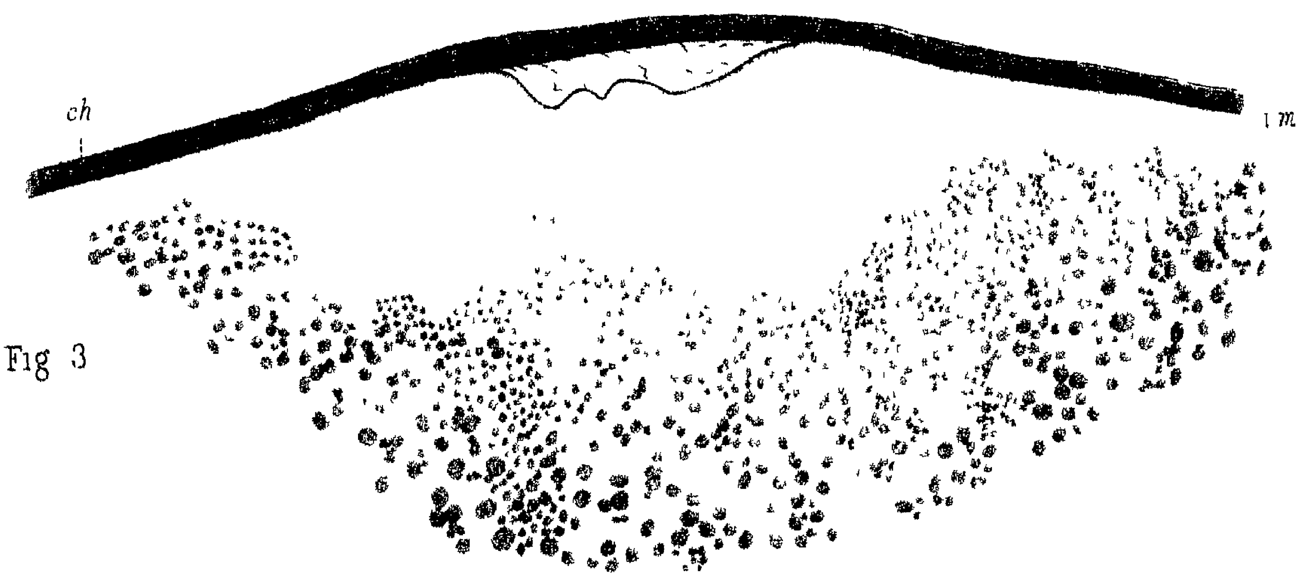

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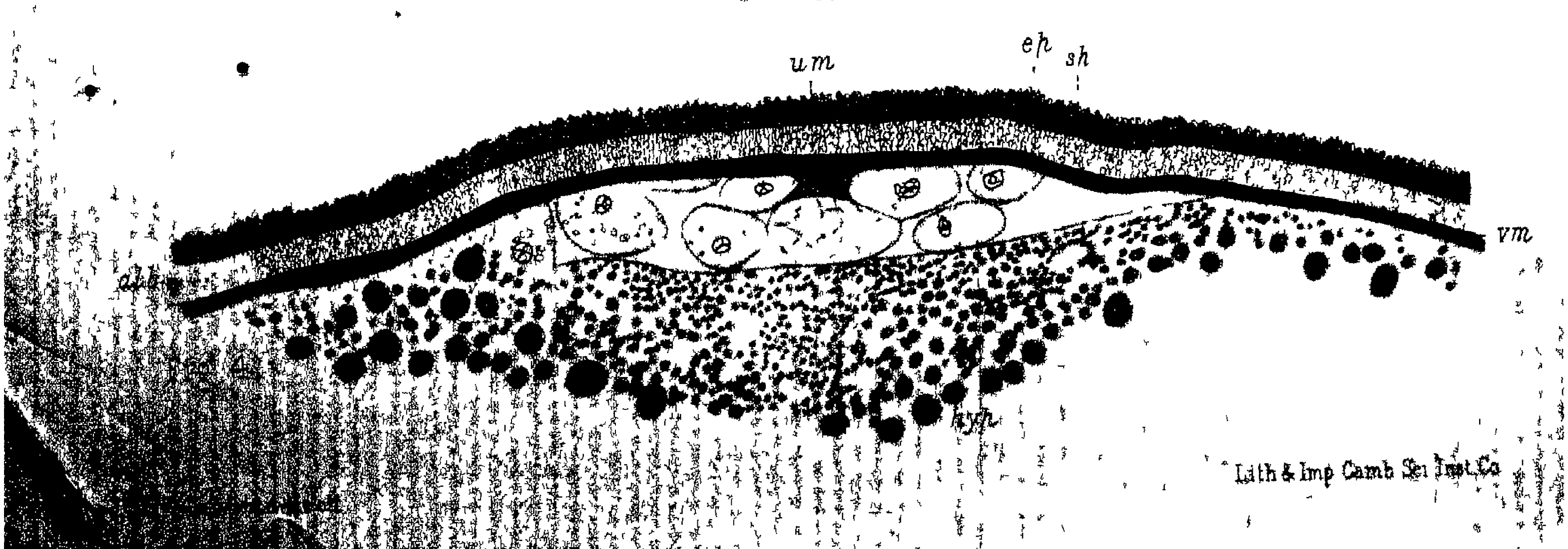

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Fig 1

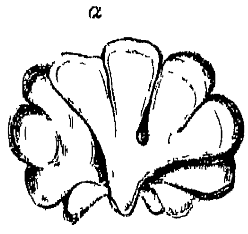


Fig 2

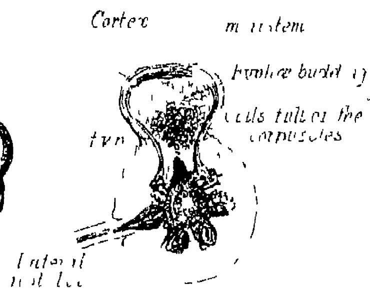
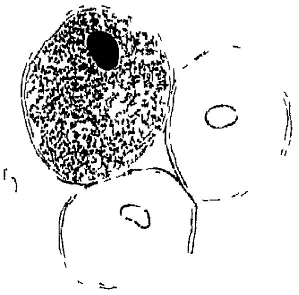
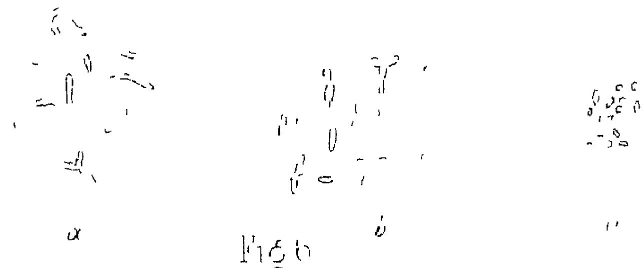


Fig 3

Fig 4

Fig 5





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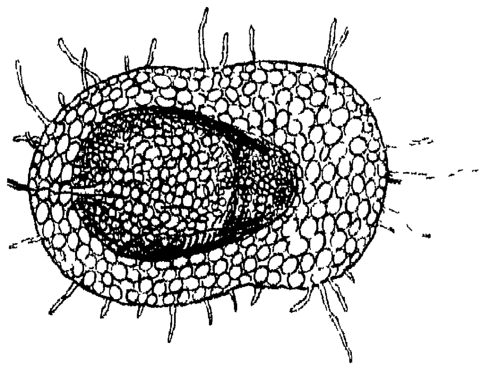

Fig 7